SEVERAL INEQUALITIES FOR AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES WITH APPLICATIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a postive operator on a complex Hilbert space H.

We show among others that, if $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right] \\
&\leq \frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1} \\
&\leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\
&\leq \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right].
\end{aligned}$$

Some examples for operator monotone and operator convex functions as well as for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [6], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.1}$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator convex functions, Operator inequalities, Löwner-Heinz inequality. Logarithmic operator inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 72, 16 pp. Received 15/06/20

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.2}$$

If f is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.1).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda) A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f : (0, \infty) \to \mathbb{R}$ is operator convex in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t+\lambda} d\mu(\lambda), \tag{1.3}$$

where a, $b \in \mathbb{R}$, $c \geq 0$ and a positive measure μ on $(0, \infty)$ such that (1.2) holds. If f is operator convex in $[0, \infty)$, then $a = f(0)$ and $b = f'_+(0)$, the right derivative, in (1.1).

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.4}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.5}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.6}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.6) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.7}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t), \; t > 0. \tag{1.8}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t), \; t > 0. \tag{1.9}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda), \tag{1.10}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.11}$$

for $T > 0$.

From (1.8) we have the representation

$$T^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(T) \tag{1.12}$$

where $T > 0$ and from (1.9)

$$(T-1)^{-1}\ln T = \mathcal{D}(w_{\ln})(T) \tag{1.13}$$

provided $T > 0$ and $T - 1$ is invertible.

In this paper we show among others that, if $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right] \\
&\leq \frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1} \\
&\leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\
&\leq \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right].
\end{aligned}$$

Some examples for operator monotone and operator convex functions as well as for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (1.6) exists and is finite for all $t > 0$.

Theorem 3. *For all A, $B>0$ with $B-A\geq 0$ we have the representation*

$$0\leq (B-A)^{1/2}\left[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\right](B-A)^{1/2} \tag{2.1}$$
$$=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)$$
$$\times w(\lambda)\,d\mu(\lambda).$$

Proof. Observe that, for all A, $B>0$

$$\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)=\int_0^\infty w(\lambda)\left[(\lambda+B)^{-1}-(\lambda+A)^{-1}\right]d\mu(\lambda). \tag{2.2}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.3}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.4}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.5}$$

Now, if we take in (2.5) $C=\lambda+B$, $D=\lambda+A$, then

$$(\lambda+B)^{-1}-(\lambda+A)^{-1} \tag{2.6}$$
$$=\int_0^1((1-t)(\lambda+B)+t(\lambda+A))^{-1}(A-B)$$
$$\times((1-t)(\lambda+B)+t(\lambda+A))^{-1}\,dt$$
$$=\int_0^1(\lambda+(1-t)B+tA)^{-1}(A-B)(\lambda+(1-t)B+tA)^{-1}\,dt$$

and by (2.2) we derive

$$\begin{aligned}(2.7)\qquad & \mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B) \\ &= \int_0^\infty w(\lambda)\left(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A)\right. \\ &\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\left(\int_0^1 (\lambda+sB+(1-s)A)^{-1}(B-A)\right. \\ &\left.\times(\lambda+sB+(1-s)A)^{-1}ds\right)d\mu(\lambda)\end{aligned}$$

for all A, $B>0$, where for the last equality we used the change of variable $s=1-t$, $t\in[0,1]$.

Now, since $B-A\geq 0$, hence by multiplying both sides with $(B-A)^{1/2}$ we get

$$\begin{aligned}(2.8)\qquad & (B-A)^{1/2}[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)](B-A)^{1/2} \\ &= \int_0^\infty w(\lambda)\left(\int_0^1 (B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)\right. \\ &\left.\times(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}ds\right)d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\left(\int_0^1 (B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right. \\ &\left.\times(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}ds\right)d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \\ &\times\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2ds\right)d\mu(\lambda),\end{aligned}$$

which proves the identity in (2.1).

Since

$$\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2\geq 0$$

then by integrating over s on $[0,1]$, multiplying by $w(\lambda)\geq 0$ and integrating over $d\mu(\lambda)$, we deduce the inequality in (2.1). □

The case of operator monotone functions is as follows:

Corollary 1. *Assume that f is operator monotone on $[0,\infty)$, then all A, $B>0$ with $B-A\geq 0$ we have the equality*

$$\begin{aligned}(2.9)\qquad & 0\leq(B-A)^{1/2}\left[f(A)A^{-1}-f(B)B^{-1}\right](B-A)^{1/2} \\ &-f(0)(B-A)^{1/2}\left(A^{-1}-B^{-1}\right)(B-A)^{1/2} \\ &= \int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2ds\right) \\ &\times\lambda d\mu(\lambda)\end{aligned}$$

for some positive measure $\mu(\lambda)$.

If $f(0)=0$, *then*

$$0\le (B-A)^{1/2}\left[f(A)A^{-1}-f(B)B^{-1}\right](B-A)^{1/2} \tag{2.10}$$
$$=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)$$
$$\times\lambda d\mu(\lambda).$$

Proof. From (1.1) we have the representation

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t), \tag{2.11}$$

with $\ell(\lambda)=\lambda$, for some positive measure $\mu(\lambda)$ and nonnegative number b.

Since

$$\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)=[f(A)-f(0)]A^{-1}-[f(B)-f(0)]B^{-1}$$
$$=f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right),$$

hence by (2.1) we get (2.9). □

The case of operator convex functions is as follows:

Corollary 2. *Assume that* f *is operator convex on* $[0,\infty)$, *then all* A, $B>0$ *with* $B-A\ge 0$ *we have that*

$$0\le (B-A)^{1/2}\left[f(A)A^{-2}-f(B)B^{-2}\right](B-A)^{1/2} \tag{2.12}$$
$$-f'_+(0)(B-A)^{1/2}\left(A^{-1}-B^{-1}\right)(B-A)^{1/2}$$
$$-f(0)(B-A)^{1/2}\left(A^{-2}-B^{-2}\right)(B-A)^{1/2}$$
$$=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)$$
$$\times\lambda d\mu(\lambda)$$

for some positive measure $\mu(\lambda)$.

If $f(0)=0$, *then*

$$0\le (B-A)^{1/2}\left[f(A)A^{-2}-f(B)B^{-2}\right](B-A)^{1/2} \tag{2.13}$$
$$-f'_+(0)(B-A)^{1/2}\left(A^{-1}-B^{-1}\right)(B-A)^{1/2}$$
$$=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)$$
$$\times\lambda d\mu(\lambda).$$

Proof. From (1.3) we have that

$$\frac{f(t)-f(0)-f'_+(0)t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for $t>0$. Then for A, $B>0$,

$$\mathcal{D}(\ell,\mu)(A)-\mathcal{D}(\ell,\mu)(B)=f(A)A^{-2}-f'_+(0)A^{-1}-f(0)A^{-2}$$
$$-f(A)B^{-2}+f'_+(0)B^{-1}+f(0)B^{-2}$$
$$=f(A)A^{-2}-f(B)B^{-2}-f'_+(0)\left(A^{-1}-B^{-1}\right)$$
$$-f(0)\left(A^{-2}-B^{-2}\right)$$

and by (2.1) we derive (2.13). □

When more conditions are imposed on the operators A and B we have the following refinements and reverses of the inequality

$$0 \leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B)$$

that holds for $B - A \geq 0$.

Theorem 4. *If $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then*

$$\begin{aligned}
(2.14)\qquad 0 &\leq \frac{m^2}{M^2}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right] \\
&\leq \frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta) - \mathcal{D}(w,\mu)(M+\beta)\right](B-A)^{-1} \\
&\leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\
&\leq \frac{M^2}{m}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{D}(w,\mu)(\alpha) - \mathcal{D}(w,\mu)(m+\alpha)\right].
\end{aligned}$$

Proof. For $s \in [0,1]$ we have

$$\lambda + sB + (1-s)A = \lambda + s(B-A) + A.$$

We have

$$\lambda + s(B-A) + A \geq \lambda + sm + A \geq \lambda + sm + \alpha = \lambda + (1-s)\alpha + s(m+\alpha)$$

$s \in [0,1]$ and $\lambda \geq 0$, which implies that

$$(\lambda + sB + (1-s)A)^{-1} \leq [\lambda + (1-s)\alpha + s(m+\alpha)]^{-1}$$

and, by multiplying both sides by $(B-A)^{1/2} \geq 0$,

$$\begin{aligned}
&(B-A)^{1/2}(\lambda + sB + (1-s)A)^{-1}(B-A)^{1/2} \\
&\leq [\lambda + (1-s)\alpha + s(m+\alpha)]^{-1}(B-A) \\
&\leq M[\lambda + (1-s)\alpha + s(m+\alpha)]^{-1}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\left[(B-A)^{1/2}(\lambda + sB + (1-s)A)^{-1}(B-A)^{1/2}\right]^2 \\
&\leq M^2[\lambda + (1-s)\alpha + s(m+\alpha)]^{-2}
\end{aligned}$$

for $s \in [0,1]$ and $\lambda \geq 0$, which implies by integration that

$$\int_0^\infty w(\lambda)\left(\int_0^1 \left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right) d\mu(\lambda)$$
$$\leq M^2 \int_0^\infty w(\lambda)\left(\int_0^1 [\lambda+(1-s)\alpha+s(m+\alpha)]^{-2} ds\right) d\mu(\lambda)$$
$$= \frac{M^2}{m}\int_0^\infty w(\lambda)\left(\int_0^1 [\lambda+(1-s)\alpha+s(m+\alpha)]^{-1}(m+\alpha-\alpha)\right.$$
$$\left.\times[\lambda+(1-s)\alpha+s(m+\alpha)]^{-1} ds\right) d\mu(\lambda) \text{ (and by (2.7))}$$
$$= \frac{M^2}{m}[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(m+\alpha)].$$

Using (2.8) we get

$$(B-A)^{1/2}[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)](B-A)^{1/2}$$
$$\leq \frac{M^2}{m}[\mathcal{D}(w,\mu)(\alpha)-\mathcal{D}(w,\mu)(m+\alpha)].$$

Multiplying both sides with $(B-A)^{-1/2}$ we deduce the fourth inequality in (2.14).

We also have

$$\lambda+s(B-A)+A \leq \lambda+sM+A \leq \lambda+sM+\beta = \lambda+(1-s)\beta+s(M+\beta),$$

which implies that

$$(\lambda+sB+(1-s)A)^{-1} \geq [\lambda+(1-s)\beta+s(M+\beta)]^{-1}$$

and, by multiplying both sides by $(B-A)^{1/2} \geq 0$,

$$(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}$$
$$\geq [\lambda+(1-s)\beta+s(M+\beta)]^{-1}(B-A)$$
$$\geq m[\lambda+(1-s)\beta+s(M+\beta)]^{-1}$$

for $s \in [0,1]$ and $\lambda \geq 0$.

By taking the square, we get

$$\left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2$$
$$\geq m^2[\lambda+(1-s)\beta+s(M+\beta)]^{-2}$$

for $s \in [0,1]$ and $\lambda \geq 0$.

By taking the integrals in this inequality we obtain

$$\begin{aligned}
&\int_0^\infty w(\lambda)\left(\int_0^1 \left[(B-A)^{1/2}(\lambda+sB+(1-s)A)^{-1}(B-A)^{1/2}\right]^2 ds\right)d\mu(\lambda)\\
&\geq m^2\int_0^\infty w(\lambda)\left(\int_0^1 [\lambda+(1-s)\beta+s(M+\beta)]^{-2}ds\right)d\mu(\lambda)\\
&=\frac{m^2}{M}\int_0^\infty w(\lambda)\left(\int_0^1 [\lambda+(1-s)\beta+s(M+\beta)]^{-1}(M+\beta-\beta)\right.\\
&\left.\times[\lambda+(1-s)\beta+s(M+\beta)]^{-1}ds\right)d\mu(\lambda)\ \text{(and by (2.7))}\\
&=\frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(M+\beta)\right].
\end{aligned}$$

Using (2.8) we get

$$\begin{aligned}
&(B-A)^{1/2}\left[\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)(B)\right](B-A)^{1/2}\\
&\geq\frac{m^2}{M}\left[\mathcal{D}(w,\mu)(\beta)-\mathcal{D}(w,\mu)(M+\beta)\right].
\end{aligned}$$

Multiplying both sides with $(B-A)^{-1/2}$ we deduce the second inequality in (2.14).

The rest of the inequalities are obvious. □

Its is well known that, if $P\geq 0$, then

$$|\langle Px,y\rangle|^2\leq\langle Px,x\rangle\langle Py,y\rangle$$

for all x, $y\in H$.

Therefore, if $T>0$, then

$$\begin{aligned}
0&\leq\langle x,x\rangle^2=\left\langle T^{-1}Tx,x\right\rangle^2=\left\langle Tx,T^{-1}x\right\rangle^2\\
&\leq\langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle
\end{aligned}$$

for all $x\in H$.

If $x\in H$, $\|x\|=1$, then

$$1\leq\langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle\leq\langle Tx,x\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\langle Tx,x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1}\leq T. \tag{2.15}$$

Remark 1. *If $A>0$ and $B-A>0$, then obviously $\|A\|\geq A\geq\left\|A^{-1}\right\|^{-1}$ and $\|B-A\|\geq B-A\geq\left\|(B-A)^{-1}\right\|^{-1}$. So, if we take $\beta=\|A\|$, $\alpha=\left\|A^{-1}\right\|^{-1}$,*

$M = \|B - A\|$ *and* $m = \left\|(B-A)^{-1}\right\|^{-1}$ *in (2.14), then we get*

$$
\begin{aligned}
(2.16)\qquad 0 &\leq \frac{\mathcal{D}(w,\mu)(\|A\|) - \mathcal{D}(w,\mu)(M+\|A\|)}{\|B-A\|^2 \left\|(B-A)^{-1}\right\|^2} \\
&\leq \frac{\mathcal{D}(w,\mu)(\|A\|) - \mathcal{D}(w,\mu)(M+\|A\|)}{\|B-A\| \left\|(B-A)^{-1}\right\|^2}(B-A)^{-1} \\
&\leq \mathcal{D}(w,\mu)(A) - \mathcal{D}(w,\mu)(B) \\
&\leq \|B-A\|^2 \left\|(B-A)^{-1}\right\| \\
&\times \left[\mathcal{D}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right) - \mathcal{D}(w,\mu)\left(\left\|(B-A)^{-1}\right\|^{-1} + \left\|A^{-1}\right\|^{-1}\right)\right] \\
&\times (B-A)^{-1} \\
&\leq \|B-A\|^2 \left\|(B-A)^{-1}\right\|^2 \\
&\times \left[\mathcal{D}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right) - \mathcal{D}(w,\mu)\left(\left\|(B-A)^{-1}\right\|^{-1} + \left\|A^{-1}\right\|^{-1}\right)\right].
\end{aligned}
$$

Corollary 3. *Assume that* f *is operator monotone on* $[0,\infty)$. *If* $\beta \geq A \geq \alpha > 0$, $B > 0$ *with* $M \geq B - A \geq m > 0$ *for some constants* α, β, m, M, *then*

$$
\begin{aligned}
(2.17)\qquad 0 &\leq \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta} - \frac{M}{\beta(M+\beta)}f(0)\right] \\
&\leq \frac{m^2}{M}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta} - \frac{M}{\beta(M+\beta)}f(0)\right](B-A)^{-1} \\
&\leq f(A)A^{-1} - f(B)B^{-1} - f(0)\left(A^{-1} - B^{-1}\right) \\
&\leq \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha} - \frac{m}{\alpha(m+\alpha)}f(0)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha} - \frac{m}{\alpha(m+\alpha)}f(0)\right].
\end{aligned}
$$

If $f(0) = 0$, *then*

$$
\begin{aligned}
(2.18)\qquad 0 &\leq \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta}\right] \leq \frac{m^2}{M}\left[\frac{f(\beta)}{\beta} - \frac{f(M+\beta)}{M+\beta}\right](B-A)^{-1} \\
&\leq f(A)A^{-1} - f(B)B^{-1} \\
&\leq \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha}\right](B-A)^{-1} \leq \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha} - \frac{f(m+\alpha)}{m+\alpha}\right].
\end{aligned}
$$

The proof follows by (2.14) and the representation (2.11).

Remark 2. *If $A > 0$ and $B - A > 0$, then for f an operator monotone function on $[0, \infty)$ with $f(0) = 0$, we obtain from (2.18) that*

$$
\begin{aligned}
(2.19)\qquad 0 &\le \frac{1}{\|B-A\|^2 \left\|(B-A)^{-1}\right\|^2}\left[\frac{f(\|A\|)}{\|A\|} - \frac{f(\|B-A\| + \|A\|)}{\|B-A\| + \|A\|}\right] \\
&\le \frac{1}{\|B-A\| \left\|(B-A)^{-1}\right\|^2}\left[\frac{f(\|A\|)}{\|A\|} - \frac{f(\|B-A\| + \|A\|)}{\|B-A\| + \|A\|}\right] \\
&\times (B-A)^{-1} \\
&\le f(A)A^{-1} - f(B)B^{-1} \\
&\le \|B-A\|^2 \left\|(B-A)^{-1}\right\| \\
&\times \left[\frac{f\left(\|A^{-1}\|^{-1}\right)}{\|A^{-1}\|^{-1}} - \frac{f\left(\left\|(B-A)^{-1}\right\|^{-1} + \|A^{-1}\|^{-1}\right)}{\left\|(B-A)^{-1}\right\|^{-1} + \|A^{-1}\|^{-1}}\right] \\
&\times (B-A)^{-1} \\
&\le \|B-A\|^2 \left\|(B-A)^{-1}\right\|^2 \\
&\left[\frac{f\left(\|A^{-1}\|^{-1}\right)}{\|A^{-1}\|^{-1}} - \frac{f\left(\left\|(B-A)^{-1}\right\|^{-1} + \|A^{-1}\|^{-1}\right)}{\left\|(B-A)^{-1}\right\|^{-1} + \|A^{-1}\|^{-1}}\right].
\end{aligned}
$$

The case of operator convex functions is as follows:

Corollary 4. *Assume that f is operator convex on $[0, \infty)$.If $\beta \ge A \ge \alpha > 0$, $B > 0$ with $M \ge B - A \ge m > 0$ for some constants α, β, m, M, then*

$$
\begin{aligned}
(2.20)\qquad 0 &\le \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta^2} - \frac{f(M+\beta)}{(M+\beta)^2} - f'_+(0)\frac{M}{\beta(M+\beta)} - f(0)\frac{M(M+2\beta)}{\beta^2(M+\beta)^2}\right] \\
&\le \frac{m^2}{M}\left[\frac{f(\beta)}{\beta^2} - \frac{f(M+\beta)}{(M+\beta)^2} - f'_+(0)\frac{M}{\beta(M+\beta)} - f(0)\frac{M(M+2\beta)}{\beta^2(M+\beta)^2}\right] \\
&\times (B-A)^{-1} \\
&\le f(A)A^{-2} - f(B)B^{-2} - f'_+(0)\left(A^{-1} - B^{-1}\right) - f(0)\left(A^{-2} - B^{-2}\right) \\
&\le \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha^2} - \frac{f(m+\alpha)}{(m+\alpha)^2} - f'_+(0)\frac{m}{\alpha(m+\alpha)} - f(0)\frac{m(m+2\alpha)}{\alpha^2(m+\alpha)^2}\right] \\
&\times (B-A)^{-1} \\
&\le \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha^2} - \frac{f(m+\alpha)}{(m+\alpha)^2} - f'_+(0)\frac{m}{\alpha(m+\alpha)} - f(0)\frac{m(m+2\alpha)}{\alpha^2(m+\alpha)^2}\right].
\end{aligned}
$$

If $f(0)=0$, then

$$0\le \frac{m^2}{M^2}\left[\frac{f(\beta)}{\beta^2}-\frac{f(M+\beta)}{(M+\beta)^2}-f'_+(0)\frac{M}{\beta(M+\beta)}\right] \tag{2.21}$$
$$\le \frac{m^2}{M}\left[\frac{f(\beta)}{\beta^2}-\frac{f(M+\beta)}{(M+\beta)^2}-f'_+(0)\frac{M}{\beta(M+\beta)}\right](B-A)^{-1}$$
$$\le f(A)A^{-2}-f(B)B^{-2}-f'_+(0)\left(A^{-1}-B^{-1}\right)$$
$$\le \frac{M^2}{m}\left[\frac{f(\alpha)}{\alpha^2}-\frac{f(m+\alpha)}{(m+\alpha)^2}-f'_+(0)\frac{m}{\alpha(m+\alpha)}\right](B-A)^{-1}$$
$$\le \frac{M^2}{m^2}\left[\frac{f(\alpha)}{\alpha^2}-\frac{f(m+\alpha)}{(m+\alpha)^2}-f'_+(0)\frac{m}{\alpha(m+\alpha)}\right].$$

Remark 3. *If $A>0$ and $B-A>0$, then for f an operator convex function on $[0,\infty)$ with $f(0)=0$, we obtain from (2.21) that*

$$0\le \frac{1}{\|B-A\|^2\left\|(B-A)^{-1}\right\|^2} \tag{2.22}$$
$$\times\left[\frac{f(\|A\|)}{\|A\|}-\frac{f(\|B-A\|+\|A\|)}{\|B-A\|+\|A\|}-\frac{f'_+(0)\|B-A\|}{\|A\|(\|B-A\|+\|A\|)}\right]$$
$$\le \frac{1}{\|B-A\|\left\|(B-A)^{-1}\right\|^2}$$
$$\times\left[\frac{f(\|A\|)}{\|A\|}-\frac{f(\|B-A\|+\|A\|)}{\|B-A\|+\|A\|}-\frac{f'_+(0)\|B-A\|}{\|A\|(\|B-A\|+\|A\|)}\right](B-A)^{-1}$$

$$\le f(A)A^{-2}-f(B)B^{-2}-f'_+(0)\left(A^{-1}-B^{-1}\right)$$
$$\le \|B-A\|^2\left\|(B-A)^{-1}\right\|\left[\frac{f\left(\|A^{-1}\|^{-1}\right)}{\|A^{-1}\|^{-1}}-\frac{f\left(\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}\right)}{\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}}\right.$$
$$\left.-\frac{f'_+(0)\left\|(B-A)^{-1}\right\|^{-1}}{\|A^{-1}\|^{-1}\left(\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}\right)}\right](B-A)^{-1}$$

$$\le \|B-A\|^2\left\|(B-A)^{-1}\right\|^2\left[\frac{f\left(\|A^{-1}\|^{-1}\right)}{\|A^{-1}\|^{-1}}-\frac{f\left(\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}\right)}{\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}}\right.$$
$$\left.-\frac{f'_+(0)\left\|(B-A)^{-1}\right\|^{-1}}{\|A^{-1}\|^{-1}\left(\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}\right)}\right].$$

3. Some Examples

The function $f(t) = t^r$, $r \in (0, 1]$ is operator monotone on $[0, \infty)$ and by (2.18) we obtain the power inequalities

$$\begin{aligned} (3.1) \qquad 0 &\leq \frac{m^2}{M^2}\left[\beta^{r-1} - (M+\beta)^{r-1}\right] \leq \frac{m^2}{M}\left[\beta^{r-1} - (M+\beta)^{r-1}\right](B-A)^{-1} \\ &\leq A^{r-1} - B^{r-1} \\ &\leq \frac{M^2}{m}\left[\alpha^{r-1} - (m+\alpha)^{r-1}\right](B-A)^{-1} \leq \frac{M^2}{m^2}\left[\alpha^{r-1} - (m+\alpha)^{r-1}\right], \end{aligned}$$

provided that $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M.

If $A > 0$ and $B - A > 0$, then by (2.19),

$$\begin{aligned} (3.2) \qquad 0 &\leq \frac{\|A\|^{r-1} - (\|B-A\| + \|A\|)^{r-1}}{\|B-A\|^2 \left\|(B-A)^{-1}\right\|^2} \\ &\leq \frac{\|A\|^{r-1} - (\|B-A\| + \|A\|)^{r-1}}{\|B-A\| \left\|(B-A)^{-1}\right\|^2}(B-A)^{-1} \\ &\leq A^{r-1} - B^{r-1} \\ &\leq \|B-A\|^2 \left\|(B-A)^{-1}\right\| \\ &\times \left[\left\|A^{-1}\right\|^{1-r} - \left(\left\|(B-A)^{-1}\right\|^{-1} + \left\|A^{-1}\right\|^{-1}\right)^{r-1}\right](B-A)^{-1} \\ &\leq \|B-A\|^2 \left\|(B-A)^{-1}\right\|^2 \\ &\times \left[\left\|A^{-1}\right\|^{1-r} - \left(\left\|(B-A)^{-1}\right\|^{-1} + \left\|A^{-1}\right\|^{-1}\right)^{r-1}\right]. \end{aligned}$$

The function $f(t) = \ln(t+1)$ is operator monotone on $[0, \infty)$ and by (2.18) we get

$$\begin{aligned} (3.3) \qquad 0 &\leq \frac{m^2}{M^2}\left[\frac{\ln(\beta+1)}{\beta} - \frac{\ln(M+\beta+1)}{M+\beta}\right] \\ &\leq \frac{m^2}{M}\left[\frac{\ln(\beta+1)}{\beta} - \frac{\ln(M+\beta+1)}{M+\beta}\right](B-A)^{-1} \\ &\leq A^{-1}\ln(A+1) - B^{-1}\ln(B+1) \\ &\leq \frac{M^2}{m}\left[\frac{\ln(\alpha+1)}{\alpha} - \frac{\ln(m+\alpha+1)}{m+\alpha}\right](B-A)^{-1} \\ &\leq \frac{M^2}{m^2}\left[\frac{\ln(\alpha+1)}{\alpha} - \frac{\ln(m+\alpha+1)}{m+\alpha}\right], \end{aligned}$$

provided that $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M.

If $A>0$ and $B-A>0$, then by (2.19),

$$0\leq\frac{1}{\|B-A\|^{2}\left\|(B-A)^{-1}\right\|^{2}}\left[\frac{\ln(\|A\|+1)}{\|A\|}-\frac{\ln(\|B-A\|+\|A\|+1)}{\|B-A\|+\|A\|}\right] \tag{3.4}$$
$$\leq\frac{1}{\|B-A\|\left\|(B-A)^{-1}\right\|^{2}}\left[\frac{\ln(\|A\|+1)}{\|A\|}-\frac{\ln(\|B-A\|+\|A\|+1)}{\|B-A\|+\|A\|}\right]$$
$$\times(B-A)^{-1}$$
$$\leq A^{-1}\ln(A+1)-B^{-1}\ln(B+1)$$
$$\leq\|B-A\|^{2}\left\|(B-A)^{-1}\right\|$$
$$\times\left[\frac{\ln\left(\|A^{-1}\|^{-1}+1\right)}{\|A^{-1}\|^{-1}}-\frac{\ln\left(\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}+1\right)}{\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}}\right]$$
$$\times(B-A)^{-1}$$
$$\leq\|B-A\|^{2}\left\|(B-A)^{-1}\right\|^{2}$$
$$\times\left[\frac{\ln\left(\|A^{-1}\|^{-1}+1\right)}{\|A^{-1}\|^{-1}}-\frac{\ln\left(\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}+1\right)}{\left\|(B-A)^{-1}\right\|^{-1}+\|A^{-1}\|^{-1}}\right].$$

The function $f(t)=-\ln(t+1)$ is operator convex, and by (2.21) we obtain

$$0\leq\frac{m^{2}}{M^{2}}\left[\frac{\ln(M+\beta+1)}{(M+\beta)^{2}}-\frac{\ln(\beta+1)}{\beta^{2}}+\frac{M}{\beta(M+\beta)}\right] \tag{3.5}$$
$$\leq\frac{m^{2}}{M}\left[\frac{\ln(M+\beta+1)}{(M+\beta)^{2}}-\frac{\ln(\beta+1)}{\beta^{2}}+\frac{M}{\beta(M+\beta)}\right](B-A)^{-1}$$
$$\leq B^{-2}\ln(B+1)-A^{-2}\ln(A+1)+A^{-1}-B^{-1}$$
$$\leq\frac{M^{2}}{m}\left[\frac{\ln(m+\alpha+1)}{(m+\alpha)^{2}}-\frac{\ln(\alpha+1)}{\alpha^{2}}+\frac{m}{\alpha(m+\alpha)}\right](B-A)^{-1}$$
$$\leq\frac{M^{2}}{m^{2}}\left[\frac{\ln(m+\alpha+1)}{(m+\alpha)^{2}}-\frac{\ln(\alpha+1)}{\alpha^{2}}+\frac{m}{\alpha(m+\alpha)}\right],$$

provided that $\beta\geq A\geq\alpha>0$, $B>0$ with $M\geq B-A\geq m>0$ for some constants α, β, m, M.

Consider the kernel $e_{-a}(\lambda):=\exp(-a\lambda)$, $\lambda\geq0$ and $a>0$. Then

$$D(e_{-a})(t):=\int_{0}^{\infty}\frac{\exp(-a\lambda)}{t+\lambda}d\lambda=E_{1}(at)\exp(at),\ t\geq0,$$

where

$$E_{1}(t):=\int_{t}^{\infty}\frac{e^{-u}}{u}du,\ t\geq0. \tag{3.6}$$

For $a = 1$ we have

$$D(e_{-1})(t) := \int_0^\infty \frac{\exp(-\lambda)}{t+\lambda} d\lambda = E_1(t)\exp(t),\ t \geq 0.$$

Let $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M. Then by (2.14) we have

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2}\left[E_1(a\beta)\exp(a\beta) - E_1(a(M+\beta))\exp(a(M+\beta))\right] \\ &\leq \frac{m^2}{M}\left[E_1(a\beta)\exp(a\beta) - E_1(a(M+\beta))\exp(a(M+\beta))\right](B-A)^{-1} \\ &\leq E_1(aA)\exp(aA) - E_1(aB)\exp(aB) \\ &\leq \frac{M^2}{m}\left[E_1(a\alpha)\exp(a\alpha) - E_1(a(m+\alpha))\exp(a(m+\alpha))\right](B-A)^{-1} \\ &\leq \frac{M^2}{m^2}\left[E_1(a\alpha)\exp(a\alpha) - E_1(a(m+\alpha))\exp(a(m+\alpha))\right] \end{aligned} \tag{3.7}$$

for $a > 0$.

For $a = 1$ we have

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2}\left[E_1(\beta)\exp(\beta) - E_1(M+\beta)\exp(M+\beta)\right] \\ &\leq \frac{m^2}{M}\left[E_1(\beta)\exp(\beta) - E_1(M+\beta)\exp(M+\beta)\right](B-A)^{-1} \\ &\leq E_1(A)\exp(A) - E_1(B)\exp(B) \\ &\leq \frac{M^2}{m}\left[E_1(\alpha)\exp(\alpha) - E_1(m+\alpha)\exp(m+\alpha)\right](B-A)^{-1} \\ &\leq \frac{M^2}{m^2}\left[E_1(\alpha)\exp(\alpha) - E_1(m+\alpha)\exp(m+\alpha)\right]. \end{aligned} \tag{3.8}$$

More examples of such transforms are

$$D(w_{1/(\ell^2+a^2)})(t) := \int_0^\infty \frac{1}{(t+\lambda)\left(\lambda^2+a^2\right)} d\lambda = \frac{\pi t - 2a\ln(t/a)}{2a\left(t^2+a^2\right)},\ t \geq 0$$

and

$$D(w_{\ell/(\ell^2+a^2)})(t) := \int_0^\infty \frac{\lambda}{(t+\lambda)\left(\lambda^2+a^2\right)} d\lambda = \frac{\pi a + 2t\ln(t/a)}{2a\left(t^2+a^2\right)},\ t \geq 0$$

for $a > 0$.

The interested reader may state other similar results by employing the examples of monotone operator functions provided in [2], [3], [4], [7] and [8].

References

[1] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[3] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.

[4] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[5] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[6] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.
[7] M. S. Moslehian, H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.
[8] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.