LIPSCHITZ TYPE INEQUALITIES FOR AN INTEGRAL TRANSFORM OF POSITIVE OPERATORS WITH APPLICATIONS

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1. Introduction

Consider a complex Hilbert space \((H, \langle \cdot , \cdot \rangle)\). An operator \(T\) is said to be positive (denoted by \(T \geq 0\)) if \(\langle Tx, x \rangle \geq 0\) for all \(x \in H\) and also an operator \(T\) is said to be strictly positive (denoted by \(T > 0\)) if \(T\) is positive and invertible. A real valued continuous function \(f\) on \((0, \infty)\) is said to be operator monotone if \(f(A) \geq f(B)\) holds for any \(A \geq B > 0\).

In 1934, K. Löwner [15] had given a definitive characterization of operator monotone functions as follows, see for instance [5, p. 144-145]:

**Theorem 1.** A function \(f : (0, \infty) \to \mathbb{R}\) is operator monotone in \((0, \infty)\) if and only if it has the representation

\[
(1.1) \quad f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda)
\]
where \( a \in \mathbb{R} \) and \( b \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that
\[
\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty.
\]

If \( f \) is operator monotone in \([0, \infty)\), then \( a = f(0) \) in (1.1).

We recall the important fact proved by Löwner and Heinz that states that the power function \( f : (0, 1) \to \mathbb{R}, f(t) = t^\alpha \) is an operator monotone function for any \( \alpha \in [0, 1], [13] \). The function \( \ln \) is also operator monotone on \((0, \infty)\). For other examples of operator monotone functions, see [10] and [12].

A real valued continuous function \( f \) on an interval \( I \) is said to be operator convex (operator concave) on \( I \) if
\[
(OC) 
\left( (1 - \lambda) A + \lambda B \right) \geq (1 - \lambda) f(A) + \lambda f(B)
\]
in the operator order, for all \( \lambda \in [0, 1] \) and for every selfadjoint operator \( A \) and \( B \) on a Hilbert space \( H \) whose spectra are contained in \( I \). Notice that a function \( f \) is operator concave if \( f \) is operator convex.

We have the following representation of operator convex functions [5, p. 147]:

**Theorem 2.** A function \( f : (0, \infty) \to \mathbb{R} \) is operator convex in \((0, \infty)\) if and only if it has the representation
\[
(1.2) 
\begin{align*}
\frac{d}{dt} f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda),
\end{align*}
\]
where \( a, b \in \mathbb{R}, c \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that (1.2) holds. If \( f \) is operator convex in \([0, \infty)\), then \( a = f(0) \) and \( b = f'(0) \), the right derivative, in (1.2).

Let \( B(H) \) be the Banach algebra of bounded linear operators on a complex Hilbert space \( H \). The absolute value of an operator \( A \) is the positive operator \( |A| := (A^*A)^{1/2} \).

It is known that [3] in the infinite-dimensional case the map \( f(A) := |A| \) is not Lipschitz continuous on \( B(H) \) with the usual operator norm, i.e. there is no constant \( L > 0 \) such that
\[
\|A - B\| \leq L \|A - B\|, 
\]
for any \( A, B \in B(H) \).

However, as shown by Farforovskaya in [7], [8] and Kato in [14], the following inequality holds
\[
(1.3) 
\|\|A\| - \|B\|\| \leq \frac{2}{\pi} \|A - B\| \left( 2 + \log \left( \frac{\|A\| + \|B\|}{\|A - B\|} \right) \right)
\]
for any \( A, B \in B(H) \) with \( A \neq B \).

If the operator norm is replaced with Hilbert-Schmidt norm \( \|C\|_{HS} := (\text{tr } C^*C)^{1/2} \) of an operator \( C \), then the following inequality is true [1]
\[
(1.4) 
\|\|A\| - \|B\|\|_{HS} \leq \sqrt{2} \|A - B\|_{HS}
\]
for any \( A, B \in B(H) \).

The coefficient \( \sqrt{2} \) is best possible for a general \( A \) and \( B \). If \( A \) and \( B \) are restricted to be selfadjoint, then the best coefficient is 1.
It has been shown in [3] that, if $A$ is an invertible operator, then for all operators $B$ in a neighborhood of $A$ we have
\begin{equation}
\|A-B\| \leq a_1 \|A-B\|^2 + a_2 \|A-B\|^3 + O(\|A-B\|^4)
\end{equation}
where
\begin{align*}
a_1 &= \|A^{-1}\| \|A\| \quad \text{and} \quad a_2 = \|A^{-1}\| + \|A^{-1}\|^3 \|A\|^2.
\end{align*}

In [2] the author also obtained the following Lipschitz type inequality
\begin{equation}
k_f(A) f(B) \leq f'(a) \|A-B\|
\end{equation}
where $f$ is an operator monotone function on $(0, \infty)$ and $A, B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $k_f(A) f(B)$ in terms of $\|A-B\|$ for different classes of measurable functions $f$ for which the function of operator can be defined. For some results on this topic, see [4], [9] and the references therein.

We have the following integral representation for the power function when $t > 0$,
\begin{equation}
t^r = \frac{\sin \left(\frac{r\pi}{2}\right)}{\pi} t^\infty \frac{\lambda^{r-1}}{\lambda+t} d\lambda.
\end{equation}

Observe that for $t > 0$, $t \neq 1$, we have
\begin{equation}
\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln \left(\frac{u+t}{u+1}\right)
\end{equation}
for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive
\begin{equation}
\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)}
\end{equation}
for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following integral transform
\begin{equation}
D(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \quad t > 0,
\end{equation}
where $\mu$ is a positive measure on $(0, \infty)$ and the integral (1.9) exists for all $t > 0$.

For $\mu$ the Lebesgue usual measure, we put
\begin{equation}
D(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \quad t > 0.
\end{equation}

If we take $\mu$ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}, \quad r \in (0, 1]$, then
\begin{equation}
t^{r-1} = \frac{\sin \left(\frac{r\pi}{2}\right)}{\pi} D(w_r)(t), \quad t > 0.
\end{equation}

For the same measure, if we take the kernel $w_{ln}(\lambda) = (\lambda + 1)^{-1}, \quad t > 0$, we have the representation
\begin{equation}
\ln t = (t-1) D(w_{ln})(t), \quad t > 0.
\end{equation}
Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

\begin{equation}
D(w, \mu)(T) := \int_{0}^{\infty} w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),
\end{equation}

where $w$ and $\mu$ are as above. Also, when $\mu$ is the usual Lebesgue measure, then

\begin{equation}
D(w)(T) := \int_{0}^{\infty} w(\lambda)(\lambda + T)^{-1} d\lambda,
\end{equation}

for $T > 0$.

In this paper we show among others that, if $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

\begin{equation}
\|D(w, \mu)(B) - D(w, \mu)(A)\| \leq \|B - A\| \left\{ \begin{array}{ll}
\frac{D(w, \mu)(m_1) - D(w, \mu)(m_2)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
(-D'(w, \mu)(m)) & \text{if } m_1 = m_2 = m,
\end{array} \right.
\end{equation}

where $D'(w, \mu)(t)$ is the derivative of $D(w, \mu)$ as a function of $t$.

If $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function with $f(0) = 0$, then

\begin{equation}
\|f(A) A^{-1} - f(B) B^{-1}\| \leq \|B - A\| \left\{ \begin{array}{ll}
\frac{f(m_1)m_1^{-1} - f(m_2)m_2^{-1}}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\frac{f(m) - f'(m)m}{m^2} & \text{if } m_1 = m_2 = m.
\end{array} \right.
\end{equation}

Similar inequalities for operator convex functions and some particular examples of interest are also given.

2. Some Preliminary Facts

In the following, whenever we write $D(w, \mu)$ we mean that the integral from (1.9) exists and is finite for all $T > 0$.

**Lemma 1.** For all $A, B > 0$ we have the representation

\begin{equation}
D(w, \mu)(B) - D(w, \mu)(A) = -\int_{0}^{\infty} w(\lambda) \left( \int_{0}^{1} (\lambda + (1 - t) B + tA)^{-1} (B - A) (\lambda + (1 - t) B + tA)^{-1} dt \right) d\mu(\lambda).
\end{equation}

**Proof.** Observe that, for all $A, B > 0$

\begin{equation}
D(w, \mu)(B) - D(w, \mu)(A) = \int_{0}^{\infty} w(\lambda) \left[ (\lambda + B)^{-1} - (\lambda + A)^{-1} \right] d\mu(\lambda).
\end{equation}

Let $T, S > 0$. The function $f(t) = -t^{-1}$ is operator monotone on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

\begin{equation}
\nabla f_{T}(S) := \lim_{t \to 0} \left[ \frac{f(T + tS) - f(T)}{t} \right] = T^{-1} ST^{-1}
\end{equation}

for $T, S > 0$.

Consider the continuous function $f$ defined on an interval $I$ for which the corresponding operator function is Gâteaux differentiable on the segment $[C, D]$ :
\{(1-t)C + tD, t \in [0,1]\} for \(C, D\) selfadjoint operators with spectra in \(I\). We consider the auxiliary function defined on \([0,1]\) by
\[
f_{C,D}(t) := f((1-t)C + tD), \; t \in [0,1].
\]
Then we have, by the properties of the Bochner integral, that
\[
(2.4) \quad f(D) - f(C) = \int_0^1 \frac{d}{dt} f_{C,D}(t) \, dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C) \, dt.
\]
If we write this equality for the function \(f(t) = -t^{-1}\) and \(C, D > 0\), then we get the representation
\[
(2.5) \quad C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D-C) ((1-t)C + tD)^{-1} \, dt.
\]
Now, if we take in (2.5) \(C = \lambda + B, \; D = \lambda + A\), then
\[
(2.6) \quad (\lambda + B)^{-1} - (\lambda + A)^{-1} = \int_0^1 ((1-t)(\lambda + B) + t(\lambda + A))^{-1} ((A - B) \quad \times ((1-t)(\lambda + B) + t(\lambda + A))^{-1} \, dt
\]
\[
= \int_0^1 (\lambda + (1-t)B + tA)^{-1} (A - B) (\lambda + (1-t)B + tA)^{-1} \, dt
\]
and by (2.2) we derive (2.1).

**Remark 1.** If \(B \geq A > 0\), then by representation (2.1) we derive that
\[
\mathcal{D}(w,\mu)(B) \leq \mathcal{D}(w,\mu)(A),
\]
which shows that \(-\mathcal{D}(w,\mu)\) is operator monotone on \((0,\infty)\). For further results related to the operator monotonicity of this integral transform see the recent paper [6].

**Remark 2.** We observe that if \(A, B > 0\) and \(r \in (0,1]\), then by (1.11) we get the identity
\[
(2.7) \quad B^{r-1} - A^{r-1} = -\frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{r-1} \left( \int_0^1 (\lambda + (1-t)B + tA)^{-1} \times (B - A) (\lambda + (1-t)B + tA)^{-1} \, dt \right) \, d\lambda.
\]
If \(A, B > 0\) with \(A - 1\) and \(B - 1\) invertible, then
\[
(2.8) \quad (B - 1)^{-1} \ln B - (A - 1)^{-1} \ln A
\]
\[
= -\int_0^\infty (\lambda + 1)^{-1}
\]
\[
\times \left( \int_0^1 (\lambda + (1-t)B + tA)^{-1} (B - A)(\lambda + (1-t)B + tA)^{-1} \, dt \right) \, d\lambda.
\]
Assume that \( f : [0, \infty) \to \mathbb{R} \) is an operator monotone function as in (1.1). Then for all \( A, B > 0 \) we have the equality
\[
(2.9) \quad B^{-1}f(B) - A^{-1}f(A) - f(0) (B^{-1} - A^{-1})
= -\int_0^\infty \lambda \left( \int_0^1 \left( \lambda + (1 - t) B + tA \right)^{-1} (B - A) \right.
\times \left( \lambda + (1 - t) B + tA \right)^{-1} dt) \, d\mu(\lambda) .
\]

If \( f(0) = 0 \), then we have the simpler equality
\[
(2.10) \quad B^{-1}f(B) - A^{-1}f(A) - f(0) (B^{-1} - A^{-1})
= -\int_0^\infty \lambda \left( \int_0^1 \left( \lambda + (1 - t) B + tA \right)^{-1} (B - A) \right.
\times \left( \lambda + (1 - t) B + tA \right)^{-1} dt) \, d\mu(\lambda) .
\]

**Proof.** From (1.1) we have that
\[
(2.11) \quad \frac{f(t) - f(0)}{t} - b = D(\ell, \mu)(t) ,
\]
where \( \ell(\lambda) = \lambda, \lambda > 0 \). Then for \( A, B > 0 \),
\[
D(\ell, \mu)(B) - D(\ell, \mu)(A) = [f(B) - f(0)] B^{-1} - [f(A) - f(0)] A^{-1}
= B^{-1}f(B) - A^{-1}f(A) - f(0) (B^{-1} - A^{-1})
\]
and by (2.1) we derive (2.9). \( \square \)

**Corollary 2.** Assume that \( f : [0, \infty) \to \mathbb{R} \) is an operator convex function as in (1.2). Then for all \( A, B > 0 \) we have the equality
\[
(2.12) \quad f(B) B^{-2} - f(A) A^{-2} - f'_+(0) (B^{-1} - A^{-1}) - f(0) (B^{-2} - A^{-2})
= -\int_0^\infty \lambda \left( \int_0^1 \left( \lambda + (1 - t) B + tA \right)^{-1} (B - A) \right.
\times \left( \lambda + (1 - t) B + tA \right)^{-1} dt) \, d\mu(\lambda) .
\]

If \( f(0) = 0 \), then we have the simpler equality
\[
(2.13) \quad f(B) B^{-2} - f(A) A^{-2} - f'_+(0) (B^{-1} - A^{-1})
= -\int_0^\infty \lambda \left( \int_0^1 \left( \lambda + (1 - t) B + tA \right)^{-1} (B - A) \right.
\times \left( \lambda + (1 - t) B + tA \right)^{-1} dt) \, d\mu(\lambda) .
\]

**Proof.** From (1.5) we have that
\[
\frac{f(t) - f(0) - f'_+(0) t}{t^2} - c = D(\ell, \mu)(t) ,
\]
for \( t > 0 \). Then for \( A, B > 0 \),
\[
D(\ell, \mu)(B) - D(\ell, \mu)(A) = f(B) B^{-2} - f(A) A^{-2} - f'_+(0) (B^{-1} - A^{-1})
- f(0) (B^{-2} - A^{-2})
\]
and by (2.1) we derive (2.12). \( \square \)
Remark 3. Let \( a > 0 \) and \( f ( t ) = (t + a)^p \) with \( p \in [-1, 0) \cup [1, 2] \). This function is operator convex and \( f (0) = a^p, f' (0) = pa^{p-1} \). Then for all \( A, B > 0 \) we have the equality

\[
(2.14) \quad (B + a)^p B^{-2} - (A + a)^p A^{-2} - pa^{p-1} (B^{-1} - A^{-1}) - a^p (B^{-2} - A^{-2}) = - \int_0^\infty \lambda \left( \int_0^1 (\lambda + (1-t) B + tA)^{-1} (B - A) \times (\lambda + (1-t) B + tA)^{-1} dt \right) d\mu (\lambda),
\]

for some positive measure \( \mu \) on \( (0, \infty) \).

3. Main Results

We have the following Lipschitz type inequality:

Theorem 3. Assume that \( A \geq m_1 > 0, B \geq m_2 > 0 \), then

\[
(3.1) \quad \| D (w, \mu) (B) - D (w, \mu) (A) \| \leq \| B - A \| \begin{cases} \frac{D (w, \mu) (m_1) - D (w, \mu) (m_2)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ (-D' (w, \mu) (m)) & \text{if } m_1 = m_2 = m, \end{cases}
\]

where \( D' (w, \mu) (m) \) is the derivative of \( D (w, \mu) \) as a function of \( m \).

Proof. From the identity (2.1) we get by taking the norm that

\[
(3.2) \quad \| D (w, \mu) (B) - D (w, \mu) (A) \| \leq \int_0^\infty w (\lambda) \left( \int_0^1 \| (\lambda + (1-t) B + tA)^{-1} (B - A) (\lambda + (1-t) B + tA)^{-1} dt \| d\mu (\lambda) \right)
\]

for all \( A, B > 0 \).

Assume that \( m_2 > m_1 \). Then

\[
(1-t) A + tB + \lambda \geq (1-t) m_1 + tm_2 + \lambda,
\]

which implies that

\[
((1-t) A + tB + \lambda)^{-1} \leq ((1-t) m_1 + tm_2 + \lambda)^{-1},
\]

and

\[
(3.3) \quad \left\| ((1-t) A + tB + \lambda)^{-1} \right\|^2 \leq ((1-t) m_1 + tm_2 + \lambda)^{-2}
\]

for all \( t \in [0, 1] \) and \( \lambda \geq 0 \).
Therefore, by integrating (3.3) we derive
\[
\int_0^\infty w(\lambda) \left( \int_0^1 \left\| (1-t) A + tB + \lambda \right\|^2 dt \right) dw(\lambda) \\
\leq \int_0^\infty w(\lambda) \left( \int_0^1 ((1-t) m_1 + tm_2 + \lambda)^{-2} dt \right) dw(\lambda) \\
= \frac{1}{m_2 - m_1} \int_0^\infty w(\lambda) \left( \int_0^1 ((1-t) m_1 + tm_2 + \lambda)^{-1} \right. \\
\times (m_2 - m_1) ((1-t) m_1 + tm_2 + \lambda)^{-1} dt \right) dw(\lambda) \\
= \frac{1}{m_2 - m_1} [D(w, \mu)(m_1) - D(w, \mu)(m_2)] (by (2.1))
\]
and by (3.2) we deduce
\[
\|D(w, \mu)(B) - D(w, \mu)(A)\| \\
\leq \frac{1}{m_2 - m_1} [D(w, \mu)(m_1) - D(w, \mu)(m_2)].
\]
The case \(m_2 < m_1\) goes in a similar way and we also obtain (3.4).

Let \(\epsilon > 0\). Then \(B + \epsilon \geq m + \epsilon > m\). From (3.4) we get
\[
\|D(w, \mu)(B + \epsilon) - D(w, \mu)(A)\| \\
\leq \frac{1}{m + \epsilon - m} [D(w, \mu)(m) - D(w, \mu)(m + \epsilon)]
\]
and by taking the limit over \(\epsilon \to 0^+\), using the continuity and differentiability of \(D(w, \mu)\) we deduce the second part of (3.1). \(\square\)

**Corollary 3.** Assume that \(f : [0, \infty) \to \mathbb{R}\) is an operator monotone function. If \(A \geq m_1 > 0, B \geq m_2 > 0\), then
\[
\|f(A)A^{-1} - f(B)B^{-1} - f(0)(A^{-1} - B^{-1})\| \\
\leq \|B - A\| \begin{cases} 
\frac{(f(m_1)m_1^{-1} - f(m_2)m_2^{-1})}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\frac{f(m) - f(0) - f'(0)m}{m^2} & \text{if } m_1 = m_2 = m.
\end{cases}
\]
If \(f(0) = 0\), then we have the simpler inequalities
\[
\|f(A)A^{-1} - f(B)B^{-1}\| \\
\leq \|B - A\| \begin{cases} 
\frac{(f(m_1)m_1^{-1} - f(m_2)m_2^{-1})}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\frac{f(m) - f'(0)m}{m^2} & \text{if } m_1 = m_2 = m.
\end{cases}
\]

**Proof.** From (1.1) we have that
\[
\frac{f(t) - f(0)}{t} - b = D(\ell, \mu)(t),
\]
where \(\ell(\lambda) = \lambda, \lambda > 0\).
Then
\[
\frac{1}{m_2 - m_1} [\mathcal{D}(w, \mu)(m_1) - \mathcal{D}(w, \mu)(m_2)]
= \frac{f(m_1) m_1^{-1} - f(m_2) m_2^{-1}}{m_2 - m_1} - \frac{f(0)}{m_2 m_1}
\]
and
\[
-\mathcal{D}'(w, \mu)(m) = \frac{f(m) - f(0) - f'(m)m}{m^2}.
\]
By making use of (3.1) we derive (3.5). \(\square\)

**Remark 4.** Let \(A \geq m_1 > 0, B \geq m_2 > 0\) and \(r \in (0, 1]\). Then by (3.6) we have the power inequalities

\[
\|A^{-1} - B^{-1}\| \leq \|B - A\| \left\{ \begin{array}{l}
m_1^{-1} - m_2^{-1} \quad \text{if } m_1 \neq m_2, \\
\frac{1 - r}{m_2^{-1}} \quad \text{if } m_1 = m_2 = m.
\end{array} \right.
\]

If we take \(f(t) = \ln(t + 1)\), then we get by (3.6) that

\[
\|A^{-1} \ln(A + 1) - B^{-1} \ln(B + 1)\| \leq \|B - A\| \left\{ \begin{array}{l}
\frac{m_1^{-1} \ln(m_1 + 1) - m_2^{-1} \ln(m_2 + 1)}{m_2^{-1} - m_1} \quad \text{if } m_1 \neq m_2, \\
\frac{(m_1 + 1) \ln(m_1 + 1) - m}{m^2} \quad \text{if } m_1 = m_2 = m.
\end{array} \right.
\]

**Corollary 4.** Assume that \(f : [0, \infty) \to \mathbb{R}\) is an operator convex function. If \(A \geq m_1 > 0, B \geq m_2 > 0\), then

\[
\|f(B) - f(A) - f'(0)(B - A)\| \leq \|B - A\| \left\{ \begin{array}{l}
\frac{f(m_1) m_1^{-2} - f(m_2) m_2^{-2}}{m_2^{-1} - m_1} - \frac{f'_0(0) m_1 + m_2}{m_1 m_2} \quad \text{if } m_1 \neq m_2, \\
2 \frac{f(m) - f(0)}{m^2} - \frac{f'(0) m^2 + f'_0(0)}{m^2} \quad \text{if } m_1 = m_2 = m.
\end{array} \right.
\]

If \(f(0) = 0\), then

\[
\|f(B) - f(A) - f'_0(0)(B - A)\| \leq \|B - A\| \left\{ \begin{array}{l}
\frac{f(m_1) m_1^{-2} - f(m_2) m_2^{-2}}{m_2^{-1} - m_1} - \frac{f'_0(0) m_1 + m_2}{m_1 m_2} \quad \text{if } m_1 \neq m_2, \\
\frac{2 f(m) - f'(0) m}{m^2} \quad \text{if } m_1 = m_2 = m.
\end{array} \right.
\]

**Proof.** From (1.5) we have that

\[
f(t) - f(0) - f'_0(0) \frac{t}{t^2} = c = \mathcal{D}(t, \mu)(t),
\]
for \(t > 0\).
Then
\[
\frac{D (w, \mu) (m_1) - D (w, \mu) (m_2)}{m_2 - m_1} = \frac{1}{m_2 - m_1} \left[ \frac{f (m_1) - f (0) - f' (0) m_1}{m_1^2} - \frac{f (m_2) - f (0) - f' (0) m_2}{m_2^2} \right] \\
= \frac{f (m_1) m_1^2 - f (m_2) m_2^2}{m_2 - m_1} - \frac{f' (0)}{m_1 m_2} - f (0) \frac{m_1 + m_2}{m_1^2 m_2^2}.
\]

Since
\[
\left( \frac{f (t) - f (0) - f' (0) t}{t^2} \right)' = \frac{f' (t) - f' (0)}{t^2} - 2 \frac{f (t) - f (0) - f' (0) t}{t^4}
\]
\[
= \frac{f' (t) + f' (0)}{t^2} - 2 \frac{f (t) - f (0)}{t^3},
\]
hence
\[
-D' (w, \mu) (m) = 2 \frac{f (m) - f (0) - f' (m) + f' (0)}{m^3} - \frac{f' (m) + f' (0)}{m^2}
\]
and by (3.1) we obtain (3.9).

Remark 5. If we take \( f (t) = -\ln (t + 1) \) in (3.10), then for \( A \geq m_1 > 0, B \geq m_2 > 0 \) we get
\[
\| B^{-2} \ln (B + 1) - A^{-2} \ln (A + 1) - B^{-1} + A^{-1} \|
\]
\[
\leq \| B - A \| \left\{ \begin{array}{ll}
\frac{m_2^2 \ln (m_2 + 1) - m_1^2 \ln (m_1 + 1)}{m_2 - m_1} + \frac{1}{m_1 m_2} & \text{if } m_1 \neq m_2, \\
\frac{m_2 + 2}{m_2 (m + 1)} - \frac{2 \ln (m + 1)}{m^2} & \text{if } m_1 = m_2 = m.
\end{array} \right.
\]

4. MIDPOINT AND TRAPEZOID INEQUALITIES

We have the following midpoint type inequalities:

Proposition 1. For all \( A, B \geq m > 0 \) we have the midpoint inequality
\[
\int_0^1 D (w, \mu) ((1 - t) A + tB) dt - D (w, \mu) \left( \frac{A + B}{2} \right) \leq -\frac{1}{4} D' (w, \mu) (m) \| B - A \|.
\]

Proof. Since \( A, B \geq m, \) hence \( \frac{A + B}{2} \geq m > 0 \) and \((1 - t) A + tB \geq m > 0 \) for all \( t \in [0, 1] \) and by (3.1)
\[
\| D (w, \mu) ((1 - t) A + tB) - D (w, \mu) \left( \frac{A + B}{2} \right) \|
\]
\[
\leq (-D' (w, \mu) (m)) \| (1 - t) A + tB - \frac{A + B}{2} \|
\]
\[
= -D' (w, \mu) (m) \left| t - \frac{1}{2} \right| \| B - A \|
\]
for all \( t \in [0, 1] \).
Taking the integral in (4.2), we get
\[
\left\| \int_0^1 \mathcal{D}(w, \mu) ((1-t)A + tB) \, dt - \mathcal{D}(w, \mu) \left( \frac{A + B}{2} \right) \right\| \\
\leq \int_0^1 \left\| \mathcal{D}(w, \mu) ((1-t)A + tB) - \mathcal{D}(w, \mu) \left( \frac{A + B}{2} \right) \right\| \, dt \\
\leq -\mathcal{D}'(w, \mu)(m) \|B - A\| \int_0^1 \left| t - \frac{1}{2} \right| \, dt = -\frac{1}{4} \mathcal{D}'(w, \mu)(m) \|B - A\|
\]
and the inequality (4.1) is proved.

The case of operator monotone functions is as follows:

**Corollary 5.** Assume that \( f : [0, \infty) \to \mathbb{R} \) is an operator monotone function. If \( A, B \geq m > 0 \), then
\[
\left(1 - t\right) A + tB \int_0^1 \left( t - \frac{1}{2} \right) \, dt = \frac{1}{4} \mathcal{D}'(w, \mu)(m) \|B - A\|.
\]

Proof. From (1.1) we have that
\[
\left( f(t) - f(0) \right) \frac{1}{t} = \mathcal{D}(\ell, \mu)(t),
\]
where \( \ell(\lambda) = \lambda, \lambda > 0 \).

Then
\[
\int_0^1 \mathcal{D}(\ell, \mu)((1-t)A + tB) \, dt = \int_0^1 \left( (1-t)A + tB \right) \left( t - \frac{1}{2} \right) \, dt = f(0) \int_0^1 \left( (1-t)A + tB \right) \, dt - b,
\]
and
\[
\mathcal{D}(\ell, \mu) \left( \frac{A + B}{2} \right) = \left( \frac{A + B}{2} \right)^{-1} f \left( \frac{A + B}{2} \right) - f(0) \left( \frac{A + B}{2} \right)^{-1} b
\]
and by (4.1) we get (4.3).

From inequality (4.4) we get the following power inequality
\[
\left( (1-t)A + tB \right)^{\alpha-1} \left( (1-t)A + tB \right) \int_0^1 \left( (1-t)A + tB \right) \, dt - f(0) \int_0^1 \left( (1-t)A + tB \right) \, dt \leq \frac{1 - r}{4m^{2-r}} \|B - A\|,
\]
where \( r \in (0, 1] \) and \( A, B \geq m > 0 \).
The following logarithmic inequality also holds

\[ \left\| \int_0^1 ((1-t)A + tB)^{-1} \ln ((1-t)A + tB + 1) \, dt \right\| \]
\[ \leq \left\| \frac{(m+1)\ln (m+1) - m}{4m^2 (m+1)} \|B - A\|, \right\| \]

where \( A, B \geq m > 0 \).

**Corollary 6.** Assume that \( f : [0, \infty) \to \mathbb{R} \) is an operator convex function. If \( A, B \geq m > 0 \), then

\[ \left\| \int_0^1 ((1-t)A + tB)^{-2} f ((1-t)A + tB) \, dt - \left( \frac{A+B}{2} \right)^{-2} f \left( \frac{A+B}{2} \right) \right\| \]
\[ - f(0) \left( \int_0^1 ((1-t)A + tB)^{-2} \, dt - \left( \frac{A+B}{2} \right)^{-2} \right) \]
\[ - f'_+(0) \left( \int_0^1 ((1-t)A + tB)^{-1} \, dt - \left( \frac{A+B}{2} \right)^{-1} \right) \]
\[ \leq \frac{1}{2m^2} \|B - A\| \left( \frac{f(m) - f(0)}{m} - \frac{f'(m) + f'_+(0)}{2} \right). \]

If \( f(0) = 0 \), then

\[ \left\| \int_0^1 ((1-t)A + tB)^{-2} f ((1-t)A + tB) \, dt - \left( \frac{A+B}{2} \right)^{-2} f \left( \frac{A+B}{2} \right) \right\| \]
\[ - f'_+(0) \left( \int_0^1 ((1-t)A + tB)^{-1} \, dt - \left( \frac{A+B}{2} \right)^{-1} \right) \]
\[ \leq \frac{1}{2m^2} \|B - A\| \left( \frac{f(m)}{m} - \frac{f'(m) + f'_+(0)}{2} \right). \]

**Proof.** From (1.5) we have that

\[ \frac{f(t) - f(0) - f'_+(0) t}{t^2} - c = D(t, \mu)(t), \]

for \( t > 0 \). We have

\[ \int_0^1 D(t, \mu)((1-t)A + tB) \, dt = \int_0^1 ((1-t)A + tB)^{-2} f ((1-t)A + tB) \, dt \]
\[ - f(0) \int_0^1 ((1-t)A + tB)^{-2} \, dt \]
\[ - f'_+(0) \int_0^1 ((1-t)A + tB)^{-1} \, dt - c \]
and

\[
\mathcal{D}(\ell, \mu) \left( \frac{A + B}{2} \right) = \left( \frac{A + B}{2} \right)^{-2} f \left( \frac{A + B}{2} \right) - f(0) \left( \frac{A + B}{2} \right)^{-2} - f'(0) \left( \frac{A + B}{2} \right)^{-1} - c
\]

and by (4.1) we get (4.7).

From (4.8) we get the logarithmic inequality

\[
\begin{align*}
\int_0^1 ((1 - t) A + tB)^{-2} \ln ((1 - t) A + tB + 1) \, dt \\
- \left( \frac{A + B}{2} \right)^{-2} \ln \left( \frac{A + B}{2} + 1 \right) \\
- \int_0^1 ((1 - t) A + tB)^{-1} \, dt + \left( \frac{A + B}{2} \right)^{-1}
\end{align*}
\]

\[
\leq \frac{1}{4m^2} \|B - A\| \left( \frac{m + 2}{m + 1} \cdot \frac{2 \ln (m + 1)}{m} \right)
\]

for \( A, B \geq m > 0 \).

We have the following midpoint type inequalities:

**Proposition 2.** For all \( A, B \geq m > 0 \) we have the trapezoid inequality

\[
\begin{align*}
\left\| \int_0^1 \mathcal{D}(w, \mu) (A) + \mathcal{D}(w, \mu) (B) \right. \\
- \frac{1}{2} \mathcal{D}(w, \mu) (1 - s) A + s \frac{A + B}{2} \\
\left. - \mathcal{D}(w, \mu) (s \frac{A + B}{2} + (1 - s) B) \right\| \\
\leq \frac{1}{2} \| - \mathcal{D}'(w, \mu) (m) \| B - A\|
\end{align*}
\]

**Proof.** Since \( A, B \geq m, \) hence \((1 - s) A + s \frac{A + B}{2}, s \frac{A + B}{2} + (1 - s) B \geq m > 0\) for all \( s \in [0, 1] \) and by (2.5) we get

\[
\begin{align*}
\left\| \mathcal{D}(w, \mu) (A) - \mathcal{D}(w, \mu) \left( (1 - s) A + s \frac{A + B}{2} \right) \right\| \\
\leq \frac{1}{2} \left( - \mathcal{D}'(w, \mu) (m) \right) \| B - A\| s
\end{align*}
\]

and

\[
\begin{align*}
\left\| \mathcal{D}(w, \mu) (B) - \mathcal{D}(w, \mu) \left( s \frac{A + B}{2} + (1 - s) B \right) \right\| \\
\leq \frac{1}{2} \left( - \mathcal{D}'(w, \mu) (m) \right) \| B - A\| s.
\end{align*}
\]

From (4.11) and (4.12) we derive by addition, division by 2 and triangle inequality that

\[
\begin{align*}
\left\| \mathcal{D}(w, \mu) (A) + \mathcal{D}(w, \mu) (B) \right. \\
- \frac{1}{2} \left[ \mathcal{D}(w, \mu) \left( (1 - s) A + s \frac{A + B}{2} \right) + \mathcal{D}(w, \mu) \left( s \frac{A + B}{2} + (1 - s) B \right) \right] \\
\left. \| B - A\| s
\end{align*}
\]
for all $s \in [0, 1]$.

By taking the integral and using its properties, we derive

\[
\frac{D (w, \mu) (A) + D (w, \mu) (B)}{2} \leq \frac{1}{2} \left( \int_0^1 D (w, \mu) (1 - s) A + s A + B \right) ds + \frac{D (w, \mu) (s A + B + (1 - s) B) ds}{2} \leq \frac{1}{2} \left( -D' (w, \mu) (m) \right) \| B - A \| \int_0^1 sds = \frac{1}{4} \left( -D' (w, \mu) (m) \right) \| B - A \|.
\]

Now, using the change of variable $t = 2s$ we have

\[
\frac{1}{2} \int_0^1 D (w, \mu) (1 - t) A + t A + B \right) dt = \int_0^{1/2} D (w, \mu) ((1 - s) A + sB) ds
\]

and by the change of variable $t = 1 - v$ we have

\[
\frac{1}{2} \int_0^1 D (w, \mu) \left( t A + B \right) dt = \frac{1}{2} \int_0^1 D (w, \mu) \left( (1 - v) A + B \right) dv.
\]

Moreover, if we make the change of variable $v = 2s - 1$ we also have

\[
\frac{1}{2} \int_0^1 D (w, \mu) \left( (1 - v) A + B \right) dv = \int_0^{1/2} D (w, \mu) ((1 - s) A + sB) ds.
\]

Therefore

\[
\frac{1}{2} \int_0^1 \left[ D (w, \mu) (1 - s) A + s A + B \right] + D (w, \mu) \left( s A + B + (1 - s) B \right) ds = \int_0^{1/2} \frac{D (w, \mu) (1 - s) A + sB) dt + \int_0^1 \frac{D (w, \mu) ((1 - s) A + sB) ds}{\frac{1}{2}}
\]

and by (4.13) we deduce the desired result (4.10).

\[\square\]

**Corollary 7.** Assume that $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function with $f (0) = 0$. If $A, B \geq m > 0$, then

\[
\left\| \frac{f (A) A^{-1} + f (B) B^{-1} - \int_0^1 ((1 - t) A + tB)^{-1} f ((1 - t) A + tB) dt}{2} \right\| \leq \frac{f (m) - f' (m) m}{4m^2 m} \| B - A \|.
\]

Assume that $A, B \geq m > 0$, then by Corollary 7 we obtain the following power inequalities

\[
\left\| A^{r-1} + B^{r-1} - \int_0^1 ((1 - t) A + tB)^{r-1} dt \right\| \leq \frac{1 - r}{4m^2 r} \| B - A \|,
\]

where $r \in (0, 1]$. 

We can also state the logarithmic inequality
\[
\frac{A^{-1}\ln(A + 1) + B^{-1}\ln(B + 1)}{2} - \int_0^1 ((1 - t) A + tB)^{-1} \ln((1 - t) A + tB + 1) \, dt \leq \frac{(m + 1) \ln(m + 1) - m}{4m^2(m + 1)} \|B - A\|,
\]
provided that \(A, B \geq m > 0\).

**Corollary 8.** Assume that \(f : [0, \infty) \to \mathbb{R}\) is an operator convex function with \(f(0) = 0\). If \(A, B \geq m > 0\), then
\[
\frac{f(A) A^{-2} + f(B) B^{-2}}{2} - \int_0^1 ((1 - t) A + tB)^{-2} f((1 - t) A + tB) \, dt
- f'_+(0) \left(\frac{A^{-1} + B^{-1}}{2} - \int_0^1 ((1 - t) A + tB)^{-1} \, dt\right)
\leq \frac{1}{2m^2} \|B - A\| \left(\frac{f(m)}{m} - \frac{f'(m) + f'_+(0)}{2}\right).
\]
Assume that \(A, B \geq m > 0\), then by Corollary 8 we obtain the following logarithmic inequalities
\[
\frac{A^{-2}\ln(A + 1) + B^{-2}\ln(B + 1)}{2} - \int_0^1 ((1 - t) A + tB)^{-2} \ln((1 - t) A + tB + 1) \, dt
- \left(\frac{A^{-1} + B^{-1}}{2} - \int_0^1 ((1 - t) A + tB)^{-1} \, dt\right)
\leq \frac{1}{4m^2} \|B - A\| \left(\frac{m + 2}{m + 1} - \frac{2 \ln(m + 1)}{m}\right).
\]

**References**


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