LIPSCHITZ TYPE INEQUALITIES FOR AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS WITH APPLICATIONS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Assume that $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A)\| \leq \|B - A\| \begin{cases} \frac{\mathcal{D}(w,\mu)(m_1) - \mathcal{D}(w,\mu)(m_2)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ (-\mathcal{D}'(w,\mu)(m)) & \text{if } m_1 = m_2 = m, \end{cases}$$

where $\mathcal{D}'(w,\mu)(t)$ is the derivative of $\mathcal{D}(w,\mu)$ as a function of t.

If $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function with $f(0) = 0$, then

$$\left\|f(A)A^{-1} - f(B)B^{-1}\right\| \leq \|B - A\| \begin{cases} \frac{f(m_1)m_1^{-1} - f(m_2)m_2^{-1}}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ \frac{f(m) - f'(m)m}{m^2} & \text{if } m_1 = m_2 = m. \end{cases}$$

Similar inequalities for operator convex functions and some particular examples of interest are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [15] had given a definitive characterization of operator monotone functions as follows, see for instance [5, p. 144-145]:

Theorem 1. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda) \tag{1.1}$$

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where $a\in\mathbb{R}$ *and* $b\geq 0$ *and a positive measure* μ *on* $(0,\infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty.$$

If f *is operator monotone in* $[0,\infty)$, *then* $a=f(0)$ *in (1.1).*

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [13]. The function ln is also operator monotone on $(0,\infty)$. For other examples of operator monotone functions, see [10] and [12].

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [5, p. 147]:

Theorem 2. *A function* $f:(0,\infty)\to\mathbb{R}$ *is operator convex in* $(0,\infty)$ *if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.2}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ *and a positive measure* μ *on* $(0,\infty)$ *such that (1.2) holds. If* f *is operator convex in* $[0,\infty)$, *then* $a=f(0)$ *and* $b=f'_+(0)$, *the right derivative, in (1.2).*

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A|:=(A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A):=|A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L>0$ such that

$$\||A|-|B|\|\leq L\|A-B\|$$

for any A, $B\in\mathcal{B}(H)$.

However, as shown by Farforovskaya in [7], [8] and Kato in [14], the following inequality holds

$$\||A|-|B|\|\leq\frac{2}{\pi}\|A-B\|\left(2+\log\left(\frac{\|A\|+\|B\|}{\|A-B\|}\right)\right) \tag{1.3}$$

for any A, $B\in\mathcal{B}(H)$ with $A\neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS}:=(\operatorname{tr}C^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\||A|-|B|\|_{HS}\leq\sqrt{2}\|A-B\|_{HS} \tag{1.4}$$

for any A, $B\in\mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\| \leq a_1 \|A-B\| + a_2 \|A-B\|^2 + O\left(\|A-B\|^3\right) \tag{1.5}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2 .$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A-B\| \tag{1.6}$$

where f is an *operator monotone function* on $(0,\infty)$ and A, $B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [9] and the references therein.

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [5, p. 145]

$$t^r = \frac{\sin(r\pi)}{\pi} t \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.7}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.8}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.9}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.9) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.10}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \ t > 0. \tag{1.11}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1) \mathcal{D}(w_{\ln})(t), \ t > 0. \tag{1.12}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.13}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.14}$$

for $T>0$.

In this paper we show among others that, if $A\geq m_1>0$, $B\geq m_2>0$, then

$$\begin{aligned}&\|\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)\|\\ &\leq\|B-A\|\begin{cases}\frac{\mathcal{D}(w,\mu)(m_1)-\mathcal{D}(w,\mu)(m_2)}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ (-\mathcal{D}'(w,\mu)(m)) & \text{if } m_1=m_2=m,\end{cases}\end{aligned}$$

where $\mathcal{D}'(w,\mu)(t)$ is the derivative of $\mathcal{D}(w,\mu)$ as a function of t.

If $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function with $f(0)=0$, then

$$\begin{aligned}&\left\|f(A)A^{-1}-f(B)B^{-1}\right\|\\ &\leq\|B-A\|\begin{cases}\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ \frac{f(m)-f'(m)m}{m^2} & \text{if } m_1=m_2=m.\end{cases}\end{aligned}$$

Similar inequalities for operator convex functions and some particular examples of interest are also given.

2. Some Preliminary Facts

In the following, whenever we write $\mathcal{D}(w,\mu)$ we mean that the integral from (1.9) exists and is finite for all $t>0$.

Lemma 1. *For all A, $B>0$ we have the representation*

$$\begin{aligned}&\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)\\ &=-\int_0^\infty w(\lambda)\\ &\times\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).\end{aligned} \tag{2.1}$$

Proof. Observe that, for all A, $B>0$

$$\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)=\int_0^\infty w(\lambda)\left[(\lambda+B)^{-1}-(\lambda+A)^{-1}\right]d\mu(\lambda). \tag{2.2}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.3}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$:

$\{(1-t)C+tD,\ t\in[0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.4}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.5}$$

Now, if we take in (2.5) $C=\lambda+B$, $D=\lambda+A$, then

$$\begin{aligned}&(\lambda+B)^{-1}-(\lambda+A)^{-1}\\&=\int_0^1((1-t)(\lambda+B)+t(\lambda+A))^{-1}(A-B)\\&\times((1-t)(\lambda+B)+t(\lambda+A))^{-1}\,dt\\&=\int_0^1(\lambda+(1-t)B+tA)^{-1}(A-B)(\lambda+(1-t)B+tA)^{-1}\,dt\end{aligned} \tag{2.6}$$

and by (2.2) we derive (2.1). □

Remark 1. *If $B\geq A>0$, then by representation (2.1) we derive that*

$$\mathcal{D}(w,\mu)(B)\leq\mathcal{D}(w,\mu)(A),$$

which shows that $-\mathcal{D}(w,\mu)$ is operator monotone on $(0,\infty)$. For further results related to the operator monotonicity of this integral transform see the recent paper [6].

Remark 2. *We observe that if $A,\ B>0$ and $r\in(0,1]$, then by (1.11) we get the identity*

$$\begin{aligned}B^{r-1}-A^{r-1}=-\frac{\sin(r\pi)}{\pi}\int_0^\infty\lambda^{r-1}\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}\right.\\\left.\times(B-A)(\lambda+(1-t)B+tA)^{-1}\,dt\right)d\lambda.\end{aligned} \tag{2.7}$$

If $A,\ B>0$ with $A-1$ and $B-1$ invertible, then

$$\begin{aligned}&(B-1)^{-1}\ln B-(A-1)^{-1}\ln A\\&=-\int_0^\infty(\lambda+1)^{-1}\\&\times\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)(\lambda+(1-t)B+tA)^{-1}\,dt\right)d\lambda.\end{aligned} \tag{2.8}$$

Corollary 1. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function as in (1.1). Then for all* $A,\ B>0$ *we have the equality*

$$\begin{aligned}(2.9)\qquad & B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right)\\ &=-\int_0^\infty \lambda\left(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).\end{aligned}$$

If $f(0)=0$, *then we have the simpler equality*

$$\begin{aligned}(2.10)\qquad & B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right)\\ &=-\int_0^\infty \lambda\left(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).\end{aligned}$$

Proof. From (1.1) we have that

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t), \tag{2.11}$$

where $\ell(\lambda)=\lambda$, $\lambda>0$. Then for $A,\ B>0$,

$$\begin{aligned}\mathcal{D}(\ell,\mu)(B)-\mathcal{D}(\ell,\mu)(A)&=[f(B)-f(0)]B^{-1}-[f(A)-f(0)]A^{-1}\\ &=B^{-1}f(B)-A^{-1}f(A)-f(0)\left(B^{-1}-A^{-1}\right)\end{aligned}$$

and by (2.1) we derive (2.9). □

Corollary 2. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator convex function as in (1.2). Then for all* $A,\ B>0$ *we have the equality*

$$\begin{aligned}(2.12)\qquad & f(B)B^{-2}-f(A)A^{-2}-f'_+(0)\left(B^{-1}-A^{-1}\right)-f(0)\left(B^{-2}-A^{-2}\right)\\ &=-\int_0^\infty \lambda\left(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).\end{aligned}$$

If $f(0)=0$, *then we have the simpler equality*

$$\begin{aligned}(2.13)\qquad & f(B)B^{-2}-f(A)A^{-2}-f'_+(0)\left(B^{-1}-A^{-1}\right)\\ &=-\int_0^\infty \lambda\left(\int_0^1 (\lambda+(1-t)B+tA)^{-1}(B-A)\right.\\ &\left.\times(\lambda+(1-t)B+tA)^{-1}dt\right)d\mu(\lambda).\end{aligned}$$

Proof. From (1.5) we have that

$$\frac{f(t)-f(0)-f'_+(0)t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for $t>0$. Then for $A,\ B>0$,

$$\begin{aligned}\mathcal{D}(\ell,\mu)(B)-\mathcal{D}(\ell,\mu)(A)&=f(B)B^{-2}-f(A)A^{-2}-f'_+(0)\left(B^{-1}-A^{-1}\right)\\ &\quad-f(0)\left(B^{-2}-A^{-2}\right)\end{aligned}$$

and by (2.1) we derive (2.12). □

Remark 3. *Let $a > 0$ and $f(t) = (t+a)^p$ with $p \in [-1,0) \cup [1,2]$. This function is operator convex and $f(0) = a^p$, $f'(0) = pa^{p-1}$. Then for all A, $B > 0$ we have the equality*

$$(B+a)^p B^{-2} - (A+a)^p A^{-2} - pa^{p-1}\left(B^{-1} - A^{-1}\right) - a^p\left(B^{-2} - A^{-2}\right) = -\int_0^\infty \lambda \left(\int_0^1 (\lambda + (1-t)B + tA)^{-1}(B-A) \times (\lambda + (1-t)B + tA)^{-1} dt\right) d\mu(\lambda), \tag{2.14}$$

for some positive measure μ on $(0,\infty)$.

3. Main Results

We have the following Lipschitz type inequality:

Theorem 3. *Assume that $A \geq m_1 > 0$, $B \geq m_2 > 0$, then*

$$\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A)\| \leq \|B-A\| \begin{cases} \frac{\mathcal{D}(w,\mu)(m_1) - \mathcal{D}(w,\mu)(m_2)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ (-\mathcal{D}'(w,\mu)(m)) & \text{if } m_1 = m_2 = m, \end{cases} \tag{3.1}$$

where $\mathcal{D}'(w,\mu)(t)$ is the derivative of $\mathcal{D}(w,\mu)$ as a function of t.

Proof. From the identity (2.1) we get by taking the norm that

$$\begin{aligned} &\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A)\| \\ &\leq \int_0^\infty w(\lambda) \\ &\times \left\| \int_0^1 (\lambda + (1-t)B + tA)^{-1}(B-A)(\lambda + (1-t)B + tA)^{-1} dt \right\| d\mu(\lambda) \\ &\leq \int_0^\infty w(\lambda) \\ &\times \left(\int_0^1 \left\| (\lambda + (1-t)B + tA)^{-1}(B-A)(\lambda + (1-t)B + tA)^{-1} \right\| dt \right) d\mu(\lambda) \\ &\leq \|B-A\| \int_0^\infty w(\lambda) \left(\int_0^1 \left\| (\lambda + (1-t)B + tA)^{-1} \right\|^2 dt \right) d\mu(\lambda) \end{aligned} \tag{3.2}$$

for all A, $B > 0$.

Assume that $m_2 > m_1$. Then

$$(1-t)A + tB + \lambda \geq (1-t)m_1 + tm_2 + \lambda,$$

which implies that

$$((1-t)A + tB + \lambda)^{-1} \leq ((1-t)m_1 + tm_2 + \lambda)^{-1},$$

and

$$\left\| ((1-t)A + tB + \lambda)^{-1} \right\|^2 \leq ((1-t)m_1 + tm_2 + \lambda)^{-2} \tag{3.3}$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

Therefore, by integrating (3.3) we derive

$$\begin{aligned}
&\int_0^\infty w(\lambda)\left(\int_0^1 \left\|((1-t)A+tB+\lambda)^{-1}\right\|^2 dt\right) dw(\lambda)\\
&\le \int_0^\infty w(\lambda)\left(\int_0^1 ((1-t)m_1+tm_2+\lambda)^{-2} dt\right) dw(\lambda)\\
&= \frac{1}{m_2-m_1}\int_0^\infty w(\lambda)\left(\int_0^1 ((1-t)m_1+tm_2+\lambda)^{-1}\right.\\
&\left.\times(m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1} dt\right) dw(\lambda)\\
&= \frac{1}{m_2-m_1}\left[\mathcal{D}(w,\mu)(m_1)-\mathcal{D}(w,\mu)(m_2)\right] \text{ (by (2.1))}
\end{aligned}$$

and by (3.2) we deduce

$$\begin{aligned}
&\|\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)\|\\
&\le \frac{1}{m_2-m_1}\left[\mathcal{D}(w,\mu)(m_1)-\mathcal{D}(w,\mu)(m_2)\right].
\end{aligned} \tag{3.4}$$

The case $m_2 < m_1$ goes in a similar way and we also obtain (3.4).

Let $\epsilon > 0$. Then $B+\epsilon \ge m+\epsilon > m$. From (3.4) we get

$$\begin{aligned}
&\|\mathcal{D}(w,\mu)(B+\epsilon)-\mathcal{D}(w,\mu)(A)\|\\
&\le \frac{1}{m+\epsilon-m}\left[\mathcal{D}(w,\mu)(m)-\mathcal{D}(w,\mu)(m+\epsilon)\right]
\end{aligned}$$

and by taking the limit over $\epsilon \to 0+$, using the continuity and differentiability of $\mathcal{D}(w,\mu)$ we deduce the second part of (3.1). □

Corollary 3. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function. If $A \ge m_1 > 0$, $B \ge m_2 > 0$, then*

$$\begin{aligned}
&\left\|f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\right\|\\
&\le \|B-A\|\begin{cases}\left(\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1}-\frac{f(0)}{m_2m_1}\right) & \text{if } m_1\ne m_2,\\[2ex] \frac{f(m)-f(0)-f'(m)m}{m^2} & \text{if } m_1=m_2=m.\end{cases}
\end{aligned} \tag{3.5}$$

If $f(0)=0$, then we have the simpler inequalities

$$\begin{aligned}
&\left\|f(A)A^{-1}-f(B)B^{-1}\right\|\\
&\le \|B-A\|\begin{cases}\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1} & \text{if } m_1\ne m_2,\\[2ex] \frac{f(m)-f'(m)m}{m^2} & \text{if } m_1=m_2=m.\end{cases}
\end{aligned} \tag{3.6}$$

Proof. From (1.1) we have that

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t),$$

where $\ell(\lambda)=\lambda$, $\lambda>0$.

Then

$$\frac{1}{m_2 - m_1}\left[\mathcal{D}(w,\mu)(m_1) - \mathcal{D}(w,\mu)(m_2)\right] = \frac{f(m_1)m_1^{-1} - f(m_2)m_2^{-1}}{m_2 - m_1} - \frac{f(0)}{m_2 m_1}$$

and

$$-\mathcal{D}'(w,\mu)(m) = \frac{f(m) - f(0) - f'(m)m}{m^2}.$$

By making use of (3.1) we derive (3.5). □

Remark 4. *Let $A \geq m_1 > 0$, $B \geq m_2 > 0$ and $r \in (0,1]$. Then by (3.6) we have the power inequalities*

$$\left\|A^{r-1} - B^{r-1}\right\| \leq \|B - A\| \begin{cases} \frac{m_1^{r-1} - m_2^{r-1}}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \frac{1-r}{m^{2-r}} & \text{if } m_1 = m_2 = m. \end{cases} \tag{3.7}$$

If we take $f(t) = \ln(t+1)$, then we get by (3.6) that

$$\begin{aligned} &\left\|A^{-1}\ln(A+1) - B^{-1}\ln(B+1)\right\| \\ &\leq \|B - A\| \begin{cases} \frac{m_1^{-1}\ln(m_1+1) - m_2^{-1}\ln(m_2+1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \frac{(m+1)\ln(m+1) - m}{m^2(m+1)} & \text{if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{3.8}$$

Corollary 4. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator convex function. If $A \geq m_1 > 0$, $B \geq m_2 > 0$, then*

$$\begin{aligned} &\left\|f(B)B^{-2} - f(A)A^{-2} - f'_+(0)\left(B^{-1} - A^{-1}\right) - f(0)\left(B^{-2} - A^{-2}\right)\right\| \\ &\leq \|B - A\| \\ &\times \begin{cases} \frac{f(m_1)m_1^{-2} - f(m_2)m_2^{-2}}{m_2 - m_1} - \frac{f'_+(0)}{m_1 m_2} - f(0)\frac{m_1+m_2}{m_1^2 m_1^2} & \text{if } m_1 \neq m_2, \\ 2\frac{f(m) - f(0)}{m^3} - \frac{f'(m) + f'_+(0)}{m^2} & \text{if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{3.9}$$

If $f(0) = 0$, then

$$\begin{aligned} &\left\|f(B)B^{-2} - f(A)A^{-2} - f'_+(0)\left(B^{-1} - A^{-1}\right)\right\| \\ &\leq \|B - A\| \begin{cases} \frac{f(m_1)m_1^{-2} - f(m_2)m_2^{-2}}{m_2 - m_1} - \frac{f'_+(0)}{m_1 m_2} & \text{if } m_1 \neq m_2, \\ \frac{2}{m^2}\left[\frac{f(m)}{m} - \frac{f'(m) + f'_+(0)}{2}\right] & \text{if } m_1 = m_2 = m. \end{cases} \end{aligned} \tag{3.10}$$

Proof. From (1.5) we have that

$$\frac{f(t) - f(0) - f'_+(0)t}{t^2} - c = \mathcal{D}(\ell,\mu)(t),$$

for $t > 0$.

Then

$$\begin{aligned}
&\frac{\mathcal{D}(w,\mu)(m_1)-\mathcal{D}(w,\mu)(m_2)}{m_2-m_1}\\
&=\frac{1}{m_2-m_1}\left[\frac{f(m_1)-f(0)-f'_+(0)\,m_1}{m_1^2}-\frac{f(m_2)-f(0)-f'_+(0)\,m_2}{m_2^2}\right]\\
&=\frac{f(m_1)\,m_1^{-2}-f(m_2)\,m_2^{-2}}{m_2-m_1}-\frac{f'_+(0)}{m_1m_2}-f(0)\frac{m_1+m_2}{m_1^2m_1^2}.
\end{aligned}$$

Since

$$\begin{aligned}
\left(\frac{f(t)-f(0)-f'_+(0)\,t}{t^2}\right)'&=\frac{\left(f'(t)-f'_+(0)\right)t^2-2t\left(f(t)-f(0)-f'_+(0)\,t\right)}{t^4}\\
&=\frac{f'(t)+f'_+(0)}{t^2}-2\frac{f(t)-f(0)}{t^3},
\end{aligned}$$

hence

$$-\mathcal{D}'(w,\mu)(m)=2\frac{f(m)-f(0)}{m^3}-\frac{f'(m)+f'_+(0)}{m^2}$$

and by (3.1) we obtain (3.9). □

Remark 5. *If we take $f(t)=-\ln(t+1)$ in (3.10), then for $A\geq m_1>0$, $B\geq m_2>0$ we get*

$$\begin{aligned}
&\left\|B^{-2}\ln(B+1)-A^{-2}\ln(A+1)-B^{-1}+A^{-1}\right\| \qquad (3.11)\\
&\leq\|B-A\|\begin{cases}\frac{m_2^{-2}\ln(m_2+1)-m_1^{-2}\ln(m_1+1)}{m_2-m_1}+\frac{1}{m_1m_2} & \text{if } m_1\neq m_2,\\[2mm] \frac{m+2}{m^2(m+1)}-\frac{2\ln(m+1)}{m^3} & \text{if } m_1=m_2=m.\end{cases}
\end{aligned}$$

4. Midpoint and Trapezoid Inequalities

We have the following midpoint type inequalities:

Proposition 1. *For all A, $B\geq m>0$ we have the midpoint inequality*

$$\begin{aligned}
&\left\|\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \qquad (4.1)\\
&\leq-\frac{1}{4}\mathcal{D}'(w,\mu)(m)\,\|B-A\|.
\end{aligned}$$

Proof. Since A, $B\geq m$, hence $\frac{A+B}{2}\geq m>0$ and $(1-t)A+tB\geq m>0$ for all $t\in[0,1]$ and by (3.1)

$$\begin{aligned}
&\left\|\mathcal{D}(w,\mu)((1-t)A+tB)-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \qquad (4.2)\\
&\leq(-\mathcal{D}'(w,\mu)(m))\left\|(1-t)A+tB-\frac{A+B}{2}\right\|\\
&=-\mathcal{D}'(w,\mu)(m)\left|t-\frac{1}{2}\right|\|B-A\|
\end{aligned}$$

for all $t\in[0,1]$.

Taking the integral in (4.2), we get

$$\begin{aligned}&\left\|\int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\|\\&\leq\int_0^1\left\|\mathcal{D}(w,\mu)((1-t)A+tB)-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\|dt\\&\leq-\mathcal{D}'(w,\mu)(m)\|B-A\|\int_0^1\left|t-\frac{1}{2}\right|dt=-\frac{1}{4}\mathcal{D}'(w,\mu)(m)\|B-A\|\end{aligned}$$

and the inequality (4.1) is proved. □

The case of operator monotone functions is as follows:

Corollary 5. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function. If $A,\ B\geq m>0$, then*

$$\begin{aligned}&\left\|\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)\,dt-\left(\frac{A+B}{2}\right)^{-1}f\left(\frac{A+B}{2}\right)\right.\\&\left.-f(0)\left(\int_0^1((1-t)A+tB)^{-1}dt-\left(\frac{A+B}{2}\right)^{-1}\right)\right\|\\&\leq\frac{1}{4}\|B-A\|\frac{f(m)-f(0)-f'(m)m}{m^2}.\end{aligned}\tag{4.3}$$

If $f(0)=0$, then

$$\begin{aligned}&\left\|\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)\,dt-\left(\frac{A+B}{2}\right)^{-1}f\left(\frac{A+B}{2}\right)\right\|\\&\leq\frac{1}{4}\|B-A\|\frac{f(m)-f'(m)m}{m^2}.\end{aligned}\tag{4.4}$$

Proof. From (1.1) we have that

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t),$$

where $\ell(\lambda)=\lambda,\ \lambda>0$.

Then

$$\begin{aligned}&\int_0^1\mathcal{D}(\ell,\mu)((1-t)A+tB)\,dt\\&=\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)\,dt-f(0)\int_0^1((1-t)A+tB)^{-1}dt-b,\end{aligned}$$

and

$$\mathcal{D}(\ell,\mu)\left(\frac{A+B}{2}\right)=\left(\frac{A+B}{2}\right)^{-1}f\left(\frac{A+B}{2}\right)-f(0)\left(\frac{A+B}{2}\right)^{-1}-b$$

and by (4.1) we get (4.3). □

From inequality (4.4) we get the following power inequality

$$\left\|\int_0^1((1-t)A+tB)^{r-1}dt-\left(\frac{A+B}{2}\right)^{r-1}\right\|\leq\frac{1-r}{4m^{2-r}}\|B-A\|,\tag{4.5}$$

where $r\in(0,1]$ and $A,\ B\geq m>0$.

The following logarithmic inequality also holds

$$\begin{aligned}(4.6)\qquad & \left\| \int_0^1 ((1-t)A+tB)^{-1} \ln((1-t)A+tB+1)\,dt \right.\\ & \left. - \left(\frac{A+B}{2}\right)^{-1} \ln\left(\frac{A+B}{2}+1\right) \right\| \\ & \le \frac{(m+1)\ln(m+1)-m}{4m^2(m+1)} \|B-A\|,\end{aligned}$$

where A, $B \ge m > 0$.

Corollary 6. *Assume that* $f : [0,\infty) \to \mathbb{R}$ *is an operator convex function. If* A, $B \ge m > 0$, *then*

$$\begin{aligned}(4.7)\qquad & \left\| \int_0^1 ((1-t)A+tB)^{-2} f((1-t)A+tB)\,dt - \left(\frac{A+B}{2}\right)^{-2} f\left(\frac{A+B}{2}\right) \right.\\ & - f(0)\left(\int_0^1 ((1-t)A+tB)^{-2}\,dt - \left(\frac{A+B}{2}\right)^{-2}\right) \\ & \left. - f'_+(0)\left(\int_0^1 ((1-t)A+tB)^{-1}\,dt - \left(\frac{A+B}{2}\right)^{-1}\right) \right\| \\ & \le \frac{1}{2m^2}\|B-A\| \left(\frac{f(m)-f(0)}{m} - \frac{f'(m)+f'_+(0)}{2}\right).\end{aligned}$$

If $f(0) = 0$, *then*

$$\begin{aligned}(4.8)\qquad & \left\| \int_0^1 ((1-t)A+tB)^{-2} f((1-t)A+tB)\,dt - \left(\frac{A+B}{2}\right)^{-2} f\left(\frac{A+B}{2}\right) \right.\\ & \left. - f'_+(0)\left(\int_0^1 ((1-t)A+tB)^{-1}\,dt - \left(\frac{A+B}{2}\right)^{-1}\right) \right\| \\ & \le \frac{1}{2m^2}\|B-A\| \left(\frac{f(m)}{m} - \frac{f'(m)+f'_+(0)}{2}\right).\end{aligned}$$

Proof. From (1.5) we have that

$$\frac{f(t)-f(0)-f'_+(0)\,t}{t^2} - c = \mathcal{D}(\ell,\mu)(t),$$

for $t > 0$. We have

$$\begin{aligned}\int_0^1 \mathcal{D}(\ell,\mu)((1-t)A+tB)\,dt &= \int_0^1 ((1-t)A+tB)^{-2} f((1-t)A+tB)\,dt \\ &\quad - f(0)\int_0^1 ((1-t)A+tB)^{-2}\,dt \\ &\quad - f'_+(0)\int_0^1 ((1-t)A+tB)^{-1}\,dt - c\end{aligned}$$

and

$$\mathcal{D}(\ell,\mu)\left(\frac{A+B}{2}\right)=\left(\frac{A+B}{2}\right)^{-2}f\left(\frac{A+B}{2}\right)-f(0)\left(\frac{A+B}{2}\right)^{-2}$$
$$-f'_{+}(0)\left(\frac{A+B}{2}\right)^{-1}-c$$

and by (4.1) we get (4.7). □

From (4.8) we get the logarithmic inequality

$$\begin{aligned}&\left\|\int_0^1((1-t)A+tB)^{-2}\ln((1-t)A+tB+1)\,dt\right.\\&-\left(\frac{A+B}{2}\right)^{-2}\ln\left(\frac{A+B}{2}+1\right)\\&\left.-\int_0^1((1-t)A+tB)^{-1}dt+\left(\frac{A+B}{2}\right)^{-1}\right\|\\&\leq\frac{1}{4m^2}\|B-A\|\left(\frac{m+2}{m+1}-\frac{2\ln(m+1)}{m}\right)\end{aligned}\tag{4.9}$$

for A, $B\geq m>0$.

We have the following midpoint type inequalities:

Proposition 2. *For all A, $B\geq m>0$ we have the trapezoid inequality*

$$\begin{aligned}&\left\|\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt\right\|\\&\leq-\frac{1}{4}\mathcal{D}'(w,\mu)(m)\|B-A\|.\end{aligned}\tag{4.10}$$

Proof. Since A, $B\geq m$, hence $(1-s)A+s\frac{A+B}{2}$, $s\frac{A+B}{2}+(1-s)B\geq m>0$ for all $s\in[0,1]$ and by (2.5) we get

$$\begin{aligned}&\left\|\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)\right\|\\&\leq\frac{1}{2}\left(-\mathcal{D}'(w,\mu)(m)\right)\|B-A\|s\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}&\left\|\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right\|\\&\leq\frac{1}{2}\left(-\mathcal{D}'(w,\mu)(m)\right)\|B-A\|s.\end{aligned}\tag{4.12}$$

From (4.11) and (4.12) we derive by addition, division by 2 and triangle inequality that

$$\begin{aligned}&\left\|\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}\right.\\&\left.-\frac{1}{2}\left[\mathcal{D}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)+\mathcal{D}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right]\right\|\\&\leq\frac{1}{2}\left(-\mathcal{D}'(w,\mu)(m)\right)\|B-A\|s\end{aligned}$$

for all $s \in [0,1]$.

By taking the integral and using its properties, we derive

$$\begin{aligned}(4.13)\quad & \left\| \frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2} \right. \\ & -\frac{1}{2}\left[\int_0^1 \mathcal{D}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)\right. \\ & \left.\left. +\mathcal{D}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)ds\right]\right\| \\ & \leq \frac{1}{2}\left(-\mathcal{D}'(w,\mu)(m)\right)\|B-A\|\int_0^1 sds=\frac{1}{4}\left(-\mathcal{D}'(w,\mu)(m)\right)\|B-A\|.\end{aligned}$$

Now, using the change of variable $t=2s$ we have

$$\frac{1}{2}\int_0^1 \mathcal{D}(w,\mu)\left((1-t)A+t\frac{A+B}{2}\right)dt=\int_0^{1/2}\mathcal{D}(w,\mu)((1-s)A+sB)\,ds$$

and by the change of variable $t=1-v$ we have

$$\begin{aligned}&\frac{1}{2}\int_0^1 \mathcal{D}(w,\mu)\left(t\frac{A+B}{2}+(1-t)A\right)dt \\ &=\frac{1}{2}\int_0^1 \mathcal{D}(w,\mu)\left((1-v)\frac{A+B}{2}+vB\right)dv.\end{aligned}$$

Moreover, if we make the change of variable $v=2s-1$ we also have

$$\frac{1}{2}\int_0^1 \mathcal{D}(w,\mu)\left((1-v)\frac{A+B}{2}+vB\right)dv=\int_{1/2}^1 \mathcal{D}(w,\mu)((1-s)A+sB)\,ds.$$

Therefore

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\mathcal{D}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)+\mathcal{D}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right]ds \\ &=\int_0^{1/2}\mathcal{D}(w,\mu)((1-s)A+sB)\,dt+\int_{1/2}^1\mathcal{D}(w,\mu)((1-s)A+sB)\,ds \\ &=\int_0^1 \mathcal{D}(w,\mu)((1-s)A+sB)\,ds\end{aligned}$$

and by (4.13) we deduce the desired result (4.10). □

Corollary 7. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function with* $f(0)=0$. *If* $A,\ B\geq m>0$, *then*

$$\begin{aligned}(4.14)\quad & \left\|\frac{f(A)A^{-1}+f(B)B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)\,dt\right\| \\ & \leq \frac{f(m)-f'(m)m}{4m^2}\|B-A\|.\end{aligned}$$

Assume that $A,\ B\geq m>0$, then by Corollary 7 we obtain the following power inequalities

$$(4.15)\quad \left\|\frac{A^{r-1}+B^{r-1}}{2}-\int_0^1((1-t)A+tB)^{r-1}dt\right\|\leq\frac{1-r}{4m^{2-r}}\|B-A\|,$$

where $r\in(0,1]$.

We can also state the logarithmic inequality

$$\begin{aligned}(4.16)\qquad & \left\| \frac{A^{-1}\ln(A+1)+B^{-1}\ln(B+1)}{2} \right. \\ & \left. -\int_0^1 ((1-t)A+tB)^{-1}\ln((1-t)A+tB+1)\,dt \right\| \\ & \leq \frac{(m+1)\ln(m+1)-m}{4m^2(m+1)}\|B-A\|,\end{aligned}$$

provided that $A,\ B\geq m>0$.

Corollary 8. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function with $f(0)=0$. If $A,\ B\geq m>0$, then*

$$\begin{aligned}(4.17)\qquad & \left\| \frac{f(A)A^{-2}+f(B)B^{-2}}{2}-\int_0^1((1-t)A+tB)^{-2}f((1-t)A+tB)\,dt \right. \\ & \left. -f'_+(0)\left(\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right)\right\| \\ & \leq \frac{1}{2m^2}\|B-A\|\left(\frac{f(m)}{m}-\frac{f'(m)+f'_+(0)}{2}\right).\end{aligned}$$

Assume that $A,\ B\geq m>0$, then by Corollary 8 we obtain the following logarithmic inequalities

$$\begin{aligned}(4.18)\qquad & \left\| \frac{A^{-2}\ln(A+1)+B^{-2}\ln(B+1)}{2} \right. \\ & -\int_0^1((1-t)A+tB)^{-2}\ln((1-t)A+tB+1)\,dt \\ & \left. -\left(\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right)\right\| \\ & \leq \frac{1}{4m^2}\|B-A\|\left(\frac{m+2}{m+1}-\frac{2\ln(m+1)}{m}\right).\end{aligned}$$

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