REVERSES OF DAVIS-CHOI-JENSEN'S INEQUALITY FOR AN INTEGRAL TRANSFORM WITH APPLICATIONS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty w(\lambda)(\lambda+t)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for $t>0$.

Let H and K be to Hilbert spaces. In this paper we show among others that, if the linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive, preserves the operator order and is normalised while A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$ for some scalars $m < M$, then

$$\begin{aligned} 0 &\leq \Phi[\mathcal{D}(w,\mu)(A)] - \mathcal{D}(w,\mu)(\Phi(A)) \\ &\leq \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}[\mathcal{D}(w,\mu)(m) - \mathcal{D}(w,\mu)(M)], \end{aligned}$$

$$(0 \leq)\, \Phi[\mathcal{D}(w,\mu)(A)] \leq \frac{(M+m)^2}{4mM}\mathcal{D}(w,\mu)(\Phi(A))$$

and

$$\begin{aligned} 0 &\leq \Phi[\mathcal{D}(w,\mu)(A)] - \mathcal{D}(w,\mu)(\Phi(A)) \\ &\leq \frac{\mathcal{D}(w,\mu)(m) + \mathcal{D}(w,\mu)(M)}{2} - \mathcal{D}(w,\mu)\left(\frac{m+M}{2}\right). \end{aligned}$$

Some applications for operator monotone and operator convex functions are also provided.

1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [2] (see also [6, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda\Phi(A) + \mu\Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e., if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We

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write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. *The linear map* $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ *is normalised if it preserves the identity operator, i.e.,* $\Phi(1) = 1$. *We write* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1 \leq A \leq \beta 1$, then $\alpha 1 \leq \Phi(A) \leq \beta 1$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1)\Psi\Psi^{-1/2}(1)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

The following Jensen's type result is well known [2]:

Theorem 1 (Davis-Choi-Jensen's Inequality)**.** *Let* $f : I \to \mathbb{R}$ *be an operator convex function on the interval* I *and* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, *then for any selfadjoint operator* A *whose spectrum is contained in* I *we have*

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1.1}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1)\Psi\Psi^{-1/2}(1)$ in (1.1) we get

$$f\left(\Psi^{-1/2}(1)\Psi(A)\Psi^{-1/2}(1)\right) \leq \Psi^{-1/2}(1)\Psi(f(A))\Psi^{-1/2}(1).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps:*

$$\Psi^{1/2}(1)f\left(\Psi^{-1/2}(1)\Psi(A)\Psi^{-1/2}(1)\right)\Psi^{1/2}(1) \leq \Psi(f(A)). \tag{1.2}$$

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

Theorem 2. *A function* $f : (0,\infty) \to \mathbb{R}$ *is operator monotone in* $(0,\infty)$ *if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.3}$$

where $a \in R$, $b \geq 0$ *and a positive measure* μ *on* $(0,\infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.4}$$

If f *is operator monotone in* $[0,\infty)$, *then* $a = f(0)$ *in (1.3).*

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 3. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.5}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.4) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.3).

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}d\lambda.$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.6}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.6) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.7}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.8}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.9}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda),$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.10}$$

for $T>0$.

From (1.8) we have the representation

$$T^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(T) \tag{1.11}$$

where $T>0$ and from (1.9)

$$(T-1)^{-1}\ln T=\mathcal{D}(w_{\ln})(T) \tag{1.12}$$

provided $T>0$ and $T-1$ is invertible.

2. Main Results

We recall the following reverse inequalities [6, p. 29]:

Lemma 1. *Let $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A a positive operator on H satisfying the condition $M\geq A\geq m>0$ for some scalars $m<M$. Then*

$$0\leq\Phi\left(A^{-1}\right)-[\Phi(A)]^{-1}\leq\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM} \tag{2.1}$$

and

$$\Phi\left(A^{-1}\right)\leq\frac{(M+m)^2}{4mM}[\Phi(A)]^{-1}, \tag{2.2}$$

or, equivalently

$$0\leq\Phi\left(A^{-1}\right)-[\Phi(A)]^{-1}\leq\frac{(M-m)^2}{4mM}[\Phi(A)]^{-1}. \tag{2.3}$$

We have the following main result:

Theorem 4. *Let $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A a positive operator on H satisfying the condition $M\geq A\geq m>0$ for some scalars $m<M$. Then*

$$\begin{aligned}0&\leq\Phi[\mathcal{D}(w,\mu)(A)]-\mathcal{D}(w,\mu)(\Phi(A))\\&\leq\frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}[\mathcal{D}(w,\mu)(m)-\mathcal{D}(w,\mu)(M)]\end{aligned} \tag{2.4}$$

and

$$0\leq\Phi[\mathcal{D}(w,\mu)(A)]-\mathcal{D}(w,\mu)(\Phi(A))\leq\frac{(M-m)^2}{4mM}\mathcal{D}(w,\mu)(\Phi(A)), \tag{2.5}$$

or, equivalently

$$(0\leq)\,\Phi[\mathcal{D}(w,\mu)(A)]\leq\frac{(M+m)^2}{4mM}\mathcal{D}(w,\mu)(\Phi(A)). \tag{2.6}$$

Proof. We have the properties of $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and of Bochner integral that

$$\Phi[\mathcal{D}(w,\mu)(A)] = \int_0^\infty w(\lambda)\Phi\left[(\lambda+A)^{-1}\right]d\mu(\lambda)$$

and

$$\mathcal{D}(w,\mu)(\Phi(A)) := \int_0^\infty w(\lambda)(\lambda+\Phi(A))^{-1}d\mu(\lambda),$$

which implies that

$$\Phi[\mathcal{D}(w,\mu)(A)] - \mathcal{D}(w,\mu)(\Phi(A)) = \int_0^\infty w(\lambda)\left(\Phi\left[(\lambda+T)^{-1}\right] - (\lambda+\Phi(A))^{-1}\right)d\mu(\lambda). \tag{2.7}$$

Since the function $f(t) = t^{-1}$ is operator convex, then by (1.1) we have

$$\Phi\left[(\lambda+A)^{-1}\right] - (\lambda+\Phi(A))^{-1} \geq 0$$

for all $\lambda \geq 0$, which by multiplication with $w(\lambda) \geq 0$ and integration gives by (2.7) the first inequality in (2.4).

Since $M+\lambda \geq A+\lambda \geq m+\lambda > 0$ for all $\lambda \geq 0$, then by (2.1) we get

$$\begin{aligned} 0 &\leq \Phi\left((\lambda+A)^{-1}\right) - [\Phi(\lambda+A)]^{-1} \leq \frac{\left(\sqrt{M+\lambda}-\sqrt{m+\lambda}\right)^2}{(m+\lambda)(M+\lambda)} \\ &= \frac{(M-m)^2}{\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2(m+\lambda)(M+\lambda)} \\ &\leq \frac{(M-m)^2}{\left(\sqrt{M}+\sqrt{m}\right)^2(m+\lambda)(M+\lambda)} \end{aligned} \tag{2.8}$$

for all $\lambda \geq 0$.

If we multiply by $w(\lambda) \geq 0$ and integrate, then we get by (2.7) that

$$\begin{aligned} &\Phi[\mathcal{D}(w,\mu)(A)] - \mathcal{D}(w,\mu)(\Phi(A)) \\ &\leq \frac{(M-m)^2}{\left(\sqrt{M}+\sqrt{m}\right)^2}\int_0^\infty \frac{w(\lambda)\,d\mu(\lambda)}{(m+\lambda)(M+\lambda)}. \end{aligned} \tag{2.9}$$

Now, observe that

$$\begin{aligned} \mathcal{D}(w,\mu)(m) - \mathcal{D}(w,\mu)(M) &= \int_0^\infty \left(\frac{1}{\lambda+m} - \frac{1}{\lambda+M}\right)w(\lambda)\,d\mu(\lambda) \\ &= (M-m)\int_0^\infty \frac{w(\lambda)\,d\mu(\lambda)}{(m+\lambda)(M+\lambda)}. \end{aligned} \tag{2.10}$$

By making use of (2.9) and (2.10) we obtain (2.4).

From (2.3) we get

$$\begin{aligned} 0 &\leq \Phi\left((A+\lambda)^{-1}\right) - [\Phi(A+\lambda)]^{-1} \leq \frac{(M-m)^2}{4(m+\lambda)(M+\lambda)}[\Phi(A+\lambda)]^{-1} \\ &\leq \frac{(M-m)^2}{4mM}[\Phi(A+\lambda)]^{-1} \end{aligned} \tag{2.11}$$

for all $\lambda \geq 0$, which by multiplication with $w(\lambda) \geq 0$ and integration gives

$$0 \leq \int_0^\infty w(\lambda)\left(\Phi\left[(\lambda+T)^{-1}\right] - (\lambda+\Phi(A))^{-1}\right) d\mu(\lambda) \tag{2.12}$$
$$\leq \frac{(M-m)^2}{4mM}\int_0^\infty w(\lambda)\left[\Phi(A+\lambda)\right]^{-1} d\mu(\lambda)$$
$$= \frac{(M-m)^2}{4mM}\int_0^\infty w(\lambda)\left[\Phi(A)+\lambda\right]^{-1} d\mu(\lambda)$$
$$= \frac{(M-m)^2}{4mM}\mathcal{D}(w,\mu)(\Phi(A)).$$

By utilising (2.7) and (2.12) we get (2.5). □

The case of operator monotone functions is as follows:

Corollary 1. *If f is operator monotone in $[0,\infty)$, $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$, then*

$$0 \leq \Phi\left(f(A)A^{-1}\right) - f(\Phi(A))(\Phi(A))^{-1} - f(0)\left[\Phi\left(A^{-1}\right) - (\Phi(A))^{-1}\right] \tag{2.13}$$
$$\leq \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\frac{f(m)}{m} - \frac{f(M)}{M} - f(0)\frac{M-m}{mM}\right]$$

and, if $f(0)=0$, then

$$0 \leq \Phi\left(f(A)A^{-1}\right) - f(\Phi(A))(\Phi(A))^{-1} \tag{2.14}$$
$$\leq \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\frac{f(m)}{m} - \frac{f(M)}{M}\right].$$

Also

$$(0 \leq)\, \Phi\left(f(A)A^{-1}\right) - f(0)\Phi\left(A^{-1}\right) \tag{2.15}$$
$$\leq \frac{(M+m)^2}{4mM}\left[f(\Phi(A))(\Phi(A))^{-1} - f(0)(\Phi(A))^{-1}\right].$$

If $f(0)=0$, then

$$(0 \leq)\, \Phi\left(f(A)A^{-1}\right) \leq \frac{(M+m)^2}{4mM} f(\Phi(A))(\Phi(A))^{-1}. \tag{2.16}$$

Proof. From (1.3) we have

$$\frac{f(t)-f(0)}{t} - b = \mathcal{D}(\ell,\mu)(t), \tag{2.17}$$

for some μ, a positive measure on $(0,\infty)$, where $\ell(\lambda)=\lambda$, $\lambda \geq 0$.

We have

$$\Phi\left[\mathcal{D}(\ell,\mu)(A)\right] = \Phi\left(f(A)A^{-1}\right) - f(0)\Phi\left(A^{-1}\right) - b,$$

$$\mathcal{D}(\ell,\mu)(\Phi(A)) = f(\Phi(A))(\Phi(A))^{-1} - f(0)(\Phi(A))^{-1} - b,$$

$$\begin{aligned}&\Phi\left[\mathcal{D}(\ell,\mu)(A)\right]-\mathcal{D}(\ell,\mu)(\Phi(A))\\&=\Phi\left(f(A)A^{-1}\right)-f(0)\Phi\left(A^{-1}\right)-f(\Phi(A))(\Phi(A))^{-1}+f(0)(\Phi(A))^{-1}\\&=\Phi\left(f(A)A^{-1}\right)-f(\Phi(A))(\Phi(A))^{-1}-f(0)\left[\Phi\left(A^{-1}\right)-(\Phi(A))^{-1}\right]\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}(\ell,\mu)(m)-\mathcal{D}(\ell,\mu)(M)&=\frac{f(m)-f(0)}{m}-\frac{f(M)-f(0)}{M}\\&=\frac{f(m)}{m}-\frac{f(M)}{M}-f(0)\frac{M-m}{mM}\end{aligned}$$

and by (2.4) we get (2.13).

From (2.6) we obtain

$$\begin{aligned}&\Phi\left(f(A)A^{-1}\right)-f(0)\Phi\left(A^{-1}\right)-b\\&\le\frac{(M+m)^2}{4mM}\left[f(\Phi(A))(\Phi(A))^{-1}-f(0)(\Phi(A))^{-1}-b\right],\end{aligned}$$

namely

$$\begin{aligned}&\Phi\left(f(A)A^{-1}\right)-f(0)\Phi\left(A^{-1}\right)\\&\le\frac{(M+m)^2}{4mM}\left[f(\Phi(A))(\Phi(A))^{-1}-f(0)(\Phi(A))^{-1}-b\right]+b\\&=\frac{(M+m)^2}{4mM}\left[f(\Phi(A))(\Phi(A))^{-1}-f(0)(\Phi(A))^{-1}\right]+b\left(1-\frac{(M+m)^2}{4mM}\right)\\&=\frac{(M+m)^2}{4mM}\left[f(\Phi(A))(\Phi(A))^{-1}-f(0)(\Phi(A))^{-1}\right]-b\frac{(M-m)^2}{4mM}\\&\le\frac{(M+m)^2}{4mM}\left[f(\Phi(A))(\Phi(A))^{-1}-f(0)(\Phi(A))^{-1}\right]\end{aligned}$$

since $b\ge 0$. This proves (2.15). □

Corollary 2. *If $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$, $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M\ge A\ge m>0$, then*

$$\begin{aligned}0&\le\Phi\left(f(A)A^{-2}\right)-f(\Phi(A))[\Phi(A)]^{-2}-f(0)\left[\Phi\left(A^{-2}\right)-[\Phi(A)]^{-2}\right]\\&\quad-f'_+(0)\left[\Phi\left(A^{-1}\right)-[\Phi(A)]^{-1}\right]\\&\le\frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\\&\quad\times\left[\frac{f(m)}{m^2}-\frac{f(M)}{M^2}-f(0)\frac{(M-m)(m+M)}{m^2M^2}-f'_+(0)\frac{M-m}{mM}\right].\end{aligned}\tag{2.18}$$

If $f(0)=0$, then

$$\begin{aligned}0&\le\Phi\left(f(A)A^{-2}\right)-f(\Phi(A))[\Phi(A)]^{-2}-f'_+(0)\left[\Phi\left(A^{-1}\right)-[\Phi(A)]^{-1}\right]\\&\le\frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\frac{f(m)}{m^2}-\frac{f(M)}{M^2}-f'_+(0)\frac{M-m}{mM}\right].\end{aligned}\tag{2.19}$$

Also,

$$0 \leq \Phi\left(f(A) A^{-2}\right) - f(0) \Phi\left(A^{-2}\right) - f'_{+}(0) \Phi\left(A^{-1}\right) \tag{2.20}$$
$$\leq \frac{(M+m)^2}{4mM} \left[f(\Phi(A)) [\Phi(A)]^{-2} - f(0) [\Phi(A)]^{-2} - f'_{+}(0) [\Phi(A)]^{-1} \right]$$

and if $f(0) = 0$*, then*

$$0 \leq \Phi\left(f(A) A^{-2}\right) - f'_{+}(0) \Phi\left(A^{-1}\right) \tag{2.21}$$
$$\leq \frac{(M+m)^2}{4mM} \left[f(\Phi(A)) [\Phi(A)]^{-2} - f'_{+}(0) [\Phi(A)]^{-1} \right].$$

Proof. From (1.5) we have

$$\left[f(t) - f(0) - f'_{+}(0) t\right] t^{-2} - c = \mathcal{D}(\ell, \mu)(t),$$

for some μ, a positive measure on $(0, \infty)$, where $\ell(\lambda) = \lambda$, $\lambda \geq 0$.

We have

$$\Phi[\mathcal{D}(\ell, \mu)(A)] = \Phi\left(f(A) A^{-2}\right) - f(0) \Phi\left(A^{-2}\right) - f'_{+}(0) \Phi\left(A^{-1}\right) - c,$$

$$\mathcal{D}(\ell, \mu)(\Phi(A)) = f(\Phi(A)) [\Phi(A)]^{-2} - f(0) [\Phi(A)]^{-2} - f'_{+}(0) [\Phi(A)]^{-1} - c,$$

$$\begin{aligned}
&\Phi[\mathcal{D}(\ell, \mu)(A)] - \mathcal{D}(\ell, \mu)(\Phi(A)) \\
&= \Phi\left(f(A) A^{-2}\right) - f(\Phi(A)) [\Phi(A)]^{-2} - f(0) \left[\Phi\left(A^{-2}\right) - [\Phi(A)]^{-2}\right] \\
&- f'_{+}(0) \left[\Phi\left(A^{-1}\right) - [\Phi(A)]^{-1}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{D}(\ell, \mu)(m) - \mathcal{D}(\ell, \mu)(M) \\
&= \left[f(m) - f(0) - f'_{+}(0) m\right] m^{-2} - \left[f(M) - f(0) - f'_{+}(0) M\right] M^{-2} \\
&= \frac{f(m)}{m^2} - \frac{f(M)}{M^2} - f(0) \frac{(M-m)(m+M)}{m^2 M^2} - f'_{+}(0) \frac{M-m}{mM}
\end{aligned}$$

and by (2.4) we get (2.18).

From (2.6) we get

$$\begin{aligned}
&(0 \leq) \Phi\left(f(A) A^{-2}\right) - f(0) \Phi\left(A^{-2}\right) - f'_{+}(0) \Phi\left(A^{-1}\right) - c \\
&\leq \frac{(M+m)^2}{4mM} \left[f(\Phi(A)) [\Phi(A)]^{-2} - f(0) [\Phi(A)]^{-2} - f'_{+}(0) [\Phi(A)]^{-1} - c \right] \\
&= \frac{(M+m)^2}{4mM} \left[f(\Phi(A)) [\Phi(A)]^{-2} - f(0) [\Phi(A)]^{-2} - f'_{+}(0) [\Phi(A)]^{-1} \right] \\
&- c \frac{(M+m)^2}{4mM}
\end{aligned}$$

namely

$$
\begin{aligned}
0 &\leq \Phi\left(f(A)A^{-2}\right) - f(0)\Phi\left(A^{-2}\right) - f'_+(0)\Phi\left(A^{-1}\right) \\
&\leq \frac{(M+m)^2}{4mM}\left[f(\Phi(A))[\Phi(A)]^{-2} - f(0)[\Phi(A)]^{-2} - f'_+(0)[\Phi(A)]^{-1}\right] \\
&+ c\left(1 - \frac{(M+m)^2}{4mM}\right) \\
&= \frac{(M+m)^2}{4mM}\left[f(\Phi(A))[\Phi(A)]^{-2} - f(0)[\Phi(A)]^{-2} - f'_+(0)[\Phi(A)]^{-1}\right] \\
&- c\frac{(M-m)^2}{4mM} \\
&\leq \frac{(M+m)^2}{4mM}\left[f(\Phi(A))[\Phi(A)]^{-2} - f(0)[\Phi(A)]^{-2} - f'_+(0)[\Phi(A)]^{-1}\right],
\end{aligned}
$$

which proves (2.20). □

We also have:

Theorem 5. *Let $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and A be a positive operator on H satisfying the condition $M \geq A \geq m > 0$ for some scalars $m < M$. Then*

$$
\begin{aligned}
0 &\leq \Phi[\mathcal{D}(w,\mu)(A)] - \mathcal{D}(w,\mu)(\Phi(A)) \\
&\leq \frac{1}{4}(M-m)[\mathcal{D}(\ell,\mu)(m) - \mathcal{D}(\ell,\mu)(M)][\Phi(A)]^{-1} \\
&\leq \frac{1}{4}\left(\frac{M}{m} - 1\right)[\mathcal{D}(\ell,\mu)(m) - \mathcal{D}(\ell,\mu)(M)].
\end{aligned} \tag{2.22}
$$

Proof. From (2.7) and (2.11) we have

$$
\begin{aligned}
0 &\leq \Phi[\mathcal{D}(w,\mu)(A)] - \mathcal{D}(w,\mu)(\Phi(A)) \\
&= \int_0^\infty \left[\Phi\left((A+\lambda)^{-1}\right) - [\Phi(A+\lambda)]^{-1}\right] w(\lambda)\, d\mu(\lambda) \\
&\leq \frac{(M-m)^2}{4}\int_0^\infty \frac{1}{(m+\lambda)(M+\lambda)}[\Phi(A)+\lambda]^{-1} w(\lambda)\, d\mu(\lambda).
\end{aligned} \tag{2.23}
$$

Since $\Phi(A) + \lambda \geq \Phi(A)$ for $\lambda \geq 0$, then $(\Phi(A)+\lambda)^{-1} \leq [\Phi(A)]^{-1}$. Therefore

$$
\begin{aligned}
&\int_0^\infty \frac{1}{(m+\lambda)(M+\lambda)}[\Phi(A)+\lambda]^{-1} w(\lambda)\, d\mu(\lambda) \\
&\leq \int_0^\infty \frac{1}{(m+\lambda)(M+\lambda)}[\Phi(A)]^{-1} w(\lambda)\, d\mu(\lambda) \\
&= \left(\int_0^\infty \frac{1}{(m+\lambda)(M+\lambda)} w(\lambda)\, d\mu(\lambda)\right)[\Phi(A)]^{-1} \\
&= \frac{1}{M-m}\left(\int_0^\infty \frac{M-m}{(m+\lambda)(M+\lambda)} w(\lambda)\, d\mu(\lambda)\right)[\Phi(A)]^{-1} \\
&= \frac{1}{M-m}[\mathcal{D}(\ell,\mu)(m) - \mathcal{D}(\ell,\mu)(M)][\Phi(A)]^{-1}.
\end{aligned} \tag{2.24}
$$

By using (2.23) and (2.24) we derive the first part of (2.22).

The last part is obvious since $\Phi(A) \geq m > 0$ implies that $[\Phi(A)]^{-1} \leq \frac{1}{m}$ from where we get

$$\begin{aligned}
&\frac{1}{4}(M-m)[\mathcal{D}(\ell,\mu)(m)-\mathcal{D}(\ell,\mu)(M)][\Phi(A)]^{-1} \\
&\leq \frac{1}{4}\left(\frac{M}{m}-1\right)[\mathcal{D}(\ell,\mu)(m)-\mathcal{D}(\ell,\mu)(M)]
\end{aligned}$$

and the last part in (2.22) is proved. □

The case of operator monotone functions is as follows:

Corollary 3. *If f is operator monotone in $[0,\infty)$, $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$, then*

$$\begin{aligned}
0 &\leq \Phi\left(f(A)A^{-1}\right) - f(\Phi(A))(\Phi(A))^{-1} - f(0)\left[\Phi\left(A^{-1}\right)-(\Phi(A))^{-1}\right] \\
&\leq \frac{1}{4}(M-m)\left[\frac{f(m)}{m}-\frac{f(M)}{M}-f(0)\frac{M-m}{mM}\right][\Phi(A)]^{-1} \\
&\leq \frac{1}{4}\left(\frac{M}{m}-1\right)\left[\frac{f(m)}{m}-\frac{f(M)}{M}-f(0)\frac{M-m}{mM}\right].
\end{aligned} \tag{2.25}$$

If $f(0)=0$, then

$$\begin{aligned}
0 &\leq \Phi\left(f(A)A^{-1}\right) - f(\Phi(A))(\Phi(A))^{-1} \\
&\leq \frac{1}{4}(M-m)\left[\frac{f(m)}{m}-\frac{f(M)}{M}\right][\Phi(A)]^{-1} \\
&\leq \frac{1}{4}\left(\frac{M}{m}-1\right)\left[\frac{f(m)}{m}-\frac{f(M)}{M}\right].
\end{aligned} \tag{2.26}$$

The case of operator convex functions is as follows:

Corollary 4. *If $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$, $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$, then*

$$\begin{aligned}
0 &\leq \Phi\left(f(A)A^{-2}\right) - f(\Phi(A))[\Phi(A)]^{-2} - f(0)\left[\Phi\left(A^{-2}\right)-[\Phi(A)]^{-2}\right] \\
&- f'_+(0)\left[\Phi\left(A^{-1}\right)-[\Phi(A)]^{-1}\right] \\
&\leq \frac{1}{4}(M-m) \\
&\times\left[\frac{f(m)}{m^2}-\frac{f(M)}{M^2}-f(0)\frac{(M-m)(m+M)}{m^2M^2}-f'_+(0)\frac{M-m}{mM}\right] \\
&\times[\Phi(A)]^{-1} \\
&\leq \frac{1}{4}\left(\frac{M}{m}-1\right) \\
&\times\left[\frac{f(m)}{m^2}-\frac{f(M)}{M^2}-f(0)\frac{(M-m)(m+M)}{m^2M^2}-f'_+(0)\frac{M-m}{mM}\right].
\end{aligned} \tag{2.27}$$

If $f(0)=0$, *then*

$$0\le\Phi\left(f(A)A^{-2}\right)-f(\Phi(A))[\Phi(A)]^{-2}-f'_{+}(0)\left[\Phi\left(A^{-1}\right)-[\Phi(A)]^{-1}\right]\tag{2.28}$$
$$\le\frac{1}{4}(M-m)\left[\frac{f(m)}{m^2}-\frac{f(M)}{M^2}-f'_{+}(0)\frac{M-m}{mM}\right][\Phi(A)]^{-1}$$
$$\le\frac{1}{4}\left(\frac{M}{m}-1\right)\left[\frac{f(m)}{m^2}-\frac{f(M)}{M^2}-f'_{+}(0)\frac{M-m}{mM}\right].$$

From a different perspective we also have the upper bound in terms of Jensen difference:

Theorem 6. *Let* $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ *and* A *a positive operator on* H *satisfying the condition* $M\ge A\ge m>0$ *for some scalars* $m<M$. *Then*

$$0\le\Phi[\mathcal{D}(w,\mu)(A)]-\mathcal{D}(w,\mu)(\Phi(A))\tag{2.29}$$
$$\le\frac{\mathcal{D}(w,\mu)(m)+\mathcal{D}(w,\mu)(M)}{2}-\mathcal{D}(w,\mu)\left(\frac{m+M}{2}\right).$$

Proof. As in the proof of Theorem 4, see (2.8),

$$0\le\Phi\left((\lambda+A)^{-1}\right)-[\Phi(\lambda+A)]^{-1}\le\frac{\left(\sqrt{M+\lambda}-\sqrt{m+\lambda}\right)^2}{(m+\lambda)(M+\lambda)}\tag{2.30}$$
$$=\frac{(M-m)^2}{\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2(m+\lambda)(M+\lambda)}$$

for $\lambda\ge0$.

Using the elementary inequality $\sqrt{a}+\sqrt{b}\ge\sqrt{a+b}$, $a,\ b\ge0$ we deduce that

$$\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2\ge M+m+2\lambda$$

for $\lambda\ge0$, which implies that

$$\frac{(M-m)^2}{\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2(m+\lambda)(M+\lambda)}\le\frac{(M-m)^2}{(M+m+2\lambda)(m+\lambda)(M+\lambda)}.$$

We observe that, by performing the calculations, one has the equality

$$\frac{1}{\left(\frac{M+m}{2}+\lambda\right)(m+\lambda)(M+\lambda)}$$
$$=\frac{1}{(M-m)^2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}-\frac{2}{\lambda+\frac{m+M}{2}}\right),$$

for $\lambda\ge0$.

Therefore

$$\frac{(M-m)^2}{\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2(m+\lambda)(M+\lambda)}\tag{2.31}$$
$$\le\frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}-\frac{2}{\lambda+\frac{m+M}{2}}\right)$$
$$=\frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}\right)-\frac{1}{\lambda+\frac{m+M}{2}}.$$

for $\lambda \geq 0$.

If we use (2.7) then by (2.30) and (2.31) we get

$$\begin{aligned} (2.32) \quad & \Phi\left[\mathcal{D}(w,\mu)(A)\right] - \mathcal{D}(w,\mu)(\Phi(A)) \\ &= \int_0^\infty w(\lambda)\left(\Phi\left[(\lambda+T)^{-1}\right] - (\lambda+\Phi(A))^{-1}\right) d\mu(\lambda) \\ &\leq \int_0^\infty w(\lambda)\left(\frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}\right) - \frac{1}{\lambda+\frac{m+M}{2}}\right) d\mu(\lambda) \\ &= \frac{1}{2}\left[\int_0^\infty \frac{w(\lambda)}{m+\lambda} d\mu(\lambda) + \int_0^\infty \frac{w(\lambda)}{M+\lambda} d\mu(\lambda)\right] \\ &\quad - \int_0^\infty \frac{w(\lambda)}{\lambda+\frac{m+M}{2}} d\mu(\lambda) \\ &= \frac{1}{2}\left(\mathcal{D}(w,\mu)(m) + \mathcal{D}(w,\mu)(M)\right) - \mathcal{D}(w,\mu)\left(\frac{m+M}{2}\right), \end{aligned}$$

and the inequality (2.29) is obtained. □

Corollary 5. *If f is operator monotone in $[0,\infty)$, $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$, then*

$$\begin{aligned} (2.33) \quad & 0 \leq \Phi\left(f(A)A^{-1}\right) - f(\Phi(A))(\Phi(A))^{-1} - f(0)\left[\Phi\left(A^{-1}\right) - (\Phi(A))^{-1}\right] \\ &\leq \frac{1}{2}\left(\frac{f(m)}{m}+\frac{f(M)}{M}\right) - \frac{f\left(\frac{m+M}{2}\right)}{\frac{m+M}{2}} + f(0)\frac{(M-m)^2}{2mM(M+m)}. \end{aligned}$$

In particular, if $f(0)=0$, then

$$\begin{aligned} (2.34) \quad & 0 \leq \Phi\left(f(A)A^{-1}\right) - f(\Phi(A))(\Phi(A))^{-1} \\ &\leq \frac{1}{2}\left(\frac{f(m)}{m}+\frac{f(M)}{M}\right) - \frac{f\left(\frac{m+M}{2}\right)}{\frac{m+M}{2}}. \end{aligned}$$

A similar result may be stated for operator convex functions, however the details are omitted.

3. Examples for Some Positive Maps

Let $P_j \in \mathcal{B}(H)$, $j=1,...,k$ be contractions with

$$(3.1) \quad \sum_{j=1}^{k} P_j^* P_j = 1_H.$$

The map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by [6]

$$\Phi(A) := \sum_{j=1}^{k} P_j^* A P_j$$

is a *normalized positive linear map* on $\mathcal{B}(H)$.

Assume that A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$ for some scalars $m < M$. Then by Theorem 4 we get the following reverses of operator Jensen's inequality

$$0 \le \sum_{j=1}^{k} P_j^* \mathcal{D}(w,\mu)(A) P_j - \mathcal{D}(w,\mu)\left(\sum_{j=1}^{k} P_j^* A P_j\right) \le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\mathcal{D}(w,\mu)(m) - \mathcal{D}(w,\mu)(M)\right] \tag{3.2}$$

and

$$(0 \le) \sum_{j=1}^{k} P_j^* \mathcal{D}(w,\mu)(A) P_j \le \frac{(M+m)^2}{4mM}\mathcal{D}(w,\mu)\left(\mathcal{D}(w,\mu)\left(\sum_{j=1}^{k} P_j^* A P_j\right)\right). \tag{3.3}$$

By applying Theorem 5 we obtain

$$0 \le \sum_{j=1}^{k} P_j^* \mathcal{D}(w,\mu)(A) P_j - \mathcal{D}(w,\mu)\left(\sum_{j=1}^{k} P_j^* A P_j\right) \le \frac{1}{4}(M-m)\left[\mathcal{D}(\ell,\mu)(m) - \mathcal{D}(\ell,\mu)(M)\right]\left(\sum_{j=1}^{k} P_j^* A P_j\right)^{-1}, \tag{3.4}$$

while from Theorem 6 we derive

$$0 \le \sum_{j=1}^{k} P_j^* \mathcal{D}(w,\mu)(A) P_j - \mathcal{D}(w,\mu)\left(\sum_{j=1}^{k} P_j^* A P_j\right) \le \frac{\mathcal{D}(w,\mu)(m) + \mathcal{D}(w,\mu)(M)}{2} - \mathcal{D}(w,\mu)\left(\frac{m+M}{2}\right). \tag{3.5}$$

If we consider the cases of f operator monotone or operator convex, then we can state some similar results to those in the corollaries of the previous section. However the details are not provided here.

4. Some Examples

Let $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and A a positive operator on H satisfying the condition $M \ge A \ge m > 0$ for some scalars $m < M$.

Let $f(t) = t^r$, $r \in (0,1]$. The function f is operator monotone on $(0,\infty)$ and by (2.14) we get

$$0 \le \Phi\left(A^{r-1}\right) - [\Phi(A)]^{r-1} \le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left(m^{r-1} - M^{r-1}\right). \tag{4.1}$$

Also from (2.16) we obtain

$$(0 \le)\, \Phi\left(A^{r-1}\right) \le \frac{(M+m)^2}{4mM}[\Phi(A)]^{r-1} \tag{4.2}$$

while from (2.26) we get

$$0 \le \Phi\left(A^{r-1}\right) - [\Phi(A)]^{r-1} \le \frac{1}{4}(M-m)\left(m^{r-1} - M^{r-1}\right)[\Phi(A)]^{-1} \le \frac{1}{4}\left(\frac{M}{m} - 1\right)\left(m^{r-1} - M^{r-1}\right), \tag{4.3}$$

for $M \ge A \ge m > 0$.

From (2.34) we get

$$0 \le \Phi\left(A^{r-1}\right) - [\Phi(A)]^{r-1} \le \frac{1}{2}\left(m^{r-1} + M^{r-1}\right) - \left(\frac{m+M}{2}\right)^{r-1} \tag{4.4}$$

for $M \ge A \ge m > 0$.

Let $P_j \in \mathcal{B}(H)$, $j = 1, ..., k$ be contractions with $\sum_{j=1}^{k} P_j^* P_j = 1_H$, then from (4.1)-(4.4) we derive the power inequalities

$$0 \le \sum_{j=1}^{k} P_j^* A^{r-1} P_j - \left(\sum_{j=1}^{k} P_j^* A P_j\right)^{r-1} \le \frac{M-m}{\left(\sqrt{M} + \sqrt{m}\right)^2}\left(m^{r-1} - M^{r-1}\right), \tag{4.5}$$

$$(0 \le) \sum_{j=1}^{k} P_j^* A^{r-1} P_j \le \frac{(M+m)^2}{4mM}\left(\sum_{j=1}^{k} P_j^* A P_j\right)^{r-1}, \tag{4.6}$$

$$0 \le \sum_{j=1}^{k} P_j^* A^{r-1} P_j - \left(\sum_{j=1}^{k} P_j^* A P_j\right)^{r-1} \le \frac{1}{4}(M-m)\left(m^{r-1} - M^{r-1}\right)\left(\sum_{j=1}^{k} P_j^* A P_j\right)^{-1} \le \frac{1}{4}\left(\frac{M}{m} - 1\right)\left(m^{r-1} - M^{r-1}\right), \tag{4.7}$$

and

$$0 \le \sum_{j=1}^{k} P_j^* A^{r-1} P_j - \left(\sum_{j=1}^{k} P_j^* A P_j\right)^{r-1} \le \frac{1}{2}\left(m^{r-1} + M^{r-1}\right) - \left(\frac{m+M}{2}\right)^{r-1} \tag{4.8}$$

for $M \ge A \ge m > 0$.

Consider the function $f(t) = \ln(t+1)$, $t \in [0, \infty)$, which is operator monotone. By (2.14) we have

$$0 \le \Phi\left(A^{-1}\ln(A+1)\right) - [\Phi(A)]^{-1}\ln(\Phi(A)+1) \le \frac{M-m}{\left(\sqrt{M} + \sqrt{m}\right)^2}\left[\frac{\ln(m+1)}{m} - \frac{\ln(M+1)}{M}\right], \tag{4.9}$$

by (2.16) we have

$$(0 \leq) \Phi\left(A^{-1} \ln(A+1)\right) \leq \frac{(M+m)^2}{4mM} [\Phi(A)]^{-1} \ln(\Phi(A)+1), \tag{4.10}$$

while by (2.26)

$$\begin{aligned} 0 &\leq \Phi\left(A^{-1} \ln(A+1)\right) - [\Phi(A)]^{-1} \ln(\Phi(A)+1) \\ &\leq \frac{1}{4}(M-m)\left[\frac{\ln(m+1)}{m} - \frac{\ln(M+1)}{M}\right][\Phi(A)]^{-1} \\ &\leq \frac{1}{4}\left(\frac{M}{m}-1\right)\left[\frac{\ln(m+1)}{m} - \frac{\ln(M+1)}{M}\right] \end{aligned} \tag{4.11}$$

for $M \geq A \geq m > 0$.

If we write the inequality (2.34) for $f(t) = \ln(t+1)$, $t \in [0, \infty)$, then we get

$$\begin{aligned} 0 &\leq \Phi\left(A^{-1} \ln(A+1)\right) - [\Phi(A)]^{-1} \ln(\Phi(A)+1) \\ &\leq \frac{1}{2}\left(\frac{\ln(m+1)}{m} + \frac{\ln(M+1)}{M}\right) - \frac{\ln\left(\frac{m+M}{2}+1\right)}{\frac{m+M}{2}}. \end{aligned} \tag{4.12}$$

for $M \geq A \geq m > 0$.

Consider the operator convex function $f(t) = -\ln(t+1)$, $t \in [0, \infty)$. Then by (2.19) we have

$$\begin{aligned} 0 &\leq [\Phi(A)]^{-2} \ln(\Phi(A)+1) - \Phi\left(A^{-2} \ln(A+1)\right) + \Phi\left(A^{-1}\right) - [\Phi(A)]^{-1} \\ &\leq \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\frac{\ln(M+1)}{M^2} - \frac{\ln(m+1)}{m^2} + \frac{M-m}{mM}\right], \end{aligned} \tag{4.13}$$

by (2.21),

$$\begin{aligned} 0 &\leq \Phi\left(A^{-1}\right) - \Phi\left(A^{-2} \ln(A+1)\right) \\ &\leq \frac{(M+m)^2}{4mM}\left[[\Phi(A)]^{-1} - [\Phi(A)]^{-2} \ln(\Phi(A)+1)\right] \end{aligned} \tag{4.14}$$

while by (2.28)

$$\begin{aligned} 0 &\leq [\Phi(A)]^{-2} \ln(\Phi(A)+1) - \Phi\left(A^{-2} \ln(A+1)\right) + \Phi\left(A^{-1}\right) - [\Phi(A)]^{-1} \\ &\leq \frac{1}{4}(M-m)\left[\frac{\ln(M+1)}{M^2} - \frac{\ln(m+1)}{m^2} + \frac{M-m}{mM}\right][\Phi(A)]^{-1} \\ &\leq \frac{1}{4}\left(\frac{M}{m}-1\right)\left[\frac{\ln(M+1)}{M^2} - \frac{\ln(m+1)}{m^2} + \frac{M-m}{mM}\right] \end{aligned} \tag{4.15}$$

for $M \geq A \geq m > 0$.

We define the *upper incomplete Gamma function* as [12]

$$\Gamma(a, z) := \int_z^\infty t^{a-1} e^{-t} dt,$$

which for $z = 0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1} e^{-t} dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [13]

$$\Gamma(a,z)=\frac{z^{a}e^{-z}}{\Gamma(1-a)}\int_{0}^{\infty}\frac{t^{-a}e^{-t}}{z+t}dt \tag{4.16}$$

for $\operatorname{Re} a<1$ and $|\operatorname{ph} z|<\pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda):=\lambda^{-a}e^{-\lambda}$ for $\lambda>0$. Then by (4.16) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=\int_{0}^{\infty}\frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1-a)t^{-a}e^{t}\Gamma(a,t) \tag{4.17}$$

for $a<1$ and $t>0$.

For $a=0$ in (4.17) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t)=\int_{0}^{\infty}\frac{e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1)e^{t}\Gamma(0,t)=e^{t}E_{1}(t) \tag{4.18}$$

for $t>0$, where E_1 is the *exponential integral*

$$E_{1}(t):=\int_{t}^{\infty}\frac{e^{-u}}{u}du. \tag{4.19}$$

Let $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and A a positive operator on H satisfying the condition $M\geq A\geq m>0$ for some scalars $m<M$. By making use of the inequality (2.4) we obtain

$$\begin{aligned}0&\leq\Phi\left[A^{-a}\exp(A)\Gamma(a,A)\right]-\left[\Phi(A)\right]^{-a}\exp\left(\Phi(A)\right)\Gamma(a,\Phi(A))\\&\leq\frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^{2}}\left[m^{-a}e^{m}\Gamma(a,m)-M^{-a}e^{M}\Gamma(a,M)\right]\end{aligned} \tag{4.20}$$

for $a<1$ and, in particular for $a=0$,

$$\begin{aligned}0&\leq\Phi\left[\exp(A)E_{1}(A)\right]-\exp\left(\Phi(A)\right)E_{1}\left(\Phi(A)\right)\\&\leq\frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^{2}}\left[e^{m}E_{1}(m)-e^{M}E_{1}(M)\right].\end{aligned} \tag{4.21}$$

From (2.6) we get

$$\begin{aligned}(0\leq)\,&\Phi\left[A^{-a}\exp(A)\Gamma(a,A)\right]\\&\leq\frac{(M+m)^{2}}{4mM}\left[\Phi(A)\right]^{-a}\exp\left(\Phi(A)\right)\Gamma(a,\Phi(A)),\end{aligned} \tag{4.22}$$

for $a<1$ and, in particular for $a=0$,

$$\Phi\left[\exp(A)E_{1}(A)\right]\leq\frac{(M+m)^{2}}{4mM}\exp\left(\Phi(A)\right)E_{1}\left(\Phi(A)\right). \tag{4.23}$$

From (2.22) we also derive

$$\begin{aligned}0&\leq\Phi\left[A^{-a}\exp(A)\Gamma(a,A)\right]-\left[\Phi(A)\right]^{-a}\exp\left(\Phi(A)\right)\Gamma(a,\Phi(A))\\&\leq\frac{1}{4}(M-m)\left[m^{-a}e^{m}\Gamma(a,m)-M^{-a}e^{M}\Gamma(a,M)\right]\left[\Phi(A)\right]^{-1}\\&\leq\frac{1}{4}\left(\frac{M}{m}-1\right)\left[m^{-a}e^{m}\Gamma(a,m)-M^{-a}e^{M}\Gamma(a,M)\right]\end{aligned} \tag{4.24}$$

for $a<1$ and, in particular for $a=0$,

$$0\leq\Phi\left[\exp\left(A\right)E_{1}\left(A\right)\right]-\exp\left(\Phi\left(A\right)\right)E_{1}\left(\Phi\left(A\right)\right)\leq\frac{1}{4}\left(M-m\right)\left[e^{m}E_{1}\left(m\right)-e^{M}E_{1}\left(M\right)\right]\left[\Phi\left(A\right)\right]^{-1}\leq\frac{1}{4}\left(\frac{M}{m}-1\right)\left[e^{m}E_{1}\left(m\right)-e^{M}E_{1}\left(M\right)\right]. \tag{4.25}$$

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