SOME INTEGRAL INEQUALITIES FOR OPERATOR MONOTONE FUNCTIONS

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Abstract. In this paper we prove among others that

$$\int_T p_t f(A_t) A_t d\mu(t) \geq \int_T p_s f(A_s) d\mu(s) \int_T p_t A_t d\mu(t),$$

for an operator monotone function f on $(0,\infty)$, where $(A_t)_{t\in T}$ is a bounded continuous field of positive operators in $\mathcal{B}(H)$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ that satisfy certain conditions while $(p_t)_{t\in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$. Some particular inequalities of interest are also provided.

1. Introduction

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda) f(A) + \lambda f(B) \tag{1.1}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [8] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0,\infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0,\infty)$. The entropy function $f(t) = -t\ln t$ is operator concave on $(0,\infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

Let T be a locally compact Hausdorff space. We say that a field $(A_t)_{t\in T}$ of operators in $\mathcal{B}(H)$ is continuous if the function $t \mapsto A_t$ is norm continuous on T. If in addition μ is a Radon measure on T and the function $t \mapsto \|A_t\|$ is integrable, then we can form the Bochner integral $\int_T A_t d\mu(t)$, which is the unique element in $\mathcal{B}(H)$ such that

$$\varphi\left(\int_T A_t d\mu(t)\right) = \int_T \varphi(A_t) d\mu(t)$$

for every linear functional φ in the norm dual $\mathcal{B}(H)^*$, cf. [11, Section 4.1].

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Assume furthermore that there is a field $(\phi_t)_{t\in T}$ of positive linear mappings $\phi_t : \mathcal{B}(H) \to \mathcal{B}(K)$ from $\mathcal{B}(H)$ to another C^*-algebra $\mathcal{B}(K)$, with K a Hilbert space. We say that such a field is continuous if the function $t \mapsto \phi_t(A)$ is continuous for every $A \in \mathcal{B}(H)$. If the field $t \mapsto \phi_t(\mathbf{1})$ is integrable with integral $\int_T \phi_t(\mathbf{1})\, d\mu(t) = \mathbf{1}$, we say that $(\phi_t)_{t\in T}$ is unital.

The following Jensen's integral inequality has been obtained in [10]:

Theorem 1. *Let $f : I \to \mathbb{R}$ be an operator convex function defined on an interval I. If $(\phi_t)_{t\in T}$ is a unital field of positive linear mappings $\phi_t : \mathcal{B}(H) \to \mathcal{B}(K)$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ, then the inequality*

$$f\left(\int_T \phi_t(A_t)\, d\mu(t)\right) \le \int_T \phi_t(f(A_t))\, d\mu(t) \tag{1.2}$$

holds for every bounded continuous field $(A_t)_{t\in T}$ of selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I.

The discrete case is as follows [16]:

$$f\left(\sum_{i=1}^{n} w_i \phi_i(A_i)\right) \le \sum_{i=1}^{n} w_i \phi_i(f(A_i))$$

for operator convex functions f defined on an interval I, where $\phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$, $i \in \{1, ..., n\}$ are unital positive linear maps, A_i, $i \in \{1, ..., n\}$ are selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I and $w_i \ge 0$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} w_i = 1$.

Also, if $f : I \to \mathbb{R}$ is operator convex on I and $U_i \in \mathcal{B}(H)$, $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} U^* U_i = 1$, then [11]

$$f\left(\sum_{i=1}^{n} U^* A_i U_i\right) \le \sum_{i=1}^{n} U^* f(A_i) U_i,$$

where A_i, $i \in \{1, ..., n\}$ are selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I.

We have the following reverse inequalities in terms of the Fréchet derivative $Df(\cdot)(\cdot)$, see [3]

Theorem 2. *Let $f : I \to \mathbb{R}$ be an operator convex function of class $C^1(I)$. If $(A_t)_{t\in T}$ is a bounded continuous field $(A_t)_{t\in T}$ of selfadjoint operators in $\mathcal{B}(H)$ with spectra contained in I defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $\int_T \mathbf{1} d\mu(t) = \mathbf{1}$, then we have the double inequality in terms of the Fréchet derivative $Df(\cdot)(\cdot)$*

$$\begin{aligned} & f(A) - Df(A)(A) + Df(A)\left(\int_T A_t d\mu(t)\right) \\ & \le \int_T f(A_t)\, d\mu(t) \\ & \le f(A) - \int_T Df(A_t)(A)\, d\mu(t) + \int_T Df(A_t)(A_t)\, d\mu(t) \end{aligned} \tag{1.3}$$

for all $A \in \mathcal{SA}_I(H)$.

We have the reverse Jensen's inequality

$$0 \leq \int_T f(A_t)\, d\mu(t) - f\left(\int_T A_s d\mu(s)\right) \tag{1.4}$$
$$\leq \int_T Df(A_t)(A_t)\, d\mu(t) - \int_T Df(A_t)\left(\int_T A_s d\mu(s)\right) d\mu(t).$$

If $S \in \mathcal{SA}_I(H)$ is an operator satisfying the equality

$$\int_T Df(A_t)(S)\, d\mu(t) = \int_T Df(A_t)(A_t)\, d\mu(t), \tag{Sl}$$

then we have the Slater type inequality

$$0 \leq f(S) - \int_T f(A_t)\, d\mu(t) \leq Df(S)(S) - Df(S)\left(\int_T A_t d\mu(t)\right). \tag{1.5}$$

Motivated by the above results, in this paper we prove among others that

$$\int_T p_t f(A_t) A_t d\mu(t) \geq \int_T p_s f(A_s)\, d\mu(s) \int_T p_t A_t d\mu(t),$$

for an operator monotone function f on $(0,\infty)$, where $(A_t)_{t\in T}$ is a bounded continuous field of positive operators in $\mathcal{B}(H)$ defined on a locally compact Hausdorff space T with a bounded Radon measure μ that satisfy certain conditions while $(p_t)_{t\in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$. Some particular inequalities of interest are also provided.

2. Main Results

In 1934, K. Löwner [13] had given a definitive characterization of operator monotone functions as follows, see for instance [2, p. 144-145]:

Lemma 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{ts}{t+s} dm(s) \tag{2.1}$$

where $a\in\mathbb{R}$ and $b\geq 0$ and a positive measure m on $(0,\infty)$ such that

$$\int_0^\infty \frac{s}{1+s} dm(s) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [12].

In [6], T. Furuta observed that for $\alpha_j\in[0,1]$, $j=1,...,n$ the functions

$$g(t) := \left(\sum_{j=1}^n t^{-\alpha_j}\right)^{-1} \text{ and } h(t) = \sum_{j=1}^n \left(1+t^{-1}\right)^{-\alpha_j}$$

are operator monotone in $(0,\infty)$.

Let $f(t)$ be a continuous function $(0,\infty)\to(0,\infty)$. It is known that $f(t)$ is operator monotone if and only if $g(t)=t/f(t) =: f^*(t)$ is also operator monotone, see for instance [6] or [14].

Consider the family of functions defined on $(0,\infty)$ and $p\in[-1,2]\setminus\{0,1\}$ by

$$f_p(t):=\frac{p-1}{p}\left(\frac{t^p-1}{t^{p-1}-1}\right)$$

and

$$f_0(t):=\frac{t}{1-t}\ln t,$$

$$f_1(t):=\frac{t-1}{\ln t}\text{ (logarithmic mean).}$$

We also have the functions of interest

$$f_{-1}(t)=\frac{2t}{1+t}\text{ (harmonic mean)},\ f_{1/2}(t)=\sqrt{t}\text{ (geometric mean).}$$

We have the representations [4]:

Lemma 2. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (2.1). Then for all* $A,\ B>0$ *we have*

$$\begin{aligned}&[f(B)-f(A)](B-A)-b(B-A)^2\\&=\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s1_H)^{-1}(B-A)\right]^2dt\right]dm(s)\end{aligned}\tag{2.2}$$

and

$$\begin{aligned}&(B-A)[f(B)-f(A)]-b(B-A)^2\\&=\int_0^\infty s^2\left[\int_0^1\left[(B-A)((1-t)A+tB+s1_H)^{-1}\right]^2dt\right]dm(s),\end{aligned}\tag{2.3}$$

where b *and* m *are provided in Lemma 1.*

Proof. Since the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$, then f can be written as in the equation (1.2) and for $A,\ B>0$ we have the representation

$$\begin{aligned}&f(B)-f(A)\\&=b(B-A)+\int_0^\infty s\left[B(B+s1_H)^{-1}-A(A+s1_H)^{-1}\right]dm(s).\end{aligned}\tag{2.4}$$

Observe that for $s>0$

$$\begin{aligned}&B(B+s1_H)^{-1}-A(A+s1_H)^{-1}\\&=(B+s1_H-s1_H)(B+s1_H)^{-1}-(A+s1_H-s1_H)(A+s1_H)^{-1}\\&=(B+s1_H)(B+s1_H)^{-1}-s1_H(B+s1_H)^{-1}\\&-(A+s1_H)(A+s1_H)^{-1}+s1_H(A+s1_H)^{-1}\\&=1_H-s1_H(B+s1_H)^{-1}-1_H+s1_H(A+s1_H)^{-1}\\&=s\left[(A+s1_H)^{-1}-(B+s1_H)^{-1}\right].\end{aligned}$$

Therefore, (2.4) becomes, see also [7]

$$f(B)-f(A)=b(B-A)+\int_0^\infty s^2\left[(A+s1_H)^{-1}-(B+s1_H)^{-1}\right]dm(s).\tag{2.5}$$

Let $T,\ S>0$. The function $f(t)=-t^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.6}$$

for $T,\ S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for $C,\ D$ selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t)=f((1-t)C+tD),\ t\in[0,1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D]:=\{(1-t)C+tD,\ t\in[0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.7}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.8}$$

Now, if we replace in (2.8) $C=A+s1_H$ and $D=B+s1_H$ for $s>0$, then

$$\begin{aligned}&(A+s1_H)^{-1}-(B+s1_H)^{-1}\\&=\int_0^1((1-t)A+tB+s1_H)^{-1}(B-A)((1-t)A+tB+s1_H)^{-1}\,dt.\end{aligned} \tag{2.9}$$

By the representation (2.5), we derive the following identity of interest

$$\begin{aligned}f(B)-f(A)&=b(B-A)\\&\quad+\int_0^\infty s^2\left[\int_0^1((1-t)A+tB+s1_H)^{-1}\right.\\&\quad\left.\times(B-A)((1-t)A+tB+s1_H)^{-1}\,dt\right]dm(s)\end{aligned} \tag{2.10}$$

for $A,\ B>0$.

If we multiply this identity at the right with $B-A$ we get

$$\begin{aligned}&[f(B)-f(A)](B-A)\\&=b(B-A)^2\\&+\int_0^\infty s^2\left[\int_0^1\left[((1-t)A+tB+s1_H)^{-1}(B-A)\right]^2dt\right]dm(s)\end{aligned}$$

for $A,\ B>0$, which is (2.2).

If we multiply (2.10) at the left with $B-A$ we get

$$\begin{aligned}&(B-A)[f(B)-f(A)]\\&=b(B-A)^2\\&+\int_0^\infty s^2\left[\int_0^1\left[(B-A)((1-t)A+tB+s1_H)^{-1}\right]^2dt\right]dm(s)\end{aligned}$$

for $A,\ B>0$, which is (2.3).

The positivity in both (2.2) and (2.3) is obvious. □

Some necessary and sufficient conditions for the operators A, $B>0$ such that the inequality $(f(B)-f(A))(B-A)\geq 0$ holds for any operator monotone function f on $(0,\infty)$ are included in the following result, see [4]:

Lemma 3. *Let A, $B>0$. The following statements are equivalent:*

(i) *For all $s\geq 0$,*

$$(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)\geq 2. \tag{2.11}$$

(ii) *For all $s\geq 0$,*

$$\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2 dt\geq 0.$$

(iii) *For all $s\geq 0$,*

$$(\ell_s(B)-\ell_s(A))(B-A)\geq 0,$$

where $\ell_s(t)=-(t+s)^{-1}$, $t>0$.

(iv) *For all operator monotone function f on $(0,\infty)$,*

$$(f(B)-f(A))(B-A)\geq 0. \tag{2.12}$$

(v) *For all operator monotone function f on $(0,\infty)$,*

$$(B-A)(f(B)-f(A))\geq 0. \tag{2.13}$$

We remark that if A, B are as in Lemma 2, then obviously

$$(f(B)-f(A))(B-A)=(B-A)(f(B)-f(A)),$$

namely

$$f(A)B+f(B)A=Bf(A)+Af(B).$$

We also have, see [4]:

Lemma 4. *Let A, $B>0$, then the statements (i) and*

(i') Operator $A(B+s)^{-1}+B(A+s)^{-1}$ is selfadjoint for all $s\geq 0$,

are equivalent.

We define the class of operators

$$\mathfrak{Cl}_{(0,\infty)}(H):=\{(A,B)\mid A,\ B>0 \text{ and satisfy condition } (i')\}.$$

We observe that if $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$ then $(B,A)\in\mathfrak{Cl}_{(0,\infty)}(H)$.

Also if $AB=BA$, A, $B>0$, then $U_s:=(A+s)^{-1}(B+s)$ and $U_s^{-1}=(B+s)^{-1}(A+s)$ are selfadjoint and since $U_s+U_s^{-1}\geq 2$, $s\geq 0$ we derive that $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$. Therefore, if $\mathfrak{Co}_{(0,\infty)}(H)$ is the class of *all pairs of commutative operators* A, $B>0$, then we have

$$\emptyset\neq\mathfrak{Co}_{(0,\infty)}(H)\subset\mathfrak{Cl}_{(0,\infty)}(H). \tag{2.14}$$

We have the following operator inequalities:

Theorem 3. *Let $f:(0,\infty)\to\mathbb{R}$ be an operator monotone function on $(0,\infty)$ defined by (2.1). If $(A_t)_{t\in T}$ is a bounded continuous field of positive operators defined on a locally compact Hausdorff space T with a bounded Radon measure μ and such that $(A_t,A_s)\in\mathfrak{Cl}_{(0,\infty)}(H)$ for all t, $s\in T$ while $(p_t)_{t\in T}$ are nonnegative with $\int_T p_t d\mu(t)=\mathbf{1}$, then*

$$\int_T p_t f(A_t)A_t d\mu(t)-\int_T p_s f(A_s)d\mu(s)\int_T p_t A_t d\mu(t) \tag{2.15}$$
$$\geq b\left(\int_T p_t A_t^2 d\mu(t)-\left(\int_T p_t A_t d\mu(t)\right)^2\right)\geq 0.$$

Proof. From (2.2) we get

$$[f(A_t)-f(A_s)](A_t-A_s)\geq b(A_t-A_s)^2\geq 0$$

for all t, $s\in T$.

This inequality is equivalent to

$$f(A_t)A_t-f(A_s)A_t-f(A_t)A_s+f(A_s)A_s \tag{2.16}$$
$$\geq b\left(A_t^2-A_tA_s-A_sA_t+A_s^2\right)\geq 0$$

for all t, $s\in T$.

If we multiply (2.16) by $p_tp_s\geq 0$ for all t, $s\in T$ we get

$$p_tp_sf(A_t)A_t-p_tp_sf(A_s)A_t-p_tp_sf(A_t)A_s+p_tp_sf(A_s)A_s \tag{2.17}$$
$$\geq b\left(p_tp_sA_t^2-p_tp_sA_tA_s-p_tp_sA_sA_t+p_tp_sA_s^2\right)\geq 0$$

for all t, $s\in T$.

If we take the integral over $t\in T$ in (2.17) and observe that $\int_T p_t d\mu(t)=1$, then we get

$$p_s\int_T p_tf(A_t)A_td\mu(t)-p_sf(A_s)\int_T p_tA_td\mu(t) \tag{2.18}$$
$$-\left(\int_T p_tf(A_t)d\mu(t)\right)p_sA_s+p_sf(A_s)A_s$$
$$\geq b\left(p_s\int_T p_tA_t^2d\mu(t)-\left(\int_T p_tA_td\mu(t)\right)p_sA_s\right.$$
$$\left.-p_sA_s\int_T p_tA_td\mu(t)+p_sA_s^2\right)$$
$$\geq 0$$

for all $s\in T$.

If we take the integral over $s \in T$ in (2.18) and observe that $\int_T p_s d\mu(s) = 1$, then we get

$$\begin{aligned}
&\int_T p_t f(A_t) A_t d\mu(t) - \int_T p_s f(A_s) d\mu(s) \int_T p_t A_t d\mu(t) \\
&- \left(\int_T p_t f(A_t) d\mu(t)\right) \int_T p_s A_s d\mu(s) + \int_T p_s f(A_s) A_s d\mu(s) \\
&\geq b \left(\int_T p_t A_t^2 d\mu(t) - \left(\int_T p_t A_t d\mu(t)\right) \int_T p_s A_s d\mu(s)\right. \\
&\left. - \int_T p_s A_s d\mu(s) \int_T p_t A_t d\mu(t) + \int_T p_s A_s^2 d\mu(s)\right) \\
&\geq 0,
\end{aligned}$$

namely

$$\begin{aligned}
&2\int_T p_t f(A_t) A_t d\mu(t) - 2\int_T p_s f(A_s) d\mu(s) \int_T p_t A_t d\mu(t) \\
&\geq 2b \left(\int_T p_t A_t^2 d\mu(t) - \left(\int_T p_t A_t d\mu(t)\right)^2\right) \geq 0,
\end{aligned}$$

which is equivalent to (2.15). □

Remark 1. *We observe that for an operator monotone function f on $(0, \infty)$, we have*

$$\int_T p_t f(A_t) A_t d\mu(t) \geq \int_T p_t f(A_t) d\mu(t) \int_T p_t A_t d\mu(t), \tag{2.19}$$

where $(A_t)_{t\in T}$ is as in Theorem 3 and $(p_t)_{t\in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$.

Corollary 1. *Let $g : (0, \infty) \to (0, \infty)$ be an operator monotone function on $(0, \infty)$. If $(A_t)_{t\in T}$ is as in Theorem 3 and $(p_t)_{t\in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$, then*

$$\int_T p_t [g(A_t)]^{-1} A_t^2 d\mu(t) \geq \int_T p_t [g(A_t)]^{-1} A_t d\mu(t) \int_T p_t A_t d\mu(t), \tag{2.20}$$

Since $g : (0, \infty) \to (0, \infty)$ is an operator monotone function on $(0, \infty)$, then $f(t) = t/g(t)$, $t > 0$ is operator monotone and by (2.19) we get (2.20).

The discrete case is as follows:

Corollary 2. *Let $f : (0, \infty) \to \mathbb{R}$ be an operator monotone function on $(0, \infty)$ defined by (2.1). If A_k, $k = 1, ..., n$ is an n-tuple of positive operators such that $(A_k, A_j) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for all $k, j = 1, ..., n$ and p_k are nonnegative with $\sum_{k=1}^n p_k = 1$, then*

$$\begin{aligned}
&\sum_{k=1}^n p_k f(A_k) A_k - \sum_{k=1}^n p_k f(A_k) \sum_{k=1}^n p_k A_k \\
&\geq b \left[\sum_{k=1}^n p_k A_k^2 - \left(\sum_{k=1}^n p_k A_k\right)^2\right] \geq 0.
\end{aligned} \tag{2.21}$$

For other recent inequalities for operator monotone functions, see [1] and [15]-[17].

3. Related Results

A related result is as follows:

Theorem 4. *Let $f:(0,\infty)\to\mathbb{R}$ be an operator monotone function on $(0,\infty)$ defined by (2.1) and $g:(0,\infty)\to(0,\infty)$ a continuous function. If $(A_t)_{t\in T}$ is a bounded continuous field of positive operators defined on a locally compact Hausdorff space T with a bounded Radon measure μ such that $\left(g(A_t), g\left(\int_T p_s A_s d\mu(s)\right)\right) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for $t\in T$ and $(p_t)_{t\in T}$ are nonnegative with $\int_T p_t d\mu(t) = \mathbf{1}$, then*

$$\begin{aligned}(3.1)\qquad &\int_T p_t (f\circ g)(A_t)\, g(A_t)\, d\mu(t) \\ &+ (f\circ g)\left(\int_T p_s A_s d\mu(s)\right) g\left(\int_T p_s A_s d\mu(s)\right) \\ &- (f\circ g)\left(\int_T p_s A_s d\mu(s)\right)\int_T p_t g(A_t)\, d\mu(t) \\ &- \int_T p_t (f\circ g)(A_t)\, d\mu(t)\, g\left(\int_T p_s A_s d\mu(s)\right) \\ &\geq b\left[\int_T p_t g^2(A_t)\, d\mu(t) + g^2\left(\int_T p_s A_s d\mu(s)\right)\right. \\ &- \int_T p_t g(A_t)\, d\mu(t)\, g\left(\int_T p_s A_s d\mu(s)\right) \\ &\left.-g\left(\int_T p_s A_s d\mu(s)\right)\int_T p_t g(A_t)\, d\mu(t)\right] \\ &\geq 0.\end{aligned}$$

Proof. We have from (2.2) for $A = g(A_t)$ and $B = g\left(\int_T p_s A_s d\mu(s)\right)$ that

$$\begin{aligned}(3.2)\qquad &\left[f(g(A_t)) - f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right)\right]\left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right)\right) \\ &\geq b\left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right)\right)^2 \geq 0\end{aligned}$$

for all $t\in T$.

If we multiply with $p_t \geq 0$ and integrate over $t\in T$ in (3.2), then we get

$$\begin{aligned}(3.3)\qquad &\int_T p_t\left[f(g(A_t)) - f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right)\right] \\ &\times\left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right)\right) d\mu(t) \\ &\geq b\int_T p_t\left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right)\right)^2 d\mu(t) \geq 0.\end{aligned}$$

Observe that

$$\begin{aligned}&\int_T p_t\left[f(g(A_t)) - f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right)\right] \\ &\times\left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right)\right) d\mu(t)\end{aligned}$$

$$= \int_T p_t f(g(A_t)) g(A_t) \mu(t) - f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) \int_T p_t g(A_t) d\mu(t)$$
$$- \int_T p_t f(g(A_t)) d\mu(t) g\left(\int_T p_s A_s d\mu(s)\right)$$
$$+ \int_T p_t d\mu(t) f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) g\left(\int_T p_s A_s d\mu(s)\right)$$
$$= \int_T p_t f(g(A_t)) g(A_t) d\mu(t) + f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) g\left(\int_T p_s A_s d\mu(s)\right)$$
$$- f\left(g\left(\int_T p_s A_s d\mu(s)\right)\right) \int_T p_t g(A_t) d\mu(t)$$
$$- \int_T p_t f(g(A_t)) d\mu(t) g\left(\int_T p_s A_s d\mu(s)\right)$$

and

$$\int_T p_t \left(g(A_t) - g\left(\int_T p_s A_s d\mu(s)\right)\right)^2 d\mu(t)$$
$$= \int_T p_t \left[g^2(A_t) - g(A_t) g\left(\int_T p_s A_s d\mu(s)\right)\right.$$
$$\left. -g\left(\int_T p_s A_s d\mu(s)\right) g(A_t) + g^2\left(\int_T p_s A_s d\mu(s)\right)\right] d\mu(t)$$
$$= \int_T p_t g^2(A_t) d\mu(t) + g^2\left(\int_T p_s A_s d\mu(s)\right)$$
$$- \int_T p_t g(A_t) d\mu(t) g\left(\int_T p_s A_s d\mu(s)\right)$$
$$- g\left(\int_T p_s A_s d\mu(s)\right) \int_T p_t g(A_t) d\mu(t).$$

By making use of (3.3), we derive (3.1). □

Corollary 3. *Let $f : (0,\infty) \to \mathbb{R}$ be an operator monotone function on $(0,\infty)$ defined by (2.1) and $g : (0,\infty) \to (0,\infty)$ a continuous function. If A_k, $k = 1, ..., n$ is an n-tuple of positive operators and such that $\left(g(A_k), g\left(\sum_{k=1}^n p_k A_k\right)\right) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for $k = 1, ..., n$ and p_k are nonnegative with $\sum_{k=1}^n p_k = 1$, then*

$$\sum_{k=1}^n p_k (f \circ g)(A_k) g(A_k) + (f \circ g)\left(\sum_{k=1}^n p_k A_k\right) g\left(\sum_{k=1}^n p_k A_k\right) \tag{3.4}$$
$$- (f \circ g)\left(\sum_{k=1}^n p_k A_k\right) \sum_{k=1}^n p_k g(A_k) - \sum_{k=1}^n p_k (f \circ g)(A_k) g\left(\sum_{k=1}^n p_k A_k\right)$$
$$\geq b\left[\sum_{k=1}^n p_k g^2(A_k) + g^2\left(\sum_{k=1}^n p_k A_k\right)\right.$$
$$\left. - \sum_{k=1}^n p_k g(A_k) g\left(\sum_{k=1}^n p_k A_k\right) - g\left(\sum_{k=1}^n p_k A_k\right) \sum_{k=1}^n p_k g(A_k)\right]$$
$$\geq 0.$$

4. Some Examples

Let $A,\ B > 0$, $p : [0,1] \to [0,\infty)$ be a continuous function and $d\mu(t) = dt$ the usual Lebesgue measure on $[0,1]$. In the following we consider $A_t := (1-t)A + tB$, with $(A_t, A_s) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for all $t,\ s \in T$, $p_t = p(t)$, $t \in [0,1]$ and assume that $\int_0^1 p(t) = 1$. We observe that for an operator monotone function f on $(0,\infty)$, we have by (2.19) that

$$\int_0^1 p(t)((1-t)A+tB) f((1-t)A+tB)\, dt \geq \int_0^1 p(t) f((1-t)A+tB)\, dt \int_0^1 p(t)((1-t)A+tB)\, dt. \tag{4.1}$$

We observe that, if p is symmetric on $[0,1]$ namely $p(1-t) = p(t)$, for all $t \in [0,1]$, then

$$\begin{aligned}\int_0^1 p(t)((1-t)A+tB)\, dt &= \left(\int_0^1 p(t)(1-t)\, dt\right) A + \left(\int_0^1 p(t)\, tdt\right) B \\ &= \left(\int_0^1 p(1-t)(1-t)\, dt\right) A + \left(\int_0^1 p(t)\, tdt\right) B \\ &= \left(\int_0^1 p(s)(s)\, ds\right) A + \left(\int_0^1 p(t)\, tdt\right) B \\ &= \left(\int_0^1 p(t)\, tdt\right)(A+B).\end{aligned}$$

Then by (4.1) we get

$$\int_0^1 p(t)((1-t)A+tB) f((1-t)A+tB)\, dt \geq \left(\int_0^1 p(t)\, tdt\right)\left(\int_0^1 p(t) f((1-t)A+tB)\, dt\right)(A+B). \tag{4.2}$$

Observe that

$$\int_0^1 p(t)\, tdt = \int_0^1 p(1-t)(1-t)\, dt = \int_0^1 p(t)(1-t)\, dt.$$

Then

$$\int_0^1 p(t)\, tdt + \int_0^1 p(t)(1-t)\, dt = \int_0^1 p(t)\, dt = 1,$$

which implies that $\int_0^1 p(t)\, tdt = 1/2$.

Then by (4.2) we get

$$\int_0^1 p(t)((1-t)A+tB) f((1-t)A+tB)\, dt \geq \left(\int_0^1 p(t) f((1-t)A+tB)\, dt\right)\left(\frac{A+B}{2}\right). \tag{4.3}$$

From (4.3) we derive the inequalities

$$\int_0^1 p(t)((1-t)A+tB)^{r+1}\, dt \geq \int_0^1 p(t)((1-t)A+tB)^{r}\, dt \left(\frac{A+B}{2}\right), \tag{4.4}$$

for $r \in (0,1)$ and

$$\int_0^1 p(t)((1-t)A+tB)\ln((1-t)A+tB)\,dt \geq \int_0^1 p(t)\ln((1-t)A+tB)\,dt\left(\frac{A+B}{2}\right). \tag{4.5}$$

If $A_t := (1-t)A+tB$ is defined on $[0,1]$ such that $\left(g(A_t), g\left(\int_0^1 p(t)A_s ds\right)\right) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for $t \in T$, while $g:(0,\infty)\to(0,\infty)$ is a continuous function, then from (3.1) we obtain

$$\begin{aligned} &\int_0^1 p(t)(f\circ g)((1-t)A+tB)\,g((1-t)A+tB)\,dt \\ &+(f\circ g)\left(\int_0^1 p(t)((1-t)A+tB)\,dt\right)g\left(\int_0^1 p(t)((1-t)A+tB)\,dt\right) \\ &\geq (f\circ g)\left(\int_0^1 p(t)((1-t)A+tB)\,dt\right)\int_0^1 p(t)\,g((1-t)A+tB)\,dt \\ &+\int_0^1 p(t)(f\circ g)((1-t)A+tB)\,dtg\left(\int_0^1 p(t)((1-t)A+tB)\,dt\right), \end{aligned} \tag{4.6}$$

where $f:(0,\infty)\to\mathbb{R}$ is an operator monotone function on $(0,\infty)$.

If p is symmetric on $[0,1]$, then

$$\int_0^1 p(t)((1-t)A+tB)\,dt = \left(\int_0^1 p(t)\,tdt\right)(A+B) = \frac{A+B}{2}$$

and by (4.6) we get

$$\begin{aligned} &\int_0^1 p(t)(f\circ g)((1-t)A+tB)\,g((1-t)A+tB)\,dt \\ &+(f\circ g)\left(\frac{A+B}{2}\right)g\left(\frac{A+B}{2}\right) \\ &\geq (f\circ g)\left(\frac{A+B}{2}\right)\int_0^1 p(t)\,g((1-t)A+tB)\,dt \\ &+\left(\int_0^1 p(t)(f\circ g)((1-t)A+tB)\,dt\right)g\left(\frac{A+B}{2}\right), \end{aligned} \tag{4.7}$$

where $f:(0,\infty)\to\mathbb{R}$ is an operator monotone function on $(0,\infty)$ while $g:(0,\infty)\to(0,\infty)$ is a continuous function.

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