SIMPLE OPERATOR ASYNCHRONICITY OF AN INTEGRAL TRANSFORM WITH APPLICATIONS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *integral transform*

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if B, $A > 0$, then

$$[\mathcal{D}(w, \mu)(A) - \mathcal{D}(w, \mu)(B)](B - A)$$
$$= \int_0^\infty w(\lambda) \left(\int_0^1 \left[(\lambda + (1-t) B + tA)^{-1} (B - A) \right]^2 dt \right) d\mu(\lambda).$$

We also provide some sufficient conditions for the operators A, $B > 0$ such that the inequality

$$\mathcal{D}(w, \mu)(A) B + \mathcal{D}(w, \mu)(B) A \geq A\mathcal{D}(w, \mu)(A) + B\mathcal{D}(w, \mu)(B)$$

holds. Some examples for power and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.1}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty. \tag{1.2}$$

If f is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.1).

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A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty\frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.3}$$

where a, $b\in\mathbb{R}$, $c\geq0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.1).

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^r=\frac{\sin(r\pi)}{\pi}t\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.4}$$

Observe that for $t>0$, $t\neq1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.5}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.6}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.6) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.7}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.8}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\,\mathcal{D}(w_{\ln})(t), \; t > 0. \tag{1.9}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.10}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.11}$$

for $T > 0$.

From (1.8) we have the representation

$$T^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(T) \tag{1.12}$$

where $T > 0$ and from (1.9)

$$(T-1)^{-1}\ln T = \mathcal{D}(w_{\ln})(T) \tag{1.13}$$

provided $T > 0$ and $T - 1$ is invertible.

2. Main Results

In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (1.6) exists and is finite for all $t > 0$.

Theorem 3. *For all A, $B > 0$ we have the representations*

$$\begin{aligned}&[\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)](B-A) \\ &= \int_0^\infty w(\lambda)\left(\int_0^1 \left[(\lambda + (1-t)B + tA)^{-1}(B-A)\right]^2 dt\right) d\mu(\lambda)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&(B-A)[\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)] \\ &= \int_0^\infty w(\lambda)\left(\int_0^1 \left[(B-A)(\lambda + (1-t)B + tA)^{-1}\right]^2 dt\right) d\mu(\lambda).\end{aligned} \tag{2.2}$$

Proof. Observe that, for all A, $B > 0$

$$\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) = \int_0^\infty w(\lambda)\left[(\lambda + B)^{-1} - (\lambda + A)^{-1}\right] d\mu(\lambda). \tag{2.3}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t\to 0}\left[\frac{f(T+tS) - f(T)}{t}\right] = T^{-1}ST^{-1} \tag{2.4}$$

for T, $S > 0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$:

$\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Now, if we take in (2.6) $C=\lambda+B$, $D=\lambda+A$, then

$$\begin{aligned}&(\lambda+B)^{-1}-(\lambda+A)^{-1}\\&=\int_0^1((1-t)(\lambda+B)+t(\lambda+A))^{-1}(A-B)\\&\times((1-t)(\lambda+B)+t(\lambda+A))^{-1}\,dt\\&=\int_0^1(\lambda+(1-t)B+tA)^{-1}(A-B)(\lambda+(1-t)B+tA)^{-1}\,dt\end{aligned} \tag{2.7}$$

and by (2.3) we derive

$$\begin{aligned}&\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\\&=\int_0^\infty w(\lambda)\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-t)B+tA)^{-1}\,dt\right)d\mu(\lambda),\end{aligned} \tag{2.8}$$

for all A, $B>0$.

If we multiply at the right with $B-A$ we get

$$\begin{aligned}&[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)](B-A)\\&=\int_0^\infty w(\lambda)\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-t)B+tA)^{-1}(B-A)\,dt\right)d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[\int_0^1\left[(\lambda+(1-t)B+tA)^{-1}(B-A)\right]^2dt\right]d\mu(\lambda)\geq 0\end{aligned}$$

for all A, $B>0$, and the representation (2.1) is obtained.

The identity (2.2) follows by multiplying (2.8) at the left with $B-A$. □

In the following, in order to simplify terminology, when we write $T\geq 0$ we automatically assume that the operator T is selfadjoint.

We need the following lemmas:

Lemma 1. *Let A, $B>0$. The following statements are equivalent:*

(i) *For all $s\geq 0$,*

$$(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)\geq 2. \tag{2.9}$$

(ii) *For all* $s \geq 0$,

$$\int_0^1 \left[((1-t)A + tB + s)^{-1} (B-A) \right]^2 dt \geq 0.$$

(iii) *For all* $s \geq 0$,

$$(\ell_s(B) - \ell_s(A))(B-A) \geq 0,$$

where $\ell_s(t) = -(t+s)^{-1}$, $t > 0$.

Proof. From (2.7) we have, by multiplying at right with $B-A$ that

$$\begin{aligned}
&\left[(A+s)^{-1} - (B+s)^{-1} \right] (B-A) \\
&= \int_0^1 ((1-t)A + tB + s)^{-1} (B-A) ((1-t)A + tB + s)^{-1} (B-A)\, dt \\
&= \int_0^1 \left[((1-t)A + tB + s)^{-1} (B-A) \right]^2 dt
\end{aligned}$$

for all $s \geq 0$.

Also

$$\begin{aligned}
&\left[(A+s)^{-1} - (B+s)^{-1} \right] (B-A) \\
&= \left[(A+s)^{-1} - (B+s)^{-1} \right] [B+s-(A+s)] \\
&= (A+s)^{-1} (B+s) + (B+s)^{-1} (A+s) - 2
\end{aligned}$$

for all $s \geq 0$.

Therefore

$$\begin{aligned}
&(\ell_s(B) - \ell_s(A))(B-A) \\
&= (A+s)^{-1} (B+s) + (B+s)^{-1} (A+s) - 2 \\
&= \int_0^1 \left[((1-t)A + tB + s)^{-1} (B-A) \right]^2 dt
\end{aligned} \tag{2.10}$$

for all $s \geq 0$.

The identity (2.10) reveals that the statements (i), (ii) and (iii) are equivalent. □

In the note [2] Fujii and Nakamoto proved the following inequality:

Lemma 2. *If* C, $D > 0$ *and* $CD^{-1} + DC^{-1}$ *is selfadjoint, then*

$$CD^{-1} + DC^{-1} \geq 2. \tag{2.11}$$

Proof. Indeed, as shown in [2], if we put $T = CD^{-1}$, then $V = T + T^{-1}$ is selfadjoint by the assumption. Note that the spectrum $\operatorname{Sp}(T)$ of T is included in $(0, \infty)$, because C, $D > 0$ and $\operatorname{Sp}(T) = \operatorname{Sp}\left(C^{1/2} D^{-1} C^{1/2}\right)$. Since $\operatorname{Sp}(V) = \left\{ t + \frac{1}{t},\ t \in \operatorname{Sp}(T) \right\}$ by the spectral mapping theorem for rational functions, hence we have $T + T^{-1} \geq 2$. □

As a consequence, they proved that, if

(i') Operator $A(B+s)^{-1} + B(A+s)^{-1}$ *is selfadjoint for all* $s \geq 0$,

then

$$(B-A)(f(B)-f(A)) \geq 0$$

for all f operator monotone functions on $(0,\infty)$.

Lemma 3. *Let* A, $B>0$, *then the statements (i) and (i') are equivalent.*

Proof. Notice that for all $s\geq 0$,

$$(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s) = (A+s)^{-1}B+(B+s)^{-1}A+s(A+s)^{-1}+s(B+s)^{-1}. \tag{2.12}$$

Also, the operator $s(A+s)^{-1}+s(B+s)^{-1}$ is selfadjoint for $s\geq 0$.

If the statement *(i)* holds, then $(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)$ is selfadjoint and by (2.12) we must have that $(A+s)^{-1}B+(B+s)^{-1}A$ is selfadjoint, which shows that

$$\left((A+s)^{-1}B+(B+s)^{-1}A\right)^{*}=B(A+s)^{-1}+A(B+s)^{-1}$$

is selfadjoint, namely *(i')* is true.

If the statement *(i')* holds, then by (2.12) we get

$$(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)$$

is selfadjoint and by (2.11) for $C=(A+s)^{-1}$, $D=(B+s)^{-1}$ we obtain the inequality (2.9), namely *(i)* is true. □

We define the class of operators

$$\mathfrak{Cl}_{(0,\infty)}(H):=\{(A,B)\mid A,\ B>0 \text{ and satisfy condition } (i')\}.$$

We observe that if $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$ then $(B,A)\in\mathfrak{Cl}_{(0,\infty)}(H)$.

Also if $AB=BA$, A, $B>0$, then $U_s:=(A+s)^{-1}(B+s)$ and $U_s^{-1}=(B+s)^{-1}(A+s)$ are selfadjoint and since $U_s+U_s^{-1}\geq 2$, $s\geq 0$ we derive that $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$. Therefore, if $\mathfrak{Co}_{(0,\infty)}(H)$ is the class of all pairs of commutative operators A, $B>0$, then we have

$$\emptyset\neq\mathfrak{Co}_{(0,\infty)}(H)\subset\mathfrak{Cl}_{(0,\infty)}(H). \tag{2.13}$$

Theorem 4. *Let* $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$. *Then*

$$[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)](B-A)=(B-A)[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)]\leq 0$$

and

$$\mathcal{D}(w,\mu)(A)B+\mathcal{D}(w,\mu)(B)A\geq A\mathcal{D}(w,\mu)(A)+B\mathcal{D}(w,\mu)(B). \tag{2.14}$$

Proof. By (ii) from Lemma 1 we have

$$\int_0^\infty w(\lambda)\left(\int_0^1\left[(\lambda+(1-t)B+tA)^{-1}(B-A)\right]^2dt\right)d\mu(\lambda)\geq 0$$

and by (2.1) we obtain

$$[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)](B-A)\leq 0.$$

Since $L:=[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)](B-A)$ is selfadjoint then

$$\begin{aligned} L^{*} &= \{[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)](B-A)\}^{*} \\ &= (B-A)^{*}[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)]^{*} \\ &= (B-A)[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)]=L \end{aligned}$$

and the theorem is proved. □

Remark 1. *By utilising the representation (1.12) we derive the following power inequalities for* $r \in (0,1]$

$$A^{r-1}B + B^{r-1}A \geq A^r + B^r$$

for all $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$.

The case of operator monotone functions is as follows:

Corollary 1. *Assume that* $f : [0,\infty) \to \mathbb{R}$ *is an operator monotone function on* $[0,\infty)$. *If* $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$, *then*

$$f(A)A^{-1}B + f(B)B^{-1}A + 2f(0) \geq f(0)\left(A^{-1}B + B^{-1}A\right) + f(B) + f(A). \tag{2.15}$$

In particular, if $f(0) = 0$, *then*

$$f(A)A^{-1}B + f(B)B^{-1}A \geq f(B) + f(A). \tag{2.16}$$

Proof. If $f : [0,\infty) \to \mathbb{R}$ is operator monotone, then by (1.1)

$$\frac{f(t) - f(0)}{t} - b = \mathcal{D}(\ell,\mu)(t), \ t > 0$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda > 0$, which proves the asynchronicity properties.

Also, by (2.1) we get

$$\begin{aligned}
0 &\leq [\mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)](B - A) \\
&= \left(f(A)A^{-1} - f(0)A^{-1} - f(B)B^{-1} + f(0)B^{-1}\right)(B - A) \\
&= f(A)A^{-1}B - f(0)A^{-1}B - f(B) + f(0) \\
&- \left(f(A) - f(0) - f(B)B^{-1}A + f(0)B^{-1}A\right) \\
&= f(A)A^{-1}B - f(0)A^{-1}B - f(B) + f(0) \\
&- f(A) + f(0) + f(B)B^{-1}A - f(0)B^{-1}A \\
&= f(A)A^{-1}B + f(B)B^{-1}A + 2f(0) - f(0)\left(A^{-1}B + B^{-1}A\right) \\
&- f(B) - f(A),
\end{aligned}$$

which implies (2.15). □

Remark 2. *If we take in Corollary 1* $f(t) = \ln(t+1)$, *then we get the logarithmic inequalities*

$$\left[A^{-1}\ln(A+1)\right]B + \left[B^{-1}\ln(B+1)\right]A \geq \ln(B+1) + \ln(A+1) \tag{2.17}$$

for all $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$.

The case of operator convex functions is as follows:

Corollary 2. *Assume that* $f : [0,\infty) \to \mathbb{R}$ *is an operator convex function on* $[0,\infty)$. *For all* $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$ *we have*

$$\begin{aligned}
&f(A)A^{-2}B + f(B)B^{-2}A + f(0)\left(B^{-1} + A^{-1}\right) + 2f'_+(0) \\
&\geq f(B)B^{-1} + f(A)A^{-1} + f(0)\left(A^{-2}B + B^{-2}A\right) \\
&+ f'_+(0)\left(A^{-1}B + B^{-1}A\right).
\end{aligned} \tag{2.18}$$

In particular, if $f(0)=0$, *then*

$$f(A)A^{-2}B+f(B)B^{-2}A+2f'_{+}(0) \geq f'_{+}(0)\left(A^{-1}B+B^{-1}A\right)+f(B)B^{-1}+f(A)A^{-1}. \tag{2.19}$$

Proof. If $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$, then by (1.3) we have that

$$\frac{f(t)-f(0)-f'_{+}(0)t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda>0$, which proves the asynchronicity properties.

Also, by (2.1) we get

$$\begin{aligned}
0 &\leq [\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)](B-A)\\
&= \left(f(A)A^{-2}-f(0)A^{-2}-f'_{+}(0)A^{-1}-f(B)B^{-2}+f(0)B^{-2}+f'_{+}(0)B^{-1}\right)\\
&\times(B-A)\\
&= f(A)A^{-2}B-f(0)A^{-2}B-f'_{+}(0)A^{-1}B-f(B)B^{-1}+f(0)B^{-1}+f'_{+}(0)\\
&-\left(f(A)A^{-1}-f(0)A^{-1}-f'_{+}(0)-f(B)B^{-2}A+f(0)B^{-2}A+f'_{+}(0)B^{-1}A\right)\\
&= f(A)A^{-2}B-f(0)A^{-2}B-f'_{+}(0)A^{-1}B-f(b)B^{-1}+f(0)B^{-1}+f'_{+}(0)\\
&-f(A)A^{-1}+f(0)A^{-1}+f'_{+}(0)+f(B)B^{-2}A-f(0)B^{-2}A-f'_{+}(0)B^{-1}A\\
&= f(A)A^{-2}B+f(B)B^{-2}A+f(0)B^{-1}+f(0)A^{-1}+f'_{+}(0)+f'_{+}(0)\\
&-f(0)A^{-2}B-f(0)B^{-2}A-f'_{+}(0)A^{-1}B-f'_{+}(0)B^{-1}A\\
&-f(B)B^{-1}-f(A)A^{-1},
\end{aligned}$$

which is equivalent to (2.18). □

Remark 3. *If we take in Corollary 2* $f(t)=-\ln(t+1)$, *then we get the logarithmic inequalities*

$$\left[A^{-2}\ln(A+1)\right]B+\left[B^{-2}\ln(B+1)\right]A+2 \leq B^{-1}\ln(B+1)+A^{-1}\ln(A+1)+A^{-1}B+B^{-1}A \tag{2.20}$$

for all $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$.

3. More Examples

We define the *upper incomplete Gamma function* as [10]

$$\Gamma(a,z):=\int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a):=\int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re}a>0.$$

We have the integral representation [11]

$$\Gamma(a,z)=\frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{3.1}$$

for $\operatorname{Re}a<1$ and $|\operatorname{ph}z|<\pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (3.1) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{3.2}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (3.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^tE_1(t) \tag{3.3}$$

for $t > 0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du. \tag{3.4}$$

Let $a = 1-n$, with n a natural number with $n \geq 0$, then by (3.2) we have

$$\begin{aligned} \mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) \\ &= (n-1)!t^{n-1}e^t\Gamma(1-n,t). \end{aligned} \tag{3.5}$$

If we define the generalized exponential integral [12] by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [12, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}E_1(z) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-z)^k,$$

we get

$$\begin{aligned} &\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) \\ &= (n-1)!e^tE_n(t) \\ &= (n-1)!e^t\left[\frac{(-t)^{n-1}}{(n-1)!}E_1(t) + \frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-t)^k\right] \\ &= \sum_{k=0}^{n-2}(-1)^k(n-k-2)!t^k + (-1)^{n-1}t^{n-1}e^tE_1(t) \end{aligned} \tag{3.6}$$

for $n \geq 2$ and $t > 0$.

If $T > 0$, then we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T) = \int_0^\infty \lambda^{-a}e^{-\lambda}(t+\lambda)^{-1}d\lambda = \Gamma(1-a)T^{-a}\exp(T)\Gamma(a,T) \tag{3.7}$$

for $a < 1$.

In particular,

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(T) = \int_0^\infty e^{-\lambda}(T+\lambda)^{-1}d\lambda = \exp(T)E_1(T) \tag{3.8}$$

and, for $n \geq 2$

$$\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) = \int_0^\infty \lambda^{n-1} e^{-\lambda} (T+\lambda)^{-1} d\lambda = \sum_{k=0}^{n-2} (-1)^k (n-k-2)! T^k + (-1)^{n-1} T^{n-1} \exp(T) E_1(T), \tag{3.9}$$

where $T > 0$.

For $n = 2$, we also get

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(T) = \int_0^\infty \lambda e^{-\lambda} (T+\lambda)^{-1} d\lambda = 1 - T\exp(T) E_1(T) \tag{3.10}$$

for $T > 0$.

Proposition 1. *Let $a < 1$. For all $(A, B) \in \mathfrak{Cl}_{(0,\infty)}(H)$ we have*

$$A^{-a} \exp(A) \Gamma(a, A) B + B^{-a} \exp(B) \Gamma(a, B) A \geq A^{1-a} \exp(A) \Gamma(a, A) + B^{1-a} \exp(B) \Gamma(a, B). \tag{3.11}$$

The proof follows by Theorem 4 and by the representation (3.7).

In particular, we have for all $(A, B) \in \mathfrak{Cl}_{(0,\infty)}(H)$ that

$$\exp(A) E_1(A) B + \exp(B) E_1(B) A \geq A \exp(A) \exp(A) E_1(A) + B \exp(B) E_1(B). \tag{3.12}$$

We also have by (3.10) that

$$(1 - A\exp(A) E_1(A)) B + (1 - B\exp(B) E_1(B)) A \geq (1 - A\exp(A) E_1(A)) A + (1 - B\exp(B) E_1(B)) B$$

namely

$$A^2 \exp(A) E_1(A) + B^2 \exp(B) E_1(B) \geq A\exp(A) E_1(A) B + B\exp(B) E_1(B) A \tag{3.13}$$

for all $(A, B) \in \mathfrak{Cl}_{(0,\infty)}(H)$.

We consider the weight $w_{(\cdot+a)^{-1}}(\lambda) := \frac{1}{\lambda+a}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)(\lambda+a)} d\lambda = \frac{\ln t - \ln a}{t - a} \tag{3.14}$$

for all $a > 0$ and $t > 0$ with $t \neq a$.

From this, we get

$$\ln t = \ln a + (t-a)\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t)$$

for all t, $a > 0$.

If $T > 0$, then

$$\ln T = \ln a + (T-a)\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(t) = \ln a + (T-a)\int_0^\infty \frac{1}{(\lambda+a)} (\lambda+T)^{-1} d\lambda. \tag{3.15}$$

Let $a > 0$. Assume that either $0 < T < a$ or $T > a$, then by (3.16) we get

$$\mathcal{D}\left(w_{(\cdot+a)^{-1}}\right)(T) = \int_0^\infty \frac{1}{(\lambda + a)} (\lambda + T)^{-1} d\lambda = (\ln T - \ln a)(T - a)^{-1}. \tag{3.16}$$

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda + t)\left(\lambda^2 + a^2\right)} d\lambda = \frac{\pi t}{2a\left(t^2 + a^2\right)} - \frac{\ln t - \ln a}{t^2 + a^2}$$

for $t > 0$ and $a > 0$.

For $a = 1$ we also have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda + t)\left(\lambda^2 + 1\right)} d\lambda = \frac{\pi t}{2\left(t^2 + 1\right)} - \frac{\ln t}{t^2 + 1}$$

for $t > 0$.

If $T > 0$ and $a > 0$, then

$$\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(T) = \int_0^\infty \frac{1}{\left(\lambda^2 + a^2\right)} (\lambda + T)^{-1} d\lambda = \left(T^2 + a^2\right)^{-1} \left[\frac{\pi}{2a} T - (\ln T - \ln a)\right] \tag{3.17}$$

and, in particular,

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(T) = \int_0^\infty \frac{1}{\left(\lambda^2 + 1\right)} (\lambda + T)^{-1} d\lambda = \left(T^2 + 1\right)^{-1} \left(\frac{\pi}{2} T - \ln T\right). \tag{3.18}$$

By employing Theorem 4 and the transforms given in (3.16)-(3.18), the interested reader may state other similar operator inequalities. The details are omitted.

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