OPERATOR INEQUALITIES AND CHARACTERIZATIONS

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Abstract. Let $\mathfrak{B}(H)$ be the C*-algebra of all bounded linear operators acting on a complex separable Hilbert space H. In this survey, we shall present characterizations of some distinguished classes of $\mathfrak{B}(H)$ (namely, normal operators, selfadjoint operators, and unitary operators) in terms of operator inequalities related to the arithmetic-geometric mean inequality.

For the class of all normal operators, we shall present new four general characterizations given as follows:

(i) $\left|S^2\right| = |S|^2,\ \left|S^{*2}\right| = |S^*|^2,\quad (S \in \mathfrak{B}(H)),$

(ii) $\left|S^2\right|^2 \geq |S|^4,\ \left|S^{*2}\right|^2 \geq |S^*|^4,\quad (S \in \mathfrak{B}(H)),$

(iii) S and S^* belong to class $\mathbf{A}$, $(S \in \mathfrak{B}(H))$,

(iv) S and S^* are paranormal, $(S \in \mathfrak{B}(H))$.

Note that the characterization (iv) was given by Ando (see [1]) but with the kernel assumption $\ker S = \ker S^*$.

1. Introduction and preliminaries

Let $\mathfrak{B}(H)$ be the C*-algebra of all bounded linear operators acting on a complex separable Hilbert space H.

We denote by:

- $u \otimes v$ (where $u,\ v \in H$), the operator of rank less or equal to one on H defined by $(u \otimes v)\, x = \langle x, v\rangle\, u$, for every $x \in H$,
- $\mathcal{F}_1(H) = \{x \otimes y : x,\ y \in H\}$, the set of all operators of rank less or equal to one on H,
- $|S|$, the positive square root of the positive operator S^*S (where $S \in \mathfrak{B}(H)$),
- $\{S\}'$ and $\{S\}''$, the commutant and the bicommutant of S, respectively (where $S \in \mathfrak{B}(H)$),
- $(M)_1 = \{x \in M : \|x\| = 1\}$, for M be a subset of some normed space,
- $K \circ L = \left\{\sum_{i=1}^n \alpha_i \beta_i : (\alpha_1, ..., \alpha_n) \in K,\ (\beta_1, ..., \beta_n) \in L\right\}$, for $L,\ K \subset \mathbb{C}^n$, $n \geq 1$,
- $|\Gamma| = \sup\limits_{\gamma \in \Gamma} |\gamma|$, where Γ is a bounded subset of the field of scalars,
- $\Gamma M = \{\lambda m : \lambda \in \Gamma,\ m \in M\}$, where M is a subspace of some vector space, and Γ is a subset of the field of scalars.

For $S, T \in \mathfrak{B}(H)$:

- we say that S and T are unitarily equivalent, if there exists a unitary operator $U \in \mathfrak{B}(H)$ such that $S = U^*TU$,

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- S is paranormal if, $\|x\| \left\|S^2x\right\| \geq \|Sx\|^2$, for every $x \in H$,
- we say that S belongs to class **A**, if $\left|S^2\right| \geq |S|^2$,
- if $S \geq 0,\ T \geq 0$, and $S \geq T$, then $S^\alpha \geq T^\alpha$, for every $\alpha \in [0,1]$ (Lŏwner-Heinz inequality ([8]),
- if S belongs to class **A**, then it is paranormal (see [6]).

If $\mathcal{A}$ is a (real or complex) unital normed algebra, and $A \in \mathcal{A}$, then

- we denote by $\sigma(A)$ and $r(A)$, the spectrum and the spectral radius of A, respectively,
- we denote by $V(A)$ and $w(A)$, the algebraic numerical range and the numerical radius of A, respectively,
- A is called convexoid if $V(A) = co\sigma(A)$,
- if $\mathcal{A} = \mathfrak{B}(H)$, then $V(A) = \overline{W(A)}$ (where $\overline{W(A)}$ is the closure of the usual numerical range of A).

For $S \in \mathfrak{B}(H)$, let $R(S)$ and $\ker S$ denote the range and the kernel of S, respectively.

It is known that for $S \in \mathfrak{B}(H)$,

- then S is of closed range if and only if there exits an operator $S^+ \in \mathcal{R}(H)$ satisfying the four following equations
$$SS^+S = S,\ S^+SS^+ = S^+,\ (SS^+)^* = SS^+,\ (S^+S)^* = S^+S,$$
- the operator S^+ if exists is unique, and it is called the Moore-Penrose inverse of S, and it satisfies that SS^+ and S^+S are orthogonal projections onto $R(S)$ and $R(S^*)$, respectively,
- if S is invertible, then $S^+ = S^{-1}$, and if $S \in \mathfrak{B}(H)$ is a surjective operator (resp. injective with closed range), then $SS^+ = I$ (rep. $S^+S = I$).

For every S in $\mathfrak{B}(H)$ with closed range:

- we associate the 2×2 matrix representation $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$ with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$,
- the operator S is called an EP operator if $R(S^*) = R(S)$, or equivalently $S_2 = 0$ and S_1 is invertible; in this case $S^+ = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$,
- if S is normal, then it is an EP operator.

My main purpose of this survey paper is to present our characterizations (presented in several papers) of some distinguished classes of $\mathfrak{B}(H)$, namely, the selfadjoint operators, the normal operators, and the unitary operators, in terms of operator inequalities. Our idea is to make a connection between some operator inequalities related to the known arithmetic-geometric mean inequality and some remarkable classes of operators in $\mathfrak{B}(H)$.

Part 1. Selfadjoint operators and S – AGMI.

In [8], Heinz proved that for every two positive operators P and Q in $\mathfrak{B}(H)$, and for every $\alpha \in [0,1]$, the following operator inequality holds

$$(HI) \qquad \forall X \in \mathfrak{B}(H),\ \|PX + XQ\| \geq \left\|P^\alpha XQ^{1-\alpha} + P^{1-\alpha}XQ^\alpha\right\|.$$

As a particular case of this, for $\alpha = \frac{1}{2}$, is the well known arithmetic-geometric mean inequality given by

$$(S - AGMI) \qquad \forall A, B, X \in \mathfrak{B}(H),\ \|A^*AX + XBB^*\| \geq 2\,\|AXB\|.$$

Note that the proof of (HI) given by Heinz is somewhat complicated. For this reason, McIntosh [11] with an elegant proof, proved that the operator inequality $(S-AGMI)$ holds, and deduced from it the Heinz inequality.

Independently of the work of Heinz and McIntosh, Corach et al. proved in [4], that for every invertible selfadjoint operator S in $\mathfrak{B}(H)$, the following inequality holds

$$(S1) \qquad \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\| \geq 2\left\|X\right\|,$$

In [5, 1993], Fujji et al. had proved that the three above inequalities are mutually equivalent, and in [3, 2018], it was presented other operator inequalities that are also equivalent to $(S-AGMI)$, and here we cite two of them given by:

$$(S2) \qquad \forall X \in \mathfrak{B}(H),\ \left\|SXS^{+}+S^{+}XS\right\| \geq 2\left\|SS^{+}XS^{+}S\right\|,$$

for every selfadjoint operator with closed range $S \in \mathfrak{B}(H)$,

$$(S3) \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^{2}X+XS^{2}\right\| \geq 2\left\|SXS\right\|,$$

for every selfadjoint operator $S \in \mathfrak{B}(H)$.

Note that $(S2)$ (resp. $(S3)$) is a general form of the Corach-Porta-Recht inequality from the invertible case to closed range case (resp. the general situation). This first family of operator inequalities $(S1)$, $(S2)$, and $(S3)$ that are equivalent to $(S-AGMI)$ is generated by a selfadjoint operator (invertible, with closed range, and any).

In [5], Fujji et al. had shown that $(S1)$ holds with an easy proof, this gives us an easier proof of Heinz inequality. Without forgetting that these three inequalities have been generalized from the usual norm to unitarily invariant norms. For us, we have followed a different way in introducing in [12, 2001] our first characterization of the class of all invertible operators $S \in \mathfrak{B}(H)$ satisfying the operator inequality $(S1)$. We have showed that this class is exactly the class of all invertible selfadjoint operators in $\mathfrak{B}(H)$ multiplied by nonzero scalars.

Following this kind of problem from the invertible case to the closed range case, and to the general situation, we have asked to finding the class:

(**i**) of all operators with closed ranges $S \in \mathfrak{B}(H)$ satisfying $(S2)$,

(**ii**) of all operators $S \in \mathfrak{B}(H)$ satisfying $(S3)$.

We have showed that the class

- (**i**) is exactly the class of all selfadjoint operators with closed ranges in $\mathfrak{B}(H)$ multiplied by nonzero scalars (see [18, 2015]),
- (**ii**) is exactly the class of all selfadjoint operators in $\mathfrak{B}(H)$ multiplied by nonzero scalars (see [20, 2019]).

Part 2. Normal operators and N − AGMI.

We consider a second version of the arithmetic-geometric mean inequality which follows immediately from $(S-AGMI)$ given as follows:

$$(N-AGMI) \qquad \forall A, B, X \in \mathfrak{B}(H),\ \|A^{*}AX\|+\|XBB^{*}\| \geq 2\|AXB\|$$

In [3, 2018], we have showed that $(N-AGMI)$ is equivalent to each of the three following inequalities:

$$(N1) \qquad \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\| \geq 2\left\|X\right\|.$$

for every invertible normal operator $S \in \mathfrak{B}(H)$,

$$(N2)\qquad \forall X\in\mathfrak{B}(H),\ \left\|SXS^{+}\right\|+\left\|S^{+}XS\right\|\geq 2\left\|SS^{+}XS^{+}S\right\|,$$

for every normal operator with closed range $S\in\mathfrak{B}(H)$,

$$(N3)\qquad \forall X\in\mathfrak{B}(H),\ \left\|S^{2}X\right\|+\left\|XS^{2}\right\|\geq 2\left\|SXS\right\|,$$

for every normal operators $S\in\mathfrak{B}(H)$.

Note that the inequality $(N2)$ (resp. $(N3)$) is a general form of $(N1)$ from the invertible case to the closed range case (resp. the general situation). In [3, 2018], we have proved $(N-AGMI)$ independently to $(S-AGMI)$.

This second family of operator inequalities $(N1)$, $(N2)$, and $(N3)$ that are equivalent to $(N-AGMI)$ is generated by a normal operator (invertible, with closed range, and any).

As we have done for the characterizations with the above first family, also it is interesting to describe the largest class

(**i**) of all invertible operators $S\in\mathfrak{B}(H)$ satisfying $(N1)$,

(**ii**) of all operators with closed ranges $S\in\mathfrak{B}(H)$ satisfying $(N2)$,

(**iii**) of all operators $S\in\mathfrak{B}(H)$ satisfying $(N3)$.

We have showed that the class

- (**i**) is exactly the class of all invertible normal operators in $\mathfrak{B}(H)$ (see [15, 2009]),
- (**ii**) is exactly the class of all normal operators with closed ranges in $\mathfrak{B}(H)$ (see [18, 2015]),
- (**iii**) is exactly the class of all normal operators in $\mathfrak{B}(H)$ (see [20, 2019]).

In this work, we shall present four new general characterizations of the class of all normal operators in $\mathfrak{B}(H)$, that are given as follows:

(**i**) $\left|S^{2}\right|=\left|S\right|^{2},\ \left|S^{*2}\right|=\left|S^{*}\right|^{2},\ \ (S\in\mathfrak{B}(H))$,

(**ii**) $\left|S^{2}\right|^{2}\geq\left|S\right|^{4},\ \left|S^{*2}\right|^{2}\geq\left|S^{*}\right|^{4},\ \ (S\in\mathfrak{B}(H))$,

(**iii**) S and S^{*} belong to class $\mathbf{A}$, $\ (S\in\mathfrak{B}(H))$,

(**iv**) S and S^{*} are paranormal, $\ (S\in\mathfrak{B}(H))$.

Note that the characterization (**iv**) was given by T. Ando [1, 1972], but with the kernel assumption $\ker S=\ker S^{*}$.

Part 3. Unitary operators.

Let S be an invertible operator in $\mathfrak{B}(H)$.

It is clear that

$$\inf_{\|X\|=1}\left\|SXS^{-1}+S^{-1}XS\right\|\leq 2\leq\sup_{\|X\|=1}\left\|SXS^{-1}+S^{-1}XS\right\|.$$

Then from the above first part with the invertible case, the above infimum gets its maximal value 2 if and only if S is an invertible selfadjoint operator in $\mathfrak{B}(H)$ multiplied by a nonzero scalar.

So, what about the operator S for which the above supremum gets its minimal value 2? It was proved that:

(**i**) this supremum gets its minimal value 2 if and only if S is a unitary operator multiplied by a nonzero scalar (see [16, 2011]),

(**ii**) $\sup\limits_{\|X\|=1=rankX}\left\|SXS^{-1}+S^{-1}XS\right\|\geq 2$ (see [15, 2009]), (**iii**) this last supremum gets its minimal value 2 if and only if S is normal and $\sup\limits_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right|=2$ (see [16, 2011]).

From (**i**), the class of all unitary operators in $\mathfrak{B}(H)$ multiplied by nonzero scalars is exactly the class of all invertible operators $S\in\mathfrak{B}(H)$ satisfying the following operator inequality:

$$\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\|\leq 2\left\|X\right\|.$$

From (**iii**), the class of all invertible normal operators $S\in\mathfrak{B}(H)$ for which $\sup\limits_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right|=2$ is exactly the class of all invertible operators $S\in\mathfrak{B}(H)$ satisfying the following operator inequality:

$$\forall X\in\mathcal{F}_1(H),\ \left\|SXS^{-1}+S^{-1}XS\right\|\leq 2\left\|X\right\|.$$

This second class contains strictly the first class and contained strictly in the class of all invertible normal operators in $\mathfrak{B}(H)$.

Part 4. Some comments.

Note that for the characterization concerning $(S2)$ (resp. $(N2)$) with the closed range case is deduced from the characterization of $(S1)$ (resp. $(N1)$), and the characterization concerning $(S3)$ (resp. $(N3)$) with the general situation is deduced from the characterization of $(S2)$ (resp. $(N2)$).

Unfortunately, after the publication of the paper [18] concerning the closed range case, we have found a mistake in *Lemma* 1, and all results concerning the closed range case (for the two above parts) depend on it. So, in the corrigendum [19, 2017], we have presented a corrected proof of this lemma. Note that in the proof of this corrected lemma, we have used the *Theorem* 3.6 of [12], where one of the conditions of this theorem is an equality between spectrum of two positive operators. This condition is enough for the invertible case only, but not suffice for non-invertible case and our lemma is for non-invertible case. But, to have a complete proof of the lemma, we need *Theorem* 6.3 with inclusion between spectrum instead of equality. We have mentioned in the proof of the corrected lemma that *Theorem* 6.3 remains true with inclusion between spectrum but without argument. In this survey, we shall present this argument.

This work gives to the separately published results a certain harmony and concordance. The starting point in this work will be the largest class of normal operators, then their two subclasses of self-adjoint operators and unitary operators, and will end with the intersection of these two last subclasses, namely, the class of unitary reflections. We will then notice that each class will be linked to an operator inequality and how their forms vary from one to the other.

In section 2, we shall present a family of operator inequalities that are equivalent to $(N-AGMI)$, and the characterizations cited above in the part 2 concerning the normal operators.

In section 3, we shall present a family of operator inequalities that are equivalent to $(S-AGMI)$, and the characterizations cited above in the part 1 concerning the selfadjoint operators.

In section 4, we shall present some results concerning the injective norm of the two following operators on $\mathfrak{B}(H)$:

$$X\rightarrow SXS^{-1}+S^{-1}XS,\quad X\rightarrow S^{*}XS^{-1}+S^{-1}XS^{*}$$

(where S be an invertible operator in $\mathfrak{B}(H)$), and some characterizations that are cited in the part 3 and others.

2. N-Arithmetic-Geometric-Mean Inequality, Normal operators, and Characterizations

In this section, we shall present some characterizations of the class of all normal operators in $\mathfrak{B}(H)$ in terms of operator inequalities, and also its two subclasses of all invertible normal operators and all normal operators with closed ranges in $\mathfrak{B}(H)$. These operator inequalities are related to the N-Arithmetic-Geometric-Mean Inequality which will be introduced in the first proposition.

We start with the following remark which contains two trivial characterizations of the class of all normal operators in $\mathfrak{B}(H)$.

Remark 1. *Let $S \in \mathfrak{B}(H)$. It is easy to see that the three following properties are equivalent:*
(i) *S is normal,*
(ii) $\forall X \in \mathfrak{B}(H),\ \|S^*X\| = \|SX\|$,
(iii) $\forall X \in \mathfrak{B}(H),\ \|XS^*\| = \|XS\|$.

In this section, we consider the N-Arithmetic-Geometric Mean Inequality given by:

$$(N-AGMI) \qquad \forall A, B, X \in \mathfrak{B}(H),\ \|A^*AX\| + \|XBB^*\| \geq 2\|AXB\|.$$

This inequality follows immediately from the known Arithmetic-Geometric-Mean Inequality (which is called here $(S-AGMI)$). In the next proposition, we present a family of operator inequalities generated by normal operators that are equivalent to the $(N-AGMI)$, and we shall prove $(N-AGMI)$ independently on $(S-AGMI)$.

Proposition 1. [3] *The following operator inequalities hold and are mutually equivalent:*

$$(1) \qquad \forall X \in \mathfrak{B}(H),\ \|A^*AX\| + \|XBB^*\| \geq 2\|AXB\|,$$

for every $A,\ B \in \mathfrak{B}(H)$,

$$(2) \qquad \forall X \in \mathfrak{B}(H),\ \left\|SXR^+\right\| + \left\|S^+XR\right\| \geq 2\left\|SS^+XR^+R\right\|,$$

for every normal operators with closed ranges $S, R \in \mathfrak{B}(H)$,

$$(3) \qquad \forall X \in \mathfrak{B}(H),\ \left\|SXR^{-1}\right\| + \left\|S^{-1}XR\right\| \geq 2\|X\|,$$

for every invertible normal operators $S, R \in \mathfrak{B}(H)$,

$$(4) \qquad \forall X \in \mathfrak{B}(H), \left\|S^2X\right\| + \left\|XR^2\right\| \geq 2\|SXR\|,$$

for every normal operators $S, R \in \mathfrak{B}(H)$,

$$(1') \qquad \forall X \in \mathfrak{B}(H),\ \|A^*AX\| + \|XAA^*\| \geq 2\|AXA\|,$$

for every $A \in \mathfrak{B}(H)$,

$$(2') \qquad \forall X \in \mathfrak{B}(H),\ \left\|SXS^+\right\| + \left\|S^+XS\right\| \geq 2\left\|SS^+XS^+S\right\|,$$

for every normal operator with closed range $S \in \mathfrak{B}(H)$,

$$(3') \qquad \forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\|X\|,$$

for every invertible normal operator $S \in \mathfrak{B}(H)$,

$$(4') \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|,$$

for every normal operator $S \in \mathfrak{B}(H)$,

Proof. (1) $\Rightarrow$ (2). Assume (1) holds. Let S, R be two normal operators with closed ranges in $\mathfrak{B}(H)$, and let $X \in \mathfrak{B}(H)$. Since $S^* = S^*SS^+$ and $R^* = R^+RR^*$, then from (1) and (*Remark* 1), it follows that

$$\begin{aligned} \left\|SXR^+\right\| + \left\|S^+XR\right\| &= \left\|S^*S\left(S^+XR^+\right)\right\| + \left\|\left(S^+XR^+\right)RR^*\right\| \\ &\geq 2\left\|SS^+XR^+R\right\|. \end{aligned}$$

Hence (2) holds.

(2) $\Rightarrow$ (3). This implication is trivial.

(3) $\Rightarrow$ (4). Assume (3) holds.

let $S,\ R,\ X \in \mathfrak{B}(H)$ such that $S,\ R$ are normal. Put $P = |S|,\quad Q = |R^*|$, and let $\epsilon > 0$.

It is clear that the two operators $P + \epsilon I$ and $Q + \epsilon I$ are normal and invertible. So, from (3), we obtain

$$\forall \epsilon > 0,\ \forall X \in \mathfrak{B}(H),\ \left\|(P+\epsilon I)^2 X\right\| + \left\|X(Q+\epsilon I)^2\right\| \geq 2\left\|(P+\epsilon I)X(Q+\epsilon I)\right\|.$$

By letting $\epsilon \to 0$, we deduce that:

$$(*) \quad \forall X \in \mathfrak{B}(H),\ \left\|P^2X\right\| + \left\|XQ^2\right\| \geq 2\left\|PXQ\right\|.$$

So, we have:

$$\begin{aligned} \left\|S^2X\right\| + \left\|XR^2\right\| &= \left\|S^*SX\right\| + \left\|XRR^*\right\| \quad (\text{from } \textit{Remark } 1), \\ &= \left\|P^2X\right\| + \left\|XQ^2\right\| \\ &\geq 2\left\|PXQ\right\| \quad (\text{from } (*)), \\ &= 2\left\|SXR\right\|. \end{aligned}$$

This proves (4).

(4) $\Rightarrow$ (1). Assume (4) holds.

Let $A,\ B,\ X \in \mathfrak{B}(H)$. Put $P = |A|,\ Q = |B^*|$. Then, we have:

$$\begin{aligned} \left\|A^*AX\right\| + \left\|XBB^*\right\| &= \left\|P^2X\right\| + \left\|XQ^2\right\| \\ &\geq 2\left\|PXQ\right\| \quad (\text{from } (4)), \\ &= 2\left\|AXB\right\|. \end{aligned}$$

This proves (1).

Therefore the operator inequalities $(1) - (4)$ are equivalent.

From a pair of operators to a single operator, we deduce that the operator inequalities $(1') - (4')$ are also equivalent.

(1) $\Rightarrow$ (1′). This implication is trivial.

(1′) $\Rightarrow$ (1). Assume (1′) holds (here we use the Berberian technic).

Let $A, B, X \in \mathfrak{B}(H)$. Consider now, the bounded linear operators $C,\ Y$ defined on the Hilbert space $H \oplus H$ given by $C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$. By a simple computation, we obtain $C^*CY = \begin{bmatrix} 0 & A^*AX \\ 0 & 0 \end{bmatrix}$, $YCC^* = \begin{bmatrix} 0 & XBB^* \\ 0 & 0 \end{bmatrix}$, and

$CYC = \begin{bmatrix} 0 & AXB \\ 0 & 0 \end{bmatrix}$. Applying (1′) for the Hilbert space $H \oplus H$, we obtain $\|A^*AX\| + \|XAA^*\| = \|C^*CY\| + \|YCC^*\| \geq 2\|CYC\| = 2\|AXB\|$. This proves (1).

Therefore the inequalities $(1) - (4)$, and $(1') - (4')$ are mutually equivalent. It remains to prove that one of them holds. It is clear that (1) is an immediate consequence of the known Arithmetic-geometric mean inequality $(S - AGMI)$. But here, we shall give a direct proof of (1) independently of $(S - AGMI)$ by using the numerical arithmetic-geometric mean inequality. Let $A, B, X \in \mathfrak{B}(H)$. The following inequalities hold:

$$\begin{aligned}
\frac{1}{2}\left(\|A^*AX\| + \|XBB^*\|\right) &\geq \sqrt{\|A^*AX\|\,\|XBB^*\|} \\
&\geq \sqrt{\|BB^*X^*A^*AX\|} \\
&\geq \sqrt{r(BB^*X^*A^*AX)} \\
&= \sqrt{r\left(B^*X^*A^*AXB\right)} \\
&= \|AXB\|.
\end{aligned}$$

□

Corollary 1. *The following operator inequalities hold and are equivalent to* $(N - AGMI)$*:*

$$(5) \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^*XR^+\right\| + \left\|S^+XR^*\right\| \geq 2\left\|SS^+XR^+R\right\|,$$

for every operators with closed ranges $S, R \in \mathfrak{B}(H)$*,*

$$(6) \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^*XR^{-1}\right\| + \left\|S^{-1}XR^*\right\| \geq 2\|X\|,$$

for every invertible operators $S, R \in \mathfrak{B}(H)$*,*

$$(5') \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^*XS^+\right\| + \left\|S^+XS^*\right\| \geq 2\left\|SS^+XS^+S\right\|,$$

for every operator with closed range $S \in \mathfrak{B}(H)$*,*

$$(6') \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\| \geq 2\|X\|,$$

for every invertible operator $S \in \mathfrak{B}(H)$*.*

Proof. Assume $(N - AGMI)$ holds. Prove that (5) holds.

Let $S, R \in \mathcal{R}(H)$, and $X \in \mathfrak{B}(H)$. Since, $SS^+S = S$ and $RR^+R = R$, then we have

$$\begin{aligned}
\left\|S^*XR^+\right\| + \left\|S^+XR^*\right\| &= \left\|S^*S\left(S^+XR^+\right)\right\| + \left\|\left(S^+XR^+\right)RR^*\right\|, \\
&\geq 2\left\|SS^+XR^+R\right\|, \quad \text{(from } (N - AGMI)\text{)}.
\end{aligned}$$

This proves (5).

It is clear that (5) implies (6), (5′), (6′), and using *Remark* 1, then (5) (resp. (6), (5′), (6′)) implies (2) (resp. (3), (2′), (3′)). □

Note that the six operator inequalities $(2) - (4)$ and $(2') - (4')$ given in the last proposition are generated by a pair of normal operators and a single of normal operator, respectively.

We shall interest to describe the class of

(**i**) all invertible operators $S \in \mathfrak{B}(H)$ satisfying the operator inequality $(3^{'})$,

(**ii**) all operators with closed ranges $S \in \mathfrak{B}(H)$ satisfying the operator inequality $(2^{'})$,

(**iii**) all operators $S \in \mathfrak{B}(H)$ satisfying the operator inequality $(4^{'})$.

We shall prove that the class

- (**i**) is the class of all invertible normal operators in $\mathfrak{B}(H)$,
- (**ii**) is the class of all normal operators with closed ranges in $\mathfrak{B}(H)$,
- (**iii**) is the class of all normal operators in $\mathfrak{B}(H)$.

We shall present here all these characterizations and others. We need the following lemmas.

Lemma 1. [22]. *Let $A \in \mathfrak{B}(H)$. If $\|A - \lambda I\| = r(A - \lambda I)$, for all complex λ, then A is convexoid.*

Lemma 2. [12] *Let P, Q be two invertible positive operators in $\mathfrak{B}(H)$ satisfying the following operator inequality*

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|PXP^{-1}\right\| \geq 2\left\|QXQ^{-1}\right\|.$$

Then, we have $\{P\}' \subset \{Q\}'$.

Proof. (i). Let X be a selfadjoint operator in $\mathfrak{B}(H)$ such that $PX = XP$, and let α be an arbitrary complex number. Replace X by $X - \alpha I$ in the inequality given by the lemma, and since $X - \alpha I$ is normal, we obtain

$$\|X - \alpha I\| \geq \left\|Q(X - \alpha I)Q^{-1}\right\| \geq r\left(Q(X - \alpha I)Q^{-1}\right) = \|X - \alpha I\|.$$

Hence, $\left\|QXQ^{-1} - \alpha I\right\| = r\left(QXQ^{-1} - \alpha I\right)$, for all complex number α. Using the above lemma, we obtain that the

$$\begin{aligned} V\left(QXQ^{-1}\right) &= co\left(\sigma\left(QXQ^{-1}\right)\right), \\ &= co\sigma\left(X\right), \\ &\subset \mathbb{R}. \end{aligned}$$

This give us that QXQ^{-1} is selfadjoint. Hence, $QX = XQ$.

(ii). Now, let X be an arbitrary operator in $\mathfrak{B}(H)$. Put $X = X_1 + iX_2$, where $X_1 = \mathrm{Re}(X)$, and $X_2 = \mathrm{Im}\, X$. Assume that $PX = XP$. Then, $PX_1 = X_1P$ and $PX_2 = X_2P$. From (i), we deduce that, $QX_1 = X_1Q$ and $QX_2 = X_2Q$. Thus, $QX = XQ$. Therefore, $\{P\}' \subset \{Q\}'$. □

Lemma 3. [12] *Let P, Q be two invertible positive operators in $\mathfrak{B}(H)$ satisfying the following operator inequality*

$$\forall X \in \mathfrak{B}(H),\ \left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq 2\|X\|.$$

Then, we have $\{P\}' = \{Q\}'$.

Proof. From the inequality given in the lemma, we have

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|PQXQ^{-1}P^{-1}\right\| \geq 2\left\|QXQ^{-1}\right\|\ (*).$$

Put $M = |PQ|$. So, from this last inequality, we obtain

$$\forall X \in \mathfrak{B}(H),\ \|X\| + \left\|MXM^{-1}\right\| \geq 2\left\|QXQ^{-1}\right\|.$$

Hence, from the above lemma, we deduce that, $MQ = QM$. Then, $PQ = QP$.

Now, let X be a selfadjoint operator in $\mathfrak{B}(H)$ such that $PX = XP$, and let α be an arbitrary complex number. Replace in $(*)$, X by $X - \alpha I$, so we have, $\|X - \alpha I\| \geq \left\|Q(X - \alpha I)Q^{-1}\right\|$, for every complex number α. hence, $QX = XQ$, and thus $\{P\}^{'} \subset \{Q\}^{'}$, this follows by using the same argument as used in the proof of the above lemma.

Using again the inequality given in the lemma, we obtain also that $\{Q\}^{'} = \left\{Q^{-1}\right\}^{'} \subset \left\{P^{-1}\right\}' = \{P\}^{'}$. Therefore, $\{P\}^{'} = \{Q\}^{'}$. □

Lemma 4. *Let $\epsilon > 0$, and let $\alpha_1, ..., \alpha_n$, $\beta_1, ..., \beta_n$ (where $n \geq 1$) be positive real numbers such that $0 < \alpha_1 \leq ... \leq \alpha_n \leq 1$, $\{\alpha_1, ..., \alpha_n\} \subset \left\{\beta_1, ..., \beta_n\right\}$, and $\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \epsilon$, for every $i,\ j$. Then, we have $\left|\alpha_i - \beta_i\right| \leq \epsilon$, for $i = 1, ..., n$.*

Proof. From the hypothesis, we deduce easily that $\beta_i - \beta_j \leq \epsilon$, if $i < j$.

Let $i \in \{1, ..., n\}$ such that $\alpha_i \neq \beta_i$ (in the case $\alpha_i = \beta_i$, of course we have $\left|\alpha_i - \beta_i\right| = 0 \leq \epsilon$).

There are three cases: $i = 1,\ i = n$, and $1 < i < n$.

Case 1. $i = 1$. There exists $j \geq 2$ such that $\beta_j = \alpha_1$. So, we have, $|\alpha_1 - \beta_1| = \beta_1 - \beta_j \leq \epsilon$, since $j > 1$.

Case 2. $i = n$. There exists $j < n$ such that $\beta_j = \alpha_n$. Hence, $|\alpha_n - \beta_n| = \beta_j - \beta_n \leq \epsilon$, since $j < n$.

Case 3. $1 < i < n$.

If $\alpha_i < \beta_i$, then there exists $j > i$, such that $\beta_j \leq \alpha_i$. Hence, $\left|\alpha_i - \beta_i\right| \leq \beta_i - \beta_j \leq \epsilon$, since $i < j$.

If $\alpha_i > \beta_i$, then there exists $j < i$, such that $\beta_j \geq \alpha_i$. Hence, $\left|\alpha_i - \beta_i\right| \leq \beta_j - \beta_i \leq \epsilon$, since $i > j$. □

Remark 2. *In the original paper* [12]*, the above lemma is Lemma* 3.5*, but it is given in a particular case with equality instead of inclusion, and where the sequence $\{\alpha_1, ..., \alpha_n\}$ is increasing instead of non-decreasing; but the proof remains unchanged. In the particular case, the Lemma* 3.5 *is needed only for invertible case.*

Lemma 5. *Let P, Q be two invertible positive operators in $\mathfrak{B}(H)$ such that $\sigma(Q) \subset \sigma(P)$ or $\sigma(P) \subset \sigma(Q)$. Then, the two following properties are equivalent*

(i) $\forall X \in \mathfrak{B}(H),\ \left\|PXP^{-1}\right\| + \left\|Q^{-1}XQ\right\| \geq 2\|X\|$,

(ii) $P = Q$.

Proof. We may assume without loss of the generality that $\|P\| = \|Q\| = 1$.

$(i) \Rightarrow (ii)$. Assume (i) holds.

Decompose P and Q using their spectral measure

$$P = \int \lambda dE_\lambda, \quad Q = \int \lambda dF\lambda$$

and consider

$$P_n = \int h_n(\lambda)\, dE_\lambda = h_n(P), \quad Q_n = \int h_n(\lambda)\, dF\lambda = h_n(Q).$$

where $h_n(\lambda)$ is the function defined by

$$h_n(\lambda) = \frac{k}{n}, \text{if } \frac{k}{n} \leq \lambda < \frac{k+1}{n},\ \text{for } k = 1,\ 2,\ 3, ...$$

Case 1. $\sigma(Q) \subset \sigma(P)$. Using the spectral theorem with the function h_n, we have

$$\sigma(Q_n) = \sigma(h_n(Q)) = h_n(\sigma(Q)) \subset h_n(\sigma(P)) = \sigma(h_n(P)) = \sigma(P_n)$$

Then P_n, Q_n are invertible positive operators in $\mathfrak{B}(H)$ with finite spectrum such that $\sigma(Q_n) \subset \sigma(P_n)$, $P_n \to P$, $Q_n \to Q$ uniformly, and $P_n \in \{P\}^{''}$, $Q_n \in \{Q\}^{''}$ (where $\{P\}^{''} = \{Q\}^{''}$, from the *Lemma* 3).

Hence, $P_nQ_n = Q_nP_n$, for every $n \geq 1$. Then, $Q_n = \sum_{i=}^{p} \alpha_i E_i$, $P_n = \sum_{i=}^{p} \beta_i E_i$, where $\sigma(Q_n) = \{\alpha_1, .., \alpha_p\}$ such that $0 < \alpha_1 \leq ... \leq \alpha_p \leq 1$, $\sigma(P_n) = \left\{\beta_1, .., \beta_p\right\}$, and $E_1, ..., E_p$ are orthogonal projections in $\mathfrak{B}(H)$ such that $E_iE_j = 0$, if $i \neq j$, $\sum_{i=1}^{p} E_i = I$. Thus, $\{\alpha_1, .., \alpha_p\} \subset \left\{\beta_1, .., \beta_p\right\}$.

Let $\epsilon > 0$. Then, there exists an integer $N \geq 1$ such that

$$\forall n > N,\ \forall X \in \mathfrak{B}(H),\ \left\|P_n X P_n^{-1}\right\| + \left\|Q_n^{-1} X Q_n\right\| \geq (2-\epsilon)\|X\|.$$

Let $n > N$, and replace X by E_iXE_j (where $X \in \mathfrak{B}(H)$) in this last inequality, we deduce that

$$\frac{\alpha_i}{\alpha_j} + \frac{\beta_j}{\beta_i} \geq 2 - \epsilon, \text{ for } i, j = 1, ..., p.$$

From these last inequalities, and since $0 < \alpha_1 \leq ... \leq \alpha_p \leq 1$, $\{\alpha_1, .., \alpha_p\} \subset \left\{\beta_1, .., \beta_p\right\}$, and using the above lemma, we obtain $|\alpha_i - \beta_i| \leq \epsilon$, for $i = 1, ..., p$.

Since, $P_n = \sum_{i=}^{p} \alpha_i E_i$, and $Q_n = \sum_{i=}^{p} \beta_i E_i$, then

$$\begin{aligned} \|P_n - Q_n\| &= \max_{1 \leq i \leq p_n} |\alpha_i - \beta_i| \\ &\leq \epsilon. \end{aligned}$$

Therefore, $P = Q$.

Case 2. $\sigma(Q) \subset \sigma(P)$. Using the same argument as used before, we find also $P = Q$.

The implication $(ii) \Rightarrow (i)$ follows immediately from $(N - AGMI)$. □

Remark 3. *The above lemma in the original paper* [12] *is the Theorem* 3.6 *but with equality between spectrum of P and Q instead of the inclusion. The equality condition is enough for the invertible case. But for the non invertible case, the lemma presented here with inclusion is needed.*

In the two next propositions, we shall present some characterizations of the class of all invertible normal operators in $\mathfrak{B}(H)$.

Proposition 2. [15] *Let S be an invertible operator in $\mathfrak{B}(H)$. Then, the following properties are equivalent:*

(i) *S is normal,*

(ii) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|,$

(iii) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq \left\|S^*XS^{-1}\right\| + \left\|S^{-1}XS^*\right\|.$

(iv) $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\|X\|.$

Proof. $(i) \Rightarrow (ii)$. This follows from *Remark* 1.

$(ii) \Rightarrow (iii)$. This implication is trivial.

$(iii) \Rightarrow (iv)$. This follows immediately from *Corollary* $1.6'$.

$(iv) \Rightarrow (i)$. Assume (iv) holds.

Let $S=UP$, $S^*=U^*Q$ be the polar decompositions of S and S^*, and let $X\in\mathfrak{B}(H)$. Then, from (iv), it follows that:

$$\forall X\in\mathfrak{B}(H),\ \left\|PXP^{-1}\right\|+\left\|Q^{-1}XQ\right\|\geq 2\left\|X\right\|.$$

Since, $\sigma(P^2)=\sigma(S^*S)=\sigma(SS^*)=\sigma(Q^2)$, so from the Spectral Theorem, $\sigma(P)=\sigma(Q)$. Using the last Lemma, we obtain $P=Q$. Therefore, S is normal. □

Proposition 3. *Let S be an invertible operator in $\mathfrak{B}(H)$. The following properties are equivalent:*
(i) *S is normal,*
(ii) $\forall X\in\mathcal{F}_1(H),\ \left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|=\left\|S^*XS^{-1}\right\|+\left\|S^{-1}XS^*\right\|,$
(iii) $\forall X\in\mathcal{F}_1(H),\ \left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|\geq\left\|S^*XS^{-1}\right\|+\left\|S^{-1}XS^*\right\|,$
(iv) $\forall X\in\mathcal{F}_1(H),\ \left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|\geq 2\left\|X\right\|,$
(v) $\forall X\in\mathfrak{B}(H),\left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|\leq\left\|S^*XS^{-1}\right\|+\left\|S^{-1}XS^*\right\|,$
(vi) $\forall X\in\mathcal{F}_1(H),\left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|\leq\left\|S^*XS^{-1}\right\|+\left\|S^{-1}XS^*\right\|.$

Proof. The implication $(i)\Longrightarrow(ii)$ follows immediately from *Remark* 1, the implication $(ii)\Longrightarrow(iii)$ is trivial, and the implication $(iii)\Longrightarrow(iv)$ follows from *Corollary* 1.6′.

$(iv)\Longrightarrow(i)$. Assume (iv) holds.

Put $P=|S|$ and $Q=|S^*|$. Using (iv) and the polar decomposition of S and S^*, we deduce the following inequality:

$$\forall X\in\mathcal{F}_1(H),\ \left\|PXP^{-1}\right\|+\left\|Q^{-1}XQ\right\|\geq 2\left\|X\right\|.$$

Since P and Q are unitarily equivalent, using [10, *Theorem* 2.1], we find that $P=Q$. This proves (i).

Thus, the four conditions $(i)-(iv)$ are mutually equivalent.

$(i)\Rightarrow(v)$. This implication follows immediately from *Remark* 1.

$(v)\Rightarrow(vi)$. This implication is trivial.

$(vi)\Rightarrow(i)$. From (vi), it follows that the following inequality holds

$$\forall x,\ y\in(H)_1,\ \left\|Sx\right\|\left\|(S^*)^{-1}y\right\|+\left\|S^{-1}x\right\|\left\|S^*y\right\|\leq\left\|S^*x\right\|\left\|(S^*)^{-1}y\right\|+\left\|S^{-1}x\right\|\left\|Sy\right\|$$

Hence

$$\forall x,\ y\in(H)_1,\ (\left\|Sx\right\|-\left\|S^*x\right\|)\left\|(S^*)^{-1}y\right\|\leq(\left\|Sy\right\|-\left\|S^*y\right\|)\left\|S^{-1}x\right\|\qquad(A)$$

Thus

$$(\forall x\in(H)_1,\ \left\|Sx\right\|\geq\left\|S^*x\right\|)\vee(\forall x\in(H)_1,\ \left\|Sx\right\|\leq\left\|S^*x\right\|)$$

Assume that the inequality $\left\|Sx\right\|\geq\left\|S^*x\right\|$ holds for every $x\in(H)_1$.

Since the relation $\frac{1}{\left\|T^{-1}\right\|}\leq\left\|Tx\right\|\leq\left\|T\right\|$ holds for every invertible operator $T\in\mathfrak{B}(H)$ and for every $x\in(H)_1$, then from (A), it follows that:

$$\forall x,\ y\in(H)_1,\ \left\|Sx\right\|-\left\|S^*x\right\|\leq k\left(\left\|Sy\right\|-\left\|S^*y\right\|\right)$$

where $k=\left\|S\right\|\left\|S^{-1}\right\|$. So we have:

$$\forall x,\ y\in(H)_1,\ \left\|Sx\right\|+k\left\|S^*y\right\|\leq\left\|S^*x\right\|+k\left\|Sy\right\|$$

By taking the infimum over $y\in(H)_1$, we obtain:

$$\forall x\in H,\ \left\|Sx\right\|\leq\left\|S^*x\right\|.$$

Therefore S is normal.

With the second assumption and by the same argument, we find also that S is normal. This proves (i). □

Remark 4. *(1) The equivalences between* (i), (v), *and* (vi) *was given in* [17].

(2) For $\mathcal{R} \in \{\leq, \geq, =\}$, *and* $\mathfrak{L}(H) \in \{\mathfrak{B}(H), \mathcal{F}_1(H)\}$, *then the class of all invertible normal operators in* $\mathfrak{B}(H)$ *is exactly the class of all invertible operators* $S \in \mathfrak{B}(H)$ *satisfying each of the two following operator inequalities:*

$$\begin{aligned} \forall X &\in \mathfrak{L}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\left\|X\right\|, \\ \forall X &\in \mathfrak{L}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \ \mathcal{R}\ \left\|S^{*}XS^{-1}\right\| + \left\|S^{-1}XS^{*}\right\|. \end{aligned}$$

In the next proposition, we shall give a complete characterization of the class of all normal operators in $\mathfrak{B}(H)$ in terms of operator inequality. To prove this, we need the following results of Halmos (see [7]) that says, the set

$$\mathfrak{D}(H) = \{S \in \mathfrak{B}(H) : S \text{ is left invertible or right invertible}\}$$

is dense in $\mathfrak{B}(H)$, and from the fact that for $S \in \mathfrak{B}(H)$, we have:

(i) S is left invertible if and only if S is injective with closed range,

(ii) S is right invertible if and only if S is surjective.

Proposition 4. [19, 20] *Let* $S \in \mathfrak{B}(H)$. *Then, the following properties are equivalent*

(i) *S is normal,*

(ii) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|.$

Proof. We may assume that $S \neq 0$

$(i) \Rightarrow (ii)$. Assume (i) holds.

Let $X \in \mathfrak{B}(H)$. Then we have

$$\begin{aligned} \left\|S^2X\right\| + \left\|XS^2\right\| &= \left\|S^*SX\right\| + \left\|XSS^*\right\|, \text{ (from } Remark \text{ 1)}, \\ &\geq 2\left\|SXS\right\|, \quad \text{(from } (N-AGMI)). \end{aligned}$$

This proves (ii).

$(ii) \Rightarrow (i)$. Assume (ii) holds.

We prove (i) in three cases: S injective with closed range, S surjective, and S arbitrary.

Case 1. Assume that S is injective with closed range.

Hence, $S^+S = I$, $\ker P = \ker S = \{0\}$, and $R(P) = R(S^*S)$ is closed (since $R(S^*)$ is also closed). Thus $\ker P = \{0\}$ and $R(P) = (\ker P)^{\perp} = H$. So, P is invertible.

All the 2×2 matrices used in this proof are given with respect to the orthogonal direct sum $H = R(S) \oplus \ker S^*$. Then $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix}$. We put $P = |S|$, $Q = |S^*|$, $P_1 = |S_1|$, $P_2 = |S_2|$, $Q_1 = \left(S_1S_1^* + S_2S_2^*\right)^{\frac{1}{2}}$. So we have $S^*S = P^2 = \begin{bmatrix} P_1^2 & S_1^*S_2 \\ S_2^*S_1 & P_2^2 \end{bmatrix}$, $SS^* = Q^2 = \begin{bmatrix} Q_1^2 & 0 \\ 0 & 0 \end{bmatrix}$. It is clear that Q_1 is invertible and $Q^+ = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$.

From (ii), the two following inequalities hold

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2S^+XS^+\right\| + \left\|S^+XS\right\| \geq 2\left\|SS^+X\right\|. \tag{1}$$

$$\forall X \in \mathfrak{B}(H),\ \|XS\| + \left\|S^2XS^+\right\| \geq 2\,\|SX\|. \tag{2}$$

The proof is given in four steps.

Step 1. Prove that $\left(S^2\right)^+S = S^+$.

It is known that S^+ is the unique solution of the following four equations: $SXS = S,\ XSX = X,\ (XS)^* = XS,\ (SX)^* = SX$. It is easy to see that $\left(S^2\right)^+S$ satisfies the first three equations.

Now we prove that $\left(S^2\right)^+S$ also satisfies the last equation. Since the operator $S\left(S^2\right)^+S$ is a projection, it suffices to prove that its norm is less than or equal to one. By taking $X = \left(S^2\right)^+S$ in (2), we obtain

$$2 \geq \left\|\left(S^2\right)^+S^2\right\| + \left\|S^2\left(S^2\right)^+SS^+\right\| \geq 2\left\|S\left(S^2\right)^+S\right\|.$$

Hence $\left\|S\left(S^2\right)^+S\right\| \leq 1$. Therefore $\left(S^2\right)^+S = S^+$.

Step 2. Prove that $\left(S^2\right)^+ = (S^+)^2$.

Since $S^2\left(S^2\right)^+ = SS^+S^2\left(S^2\right)^+$, then $S^2\left(S^2\right)^+ = S^2\left(S^2\right)^+SS^+$. So from step 2, we obtain $S^2\left(S^2\right)^+ = S^2(S^+)^2$. Since S^2 is injective, we have $\left(S^2\right)^+ = (S^+)^2$.

Step 3. Prove that $\ker S^* = \{0\}$.

Since S is injective, then $\ker S^* = \{0\}$ if and only if $S_2 = 0$. Assume that $S_2 \neq 0$.

Since $\left(S^2\right)^+ = (S^+)^2$, then the two operators S^*S and SS^+ commute (see [2], [9]). Thus $P^2 = \begin{bmatrix} P_1^2 & 0 \\ 0 & P_2^2 \end{bmatrix}$. So that $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$.

Since $\ker S^* \neq \{0\}$, then $\sigma(Q^2) = \sigma(Q_1^2) \cup \{0\}$. From the fact that $\sigma(P^2) = \sigma(Q^2) - \{0\}$, we have $\sigma(P^2) = \sigma(Q_1^2)$. Then $\sigma(P_1^2) \cup \sigma(P_2^2) = \sigma(Q_1^2)$. Hence $\sigma(P_1^2) \subset \sigma(Q_1^2)$. Thus $\sigma(P_1) \subset \sigma(Q_1)$.

Using the polar decomposition of S and S^* in the inequality (1), we obtain the following inequality

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2S^+XP^{-1}\right\| + \left\|Q^+XQ\right\| \geq 2\left\|SS^+X\right\|$$

By taking $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}$ (resp. $X = \begin{bmatrix} 0 & X_2 \\ 0 & 0 \end{bmatrix}$), where $X_1 \in \mathfrak{B}(R(S))$ (resp. $X_2 \in \mathfrak{B}(\ker S^*, R(S))$) in the last inequality and since $S^2S^+ = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}$, we deduce the two following inequalities

$$\forall X_1 \in \mathfrak{B}(R(S)),\ \left\|P_1X_1P_1^{-1}\right\| + \left\|Q_1^{-1}X_1Q_1\right\| \geq 2\,\|X_1\| \tag{3}$$

$$\forall X_2 \in \mathfrak{B}(\ker S^*, R(S)),\ \left\|P_1X_2P_2^{-1}\right\| \geq 2\,\|X_2\| \tag{4}$$

By taking $X_2 = x \otimes y$ (where $x \in (R(S))_1$, $y \in \ker S^*$) in (4), we obtain

$$\forall x \in (R(S))_1, \forall y \in \ker S^*,\ \|P_1x\|\left\|P_2^{-1}y\right\| \geq 2\,\|y\|$$

So we have

$$\forall x \in (R(S))_1, \forall y \in \left(\ker S^*\right)_1,\ \|P_1x\| \geq 2\,\|P_2y\|$$

Thus $\|P_2 y\| \leq \frac{k}{2}$, for every $y \in (\ker S^*)_1$ (where $k = \inf_{\|x\|=1} \|P_1 x\| > 0$). Then $\langle P_2^2 y, y \rangle \leq \frac{k^2}{4}$, for every $y \in (\ker S^*)_1$. So we obtain $\sigma(P_2^2) \subset (0, \frac{k^2}{4}]$ and $\sigma(P_1^2) \subset [k^2, \infty)$.

Since $\sigma(P_1) \subset \sigma(Q_1)$, and P_1 , Q_1 satisfy the inequality (3), then using *Lemma* 5, we obtain $P_1 = Q_1$. Hence $\sigma(Q_1^2) = \sigma(P_1^2) = \sigma(P_1^2) \cup \sigma(P_2^2)$. Then $\sigma(P_2^2) \subset \sigma(P_1^2)$, that is impossible since $(0, \frac{k^2}{4}] \cap [k^2, \infty) = \varnothing$. Therefore $\ker S^* = \{0\}$.

Step 4. Prove that S is normal.

Since $\ker S^* = \{0\}$, then $R(S) = H$. So that S is invertible and satisfies the inequality (ii). Hence S satisfies the following inequality

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\|X\|$$

Therefore S is normal, by using *Proposition* 2.

Case 2. Assume S surjective.

Then S^* is injective with a closed range satisfying also the inequality (ii). So that from case 1, S^* is normal. Hence S is normal.

Case 3. General situation.

We may assume without loss of generality that $\|S\| = 1$.Then $\left\|S^2\right\| = \|S\|^2 = 1$. There exists a sequence $(S_n)_{n\geq 1}$ of elements in $\mathfrak{D}(H)$ such that $S_n \to S$ uniformly.

Define the real function F on the complete metric space $(\mathfrak{B}(H))_1$ by

$$\forall X \in (\mathfrak{B}(H))_1\,,\ F(X) = \left\|S^2 X\right\| + \left\|XS^2\right\| - 2\|SXS\|\,,$$

and for $n \geq 1$, define the real function F_n on $(\mathfrak{B}(H))_1$ by

$$\forall X \in (\mathfrak{B}(H))_1\,,\ F_n(X) = \left\|S_n^2 X\right\| + \left\|XS_n^2\right\| - 2\|S_n X S_n\|\,.$$

Put $D = \{X \in (\mathfrak{B}(H))_1 : F(X) > 0\}$. Then there are two cases, $D = \varnothing$, $D \neq \varnothing$.

(**1**). $D = \varnothing$. So, it follows that

$$(*)\qquad \forall X \in \mathfrak{B}(H), \left\|S^2 X\right\| + \left\|XS^2\right\| = 2\|SXS\|\,.$$

From this equality, we have

$$\forall x, y \in H,\ \left\|S^2 x\right\| \|y\| + \|x\| \left\|S^{*2} y\right\| = 2\|Sx\| \|S^* y\|\,.$$

Since $S^2 \neq 0$, and from this last inequality, we deduce easily that S and S^* are injective, and then S is with dense range.

Prove now that S is with closed range. Let (x_n) be a sequence of vectors in H such that (Sx_n) converges to a vector $y \in H$. We may choose a vector $u \in (H)_1$ such that $S^{*2} u \neq 0$. From the above inequality, we obtain

$$\forall n, m \geq 1,\ \left\|S^2 x_n - S^2 x_m\right\| + \|x_n - x_m\| \left\|S^{*2} u\right\| = 2\|Sx_n - Sx_m\| \|S^* u\|\,.$$

Hence, (x_n) is a Cauchy sequence, and then it converges to some vector $x \in H$. So that $Sx_n \to y = Sx$. This proves $R(S)$ closed. Then, S is invertible.

So, from $(*)$, it follows that

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| = 2\|X\|\,.$$

From *Proposition* 2, (i) holds.

(**2**). $D \neq \varnothing$. From the fact that F is a positive continuous map on $(\mathfrak{B}(H))_1$, it follows that

$$\overline{D} = \overline{F^{-1}((0,\infty))} = F^{-1}([0,\infty)) = \{X \in (\mathfrak{B}(H))_1 : F(X) \geq 0\} = (\mathfrak{B}(H))_1\,.$$

Let $X \in D$, and $\epsilon > 0$. Since $S_n \to S$ uniformly, then there exists an integer $N \geq 1$ (depends only in ϵ) such that

$$\forall n \geq N,\ \forall Y \in (\mathfrak{B}(H))_1\,,\ |F(Y) - F_n(Y)| \leq \epsilon.$$

If there exists $n \geq N$ such that $F_n(X) < 0$, then using this last inequality, we have $0 \leq F(X) < \epsilon$, for every $\epsilon > 0$; thus $F(X) = 0$, leading a contradiction with $X \in D$.

From this fact, it follows that

$$\forall X \in D,\ \forall n \geq N,\quad F_n(X) \geq 0.$$

Since each F_n is a continuous map on $(\mathfrak{B}(H))_1$ and L is dense in $(\mathfrak{B}(H))_1$, then

$$\forall X \in (\mathfrak{B}(H))_1\,,\ \forall n \geq N,\ F_n(X) \geq 0.$$

So, it follows that

$$\forall X \in \mathfrak{B}(H),\ \forall n \geq N,\ \left\|S_n^2 X\right\| + \left\|X S_n^2\right\| \geq 2\left\|S_n X S_n\right\|.$$

Since for each $n \geq 1$, $S_n \in \mathfrak{D}(H)$, then from the two above cases, we obtain that S_n is a normal operator, for every $n \geq N$. Since $S_n \to S$ uniformly and the class of all normal operators in $\mathfrak{B}(H)$ is closed, then S is a normal. □

Remark 5. *In the above proof, the case 1 and the case 2 are given in the lemma presented in the corrigendum* [19]*, and the general situation is presented in* [20]*. Note that in the proof of this lemma of the corrigendum, we have used the Theorem* 3.6 *of* [12] *which is given with the condition of equality between spectrum (that is not suffice), but the lemma is true with the condition of inclusion instead of equality. So, we have mentioned in the proof of the lemma that this theorem remains true with inclusion between spectrum, but without argument. In this survey, we have present this argument (see Lemma* 4 *and Lemma* 5*).*

Corollary 2. *Let $S \in \mathfrak{B}(H)$. Then, the following properties are equivalent.*
(i) *S is normal,*
(ii) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\|\left\|XS^2\right\| \geq \left\|SXS\right\|^2.$

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds, and let $X \in \mathfrak{B}(H)$. Then we have

$$\begin{aligned}\left\|S^2X\right\|\left\|XS^2\right\| &= \left\|S^*SX\right\|\left\|XSS^*\right\|, \text{ (using } Remark\ 1)\\ &\geq \left\|SXS\right\|^2, \text{ (see the proof of the } (N-AGMI) \text{ in } Proposition\ 1).\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume (ii) holds, and let $X \in \mathfrak{B}(H)$. Then we have:

$$\begin{aligned}\frac{\left\|S^2X\right\| + \left\|XS^2\right\|}{2} &\geq \sqrt{\left\|S^2X\right\|\left\|XS^2\right\|}, \quad \text{(from the numerical AGMI)},\\ &\geq \left\|SXS\right\|, \ \text{(using } (ii)).\end{aligned}$$

From the last proposition, (i) holds. □

Corollary 3. [18] *Let S be an operator with closed range in $\mathfrak{B}(H)$. Then the following properties are equivalent:*

$$\begin{array}{lllll}(i) & S \text{ is normal,} & & & \\ (ii) & \forall X \in \mathfrak{B}(H), & \left\|SXS^+\right\| + \left\|S^+XS\right\| & = & \left\|S^*XS^+\right\| + \left\|S^+XS^*\right\|, \\ (iii) & \forall X \in \mathfrak{B}(H), & \left\|SXS^+\right\| + \left\|S^+XS\right\| & \geq & \left\|S^*XS^+\right\| + \left\|S^+XS^*\right\|, \\ (iv) & \forall X \in \mathfrak{B}(H), & \left\|SXS^+\right\| + \left\|S^+XS\right\| & \geq & 2\left\|SS^+XS^+S\right\|,\end{array}$$

Proof. We may assume that $S \neq 0$.
$(i) \Rightarrow (ii)$. This follows immediately from *Remark* 1.
The implication $(ii) \Rightarrow (iii)$ is trivial.
$(iii) \Rightarrow (vi)$. This follows immediately from *Corollary* 1.5′.
$(iv) \Rightarrow (i)$. Assume (iv) holds. Then the following inequality holds

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2XSS^+\right\| + \left\|S^+SXS^2\right\| \geq 2\left\|SS^+SXSS^+S\right\|.$$

From this inequality and since $\|SS^+\| = \|S^+S\| = 1$, and $SS^+S = S$, it follows that

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|.$$

Using *Proposition* 4, S is normal. □

Remark 6. *In the original paper* [18]*, the above corollary is presented before the characterization of the class of all normal operators in $\mathfrak{B}(H)$ in its general situation, for this reason its proof was very strong. But, in this survey, we have adopt a new strategy, where this corollary follows immediately from the general situation.*

In [1, 1972], Ando proved that for $S \in \mathfrak{B}(H)$, S is normal if and only if S and S^* are paranormal, and $\ker S = \ker S^*$. In the next proposition, we present some new general characterizations of the class of all normal operators in $\mathfrak{B}(H)$, and we shall show that the Ando result remains true without the kernel assumption.

Proposition 5. *Let $S \in \mathfrak{B}(H)$. The following properties are equivalent.*
(i) *S is normal,*
(ii) $\left|S^2\right| = |S|^2,\ \left|S^{*2}\right| = |S^*|^2,$
(iii) $\left|S^2\right|^2 \geq |S|^4,\ \left|S^{*2}\right|^2 \geq |S^*|^4,$
(iv) *S and S^* belong to class* $\mathbf{A}$,
(v) *S and S^* are paranormal.*

Proof. The two implications $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (iii)$ are trivial.
$(iii) \Rightarrow (iv)$. This follows from Lŏwner-Heinz inequality with $\alpha = \frac{1}{2}$.
$(vi) \Rightarrow (v)$. This follows from [8].
$(v) \Rightarrow (i)$. Assume (v) holds. Then, we have

$$\begin{cases} (1) \quad \forall x \in H,\ \|x\|\left\|S^2s\right\| \geq \|Sx\|^2\ , \\ (2) \quad \forall x \in H,\ \|x\|\left\|(S^*)^2x\right\| \geq \|S^*x\|^2\ . \end{cases}$$

So, it follows that

$$(3) \quad \forall X \in \mathcal{F}_1(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|\ .$$

The proof is given in three cases.
Case 1. Assume that S surjective.
From (1), it follows that

$$\forall x \in H,\ \left\|S^+x\right\|\|Sx\| \geq \|x\|^2.$$

Then S is injective. Hence S is invertible. So using (3), we obtain

$$\forall X \in \mathcal{F}_1(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\|X\|.$$

So from *Proposition* 3, S is normal.
Case 2. Assume that S is injective with closed range.
Then S^* is surjective. So, from (2) and using the same argument as used in the case 1, we find that S invertible. So that S is normal.

Case 3. General situation.

From (3), and by using the same argument as used in the case 3 of the proof of *Proposition* 4 (where $\mathcal{F}_1(H)$ takes the place of $\mathfrak{B}(H)$), and using [10, *Theorem* 2.1], we obtain that S is normal. □

Remark 7. *The equivalences between* (i), (ii), *and* (iii) *in the above corollary was given in* [20], *and follow from Proposition* 4. *The equivalences between* (i), (iv), *and* (v) *are new.*

3. Arithmetic-Geometric-Mean Inequality, Selfadjoint Operators, and Characterization

In the following proposition, we shall give a family of operator inequalities that are equivalent to $(S-AGMI)$ and presenting the proof of $(S1)$ given in [5].

Proposition 6. [3, 5] *The following operator inequalities hold and are mutually equivalent:*

$$\forall X\in\mathfrak{B}(H),\ \|A^*AX+XBB^*\|\geq 2\|AXB\|, \tag{1}$$

for every $A,B\in\mathfrak{B}(H)$,

$$\forall X\in\mathfrak{B}(H),\ \left\|SXR^++S^+XR\right\|\geq 2\left\|SS^+XR^+R\right\|, \tag{2}$$

for every selfadjoint operators with closed ranges $S,R\in\mathfrak{B}(H)$,

$$\forall X\in\mathfrak{B}(H),\ \left\|SXR^{-1}+S^{-1}XR\right\|\geq 2\|X\|, \tag{3}$$

for every invertible selfadjoint operators $S,R\in\mathfrak{B}(H)$,

$$\forall X\in\mathfrak{B}(H),\ \left\|S^2X+XR^2\right\|\geq 2\|SXR\|, \tag{4}$$

for every selfadjoint operators $S,R\in\mathfrak{B}(H)$,

$$X\in\mathfrak{B}(H),\ \|A^*AX+XAA^*\|\geq 2\|AXA\|, \tag{1'}$$

for every $A\in\mathfrak{B}(H)$,

$$\forall X\in\mathfrak{B}(H),\ \left\|SXS^++S^+XS\right\|\geq 2\left\|SS^+XS^+S\right\|, \tag{2'}$$

for every selfadjoint operator with closed range $S\in\mathfrak{B}(H)$,

$$\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\|\geq 2\|X\|, \tag{3'}$$

for every invertible selfadjoint operator $S\in\mathfrak{B}(H)$,

$$\forall X\in\mathfrak{B}(H),\ \left\|S^2X+XS^2\right\|\geq 2\|SXS\|, \tag{4'}$$

for every selfadjoint operator $S\in\mathfrak{B}(H)$.

Proof. (1) $\Rightarrow$ (2). Assume (1) holds. Let $S,R\in\mathbb{S}_{cr}(H)$, $X\in\mathfrak{B}(H)$. Since $S=S^*SS^+$ and $R=R^+RR^*$, then from (1) it follows that

$$\begin{aligned}\left\|SXR^++S^+XR\right\| &= \left\|S^*S\left(S^+XR^+\right)+\left(S^+XR^+\right)RR^*\right\|\\ &\geq 2\left\|SS^+XR^+R\right\|.\end{aligned}$$

Hence (2) holds.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). Assume (3) holds. Let $S,R\in\mathbb{S}(H)$, and put $P=|S|$, $Q=|R|$.

Let $\epsilon>0$. From (4), the following inequality holds

$$\forall X\in\mathfrak{B}(H),\ \left\|(P+\epsilon I)^2X+X(Q+\epsilon I)^2\right\|\geq 2\|(P+\epsilon I)X(Q+\epsilon I)\|.$$

Letting $\epsilon \to \infty$, we obtain

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XR^2\right\| \geq 2\left\|SXR\right\|.$$

This proves (4).

(4) $\Rightarrow$ (1). This follows immediately by using the polar decomposition of an operator.

Hence, the equivalences (1) $-$ (4) hold.

From pair of operators to single operator, the equivalences $(1') - (5')$ hold.

(1) $\Rightarrow$ $(1')$. Trivial.

$(1') \Rightarrow (1)$. This follows using Berberian technic as used in *Proposition* 1.

Hence, the eight properties are mutually equivalent.

Prove now that the operator inequality $(3')$ holds.

Step1. Let $S,\ X \in \mathfrak{B}(H)$ such that S and X are selfadjoint, and S invertible.

Then, there exists $\lambda \in \sigma(X)$ such that $|\lambda| = \|X\|$. Since, $\sigma(X) = \sigma(SXS^{-1}) \subset V(SXS^{-1})$, there exists a state f on $\mathfrak{B}(H)$ such that $\lambda = f\left(SXS^{-1}\right) = f\left(S^{-1}XS\right)$. This gives us, $2\|X\| = \left|f\left(SXS^{-1} + S^{-1}XS\right)\right| \leq \left\|SXS^{-1} + S^{-1}XS\right\|$.

Step 2. Let $S,\ X \in \mathfrak{B}(H)$ such that S is selfadjoint invertible.

Let the two following operators on the Hilbert space $H \oplus H$ given by $M = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix}$. So that M and Y are selfadjoint operators in $\mathfrak{B}(H \oplus H)$ and where M is invertible. Applying step 1 for this pair of operators, so we obtain

$$\begin{aligned} \left\|SXS^{-1} + S^{-1}XS\right\| &= \left\|MXM^{-1} + M^{-1}XM\right\|, \\ &\geq 2\|Y\|, \\ &= 2\|X\|. \end{aligned}$$

$\square$

Corollary 4. *The following operator inequalities hold and are equivalent to* $(S - AGMI)$:

$$(5) \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^*XR^+ + S^+XR^*\right\| \geq 2\left\|SS^+XR^+R\right\|,$$

for every operators with closed ranges $S,\ R \in \mathfrak{B}(H)$,

$$(6) \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^*XR^{-1} + S^{-1}XR^*\right\| \geq 2\|X\|,$$

for every invertible operators $S,\ R \in \mathfrak{B}(H)$,

$$(5') \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^*XS^+ + S^+XS^*\right\| \geq 2\left\|SS^+XS^+S\right\|,$$

for every operator with closed range $S \in \mathfrak{B}(H)$,

$$(6') \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^*XS^{-1} + S^{-1}XS^*\right\| \geq 2\|X\|,$$

for every invertible operator $S \in \mathfrak{B}(H)$.

Proof. Assume $(S - AGMI)$ holds. Prove that (5) holds.

Let $S, R \in \mathcal{R}(H)$, and $X \in \mathfrak{B}(H)$. Since, $SS^+S = S$ and $RR^+R = R$, then we have

$$\begin{aligned} \left\| S^*XR^+ + S^+XR^* \right\| &= \left\| S^*S\left(S^+XR^+\right) + \left(S^+XR^+\right)RR^* \right\|, \\ &\geq 2\left\| SS^+XR^+R \right\|, \quad (\text{from } (S-AGMI)). \end{aligned}$$

This proves (5).

It is clear that (5) implies (6), (5′), (6′), and using *Remark* 1, it is clear that (5) (resp. (6), (5′), (6′)) implies (2) (resp. (3), (2′), (3′)). □

Note that the six operator inequalities (2)−(4) and $(2^{'})$−$(4^{'})$ given in *Proposition* 6 are generated by a pair of selfadjoint operators and a single selfadjoint operator, respectively.

We shall interest to describe the class of

(**i**) all invertible operators $S \in \mathfrak{B}(H)$ satisfying the operator inequality $(3^{'})$,

(**ii**) all operators with closed ranges $S \in \mathfrak{B}(H)$ satisfying the operator inequality $(2^{'})$,

(**iii**) all operators $S \in \mathfrak{B}(H)$ satisfying the operator inequality $(4^{'})$.

We shall prove that the class

- (**i**) is the class of all invertible selfadjoint operators in $\mathfrak{B}(H)$ multiplied by nonzero scalars,
- (**ii**) is the class of all selfadjoint operators with closed ranges in $\mathfrak{B}(H)$ multiplied by nonzero scalars,
- (**iii**) is the class of all selfadjoint operators in $\mathfrak{B}(H)$ multiplied by nonzero scalars..

We shall present here all these characterizations and others of the class of all invertible selfadjoint operators multiplied by nonzero scalars, the class of all selfadjoint operators with closed ranges multiplied by nonzero scalars, and the class of all selfadjoint operators multiplied by nonzero scalars,

We need the following lemma.

Lemma 6. [12] *Let* $\lambda,\ \mu \in \mathbb{C}^*$ *such that* $\frac{\lambda}{\mu} + \frac{\mu}{\lambda} \in \mathbb{R}$, *and* $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \geq 2$. *Then there exists* $\theta \in [0, \pi)$ *such that* $\lambda,\ \mu \in D_\theta$.

Proof. Let $\lambda = re^{i\alpha},\ \mu = le^{i\beta}$ be the polar decomposition of $\lambda,\ \mu$. Then we have

$$\frac{\lambda}{\mu} + \frac{\mu}{\lambda} = \left(\frac{r}{l} + \frac{l}{r}\right)\cos(\alpha - \beta) + i\left(\frac{r}{l} - \frac{l}{r}\right)\sin(\alpha - \beta).$$

Thus, $r = l$ or $\alpha - \beta \equiv 0 \pmod{.\pi}$. The case $r = l$ also gives $\alpha - \beta \equiv 0 \pmod{.\pi}$. Hence, the prof is completed. □

Proposition 7. [12] *Let* S *be an invertible operator in* $\mathfrak{B}(H)$. *Then the following properties are equivalent:*

(*i*) S *is a selfadjoint operator multiplied by a nonzero scalar,*

(*ii*) $\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} + S^{-1}XS \right\| = \left\| S^*XS^{-1} + S^{-1}XS^* \right\|,$

(*iii*) $\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} + S^{-1}XS \right\| \geq \left\| S^*XS^{-1} + S^{-1}XS^* \right\|,$

(*iv*) $\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} + S^{-1}XS \right\| \geq 2\left\| X \right\|.$

Proof. The two implications $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (iii)$ are trivial.

The implication $(iii) \Longrightarrow (iv)$ follows from *Corollary* 4.6′.

$(iv) \Rightarrow (i)$. Assume (iv) holds.

So, we have

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1}\right\| + \left\|S^{-1}XS\right\| \geq 2\left\|X\right\|.$$

Using *Proposition* 2, then S is normal. Using the spectral measure of S, there exists a sequence (S_n) of invertible normal operators with finite spectrum such that:

(a) $S_n \to S$ uniformly,

(b) for all $\lambda \in \sigma(S)$, there exists a sequence (λ_n) such that $\lambda_n \in \sigma(S_n)$, for all n and $\lambda_n \to \lambda$.

Let $\lambda,\ \mu \in \sigma(S)$ and let $\epsilon > 0$. Using (ii), (a), and (b), there exists an integer $N \geq 1$ such that

$$(1) \qquad \forall n > N,\ \forall X \in \mathfrak{B}(H),\ \left\|S_n X S_n^{-1} + S_n^{-1} X S_n\right\| \geq (2-\epsilon)\left\|X\right\|,$$

and there exist two sequences $(\lambda_n),\ (\mu_n)$ such that $\lambda_n,\ \mu_n \in \sigma(S_n)$, for all n, and $\lambda_n \to \lambda,\ \mu_n \to \mu$.

Let $n > N$ and since S_n is normal, with finite spectrum, there exist p orthogonal projections $E_1, ..., E_p$ in $\mathfrak{B}(H)$ such that $E_i E_j = 0$, if $i \neq j$, $\sum_{i=1}^{p} E_i = I$, $S_n = \sum_{i=}^{p} \alpha_i E_i$, where $\sigma(S_n) = \left\{\alpha_1, ..., \alpha_p\right\}$, $\alpha_1 = \lambda_n$, $\alpha_2 = \mu_n$.

Then by (1), and if we put $A = \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 2 \end{bmatrix}$, where $\gamma_n = \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}$, we obtain

$$(2) \qquad \forall X \in \mathfrak{B}(\mathbb{C}^2),\ \left\|A \circ X\right\| \geq (2-\epsilon)\left\|X\right\|.$$

If we put $\delta_n = \frac{1}{\gamma_n}$, and $B = \begin{bmatrix} \frac{1}{2} & \delta_n \\ \delta_n & \frac{1}{2} \end{bmatrix}$, then from the last inequality, we also have

$$(3) \qquad \forall X \in \mathfrak{B}(\mathbb{C}^2),\ \left\|B \circ X\right\| \leq \frac{\left\|X\right\|}{(2-\epsilon)}.$$

From (2), we deduce $\left|\frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}\right| \geq 2 - \epsilon$. Hence, $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \geq 2$. Put, $\beta_n = \operatorname{Im}\gamma_n$, $\gamma = \lim \gamma_n$, $\beta = \lim \beta_n$.

On the other hand, if in (3), we put $X = \begin{bmatrix} 1 & ia \\ ia & 1 \end{bmatrix}$, for $a > 0$, we obtain

$$\frac{1}{4} + a^2\left|\gamma_n\right|^2 + a\left|\beta_n\right| \leq \frac{1+a^2}{(2-\epsilon)^2}.$$

Hence,

$$\frac{1}{4} + a^2\left|\gamma\right|^2 + a\left|\beta\right| \leq \frac{1+a^2}{(2-\epsilon)^2}.$$

Thus, $a\left|\gamma\right|^2 + \left|\beta\right| \leq \frac{a}{4}$, for every $a > 0$. This gives us, $\operatorname{Im}\left(\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right) = \beta = 0$. So, from the above lemma, λ and μ belongs to a straight line through the origin. Then there exists $\theta \in [0, \pi)$ such that $\sigma(S) \subset D_\theta$. Therefore, $M = e^{-i\theta}S$ is selfadjoint, and $S = e^{i\theta}M$. This proves (i). □

In the next proposition, and from the last proposition concerning the invertible case, we conclude for the characterization of the class of all selfadjoint operators in $\mathfrak{B}(H)$ multiplied by nonzero scalars,

Proposition 8. [20] *Let $S \in \mathfrak{B}(H)$. The two following properties are equivalent*

(i) *S is a selfadjoint operator multiplied by a nonzero scalar,*

(ii) $\forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| \geq 2\left\|SXS\right\|.$

Proof. We may assume without loss of generality that $\|S\| = 1$.

$(i) \Rightarrow (ii)$. This implication follows immediately from $(S - AGMI)$.

$(ii) \Rightarrow (i)$. Assume (ii) holds.

Then, we have

$$\forall X \in \mathfrak{B}(H),\ \left\|S^2X\right\| + \left\|XS^2\right\| \geq 2\left\|SXS\right\|.$$

Hence, from *Proposition* 4, S is normal. So, we prove (i) in two cases.

Case 1. $S \in \mathfrak{D}(H)$..

Then, S is invertible. so from (ii), we obtain

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| \geq 2\left\|X\right\|.$$

Using the last proposition, we deduce (i).

Case 2. General situation.

Applying triangular inequality in (ii), we deduce that $\left\|S^2\right\| = \left\|S\right\|^2 = 1$.

Define the real function F on the complete metric space $(\mathfrak{B}(H))_1$ by

$$\forall X \in (\mathfrak{B}(H))_1,\ F(X) = \left\|S^2X + XS^2\right\| - 2\left\|SXS\right\|,$$

and for $n \geq 1$, define the real function F_n on $(\mathfrak{B}(H))_1$ by

$$\forall X \in \mathfrak{B}(H),\ F_n(X) = \left\|S_n^2X + XS_n^2\right\| - 2\left\|S_nXS_n\right\|.$$

Put $D = \{X \in (\mathfrak{B}(H))_1 : F(X) > 0\}$. Then there are two cases, $D = \varnothing$, $D \neq \varnothing$.

1. $D = \varnothing$. So, it follows that

$$(*) \qquad \forall X \in \mathfrak{B}(H),\ \left\|S^2X + XS^2\right\| = 2\left\|SXS\right\|.$$

From this equality, we have

$$\forall x, y \in H,\ \left\|S^2x \otimes y + x \otimes S^{*2}y\right\| = 2\left\|Sx\right\|\left\|S^*y\right\|.$$

Using this last equality and since $S^2 \neq 0$, we deduce that $\ker S^* = \{0\}$. Hence, S is with dense range. Using again this last equality, we obtain the following inequality,

$$\forall x, y \in (H)_1,\ \left\|S^2x\right\| + 2\left\|Sx\right\|\left\|S^*y\right\| \geq \left\|S^{*2}y\right\|.$$

By taking the supremum over $y \in (H)_1$, we obtain that $\|Sx\| \geq \frac{1}{3}\|x\|$, for every $x \in H$. Thus, S is bounded below with dense range. Hence, S is invertible. So, from $(*)$, it follows that

$$\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| = 2\left\|X\right\|.$$

Then from the last proposition, (i) holds.

2. $D \neq \varnothing$. From the fact that F is a positive continuous map on $(\mathfrak{B}(H))_1$, it follows that

$$\overline{D} = \overline{F^{-1}((0,\infty))} = F^{-1}([0,\infty)) = \{X \in (\mathfrak{B}(H))_1 : F(X) \geq 0\} = (\mathfrak{B}(H))_1.$$

Let $X \in D$, and $\epsilon > 0$. Since $S_n \to S$ uniformly, then there exists an integer $N \geq 1$ (depends only in ϵ) such that

$$\forall n \geq N,\ \forall Y \in (\mathfrak{B}(H))_1,\ |F(Y) - F_n(Y)| \leq \epsilon.$$

Using the same argument as used in *Proposition* 4, it follows that

$$\forall X \in D,\ \forall n \geq N,\quad F_n(X) \geq 0.$$

Since each F_n is a continuous map on $(\mathfrak{B}(H))_1$ and D is dense in $(\mathfrak{B}(H))_1$, then

$$\forall X \in (\mathfrak{B}(H))_1,\ \forall n \geq N,\ F_n(X) \geq 0.$$

So, it follows that

$$\forall X \in \mathfrak{B}(H),\ \forall n \geq N,\ \left\|S_n^2 X + X S_n^2\right\| \geq 2\left\|S_n X S_n\right\|.$$

Since for each $n \geq 1$, $S_n \in \mathfrak{D}(H)$, using the case 1, we obtain that S_n is a selfadjoint with closed range multiplied by nonzero scalar, for every $n \geq N$. Since $S_n \to S$ uniformly, and the class of all selfadjoint operators in $\mathfrak{B}(H)$ is closed in $\mathfrak{B}(H)$, this proves (i). □

Corollary 5. [18] *Let S be an operator with closed range in $\mathfrak{B}(H)$. Then the following properties are equivalent:*
(i) *S is a selfadjoint operator multiplied by a non zero scalar,*
(ii) $\forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| = \|S^*XS^+ + S^+XS^*\|,$
(iii) $\forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| \geq \|S^*XS^+ + S^+XS^*\|,$
(iv) $\forall X \in \mathfrak{B}(H),\ \|SXS^+ + S^+XS\| \geq 2\|SS^+XS^+S\|.$

Proof. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are trivial.
The implication $(iii) \Rightarrow (iv)$ follows immediately from *Corollary* 4.5′.
$(iv) \Rightarrow (i)$. Assume (iv) holds. Applying the triangular inequality in (iv), we obtain from *Corollary* 3, that S is normal (with a closed range). So that S is an EP operator satisfying (iv). Then, $S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R(S) \\ \ker S^* \end{bmatrix}$, where S_1 is invertible on $R(S)$. Hence, we obtain the following inequality

$$\forall X \in \mathfrak{B}(R(S)),\ \left\|S_1 X S_1^{-1} + S_1^{-1} X S_1\right\| \geq 2\|X\|.$$

Using *Proposition* 7 with the Hilbert space $R(S)$, we obtain that S_1 is a selfadjoint operator in $\mathfrak{B}(R(S))$ multiplied by a nonzero scalar. This proves (i). □

4. On the injective norm of the two operators $X \longmapsto SXS^{-1} + S^{-1}XS$ and $X \longmapsto S^*XS^{-1} + S^{-1}XS^*$, unitary operators, and characterizations

Let E be a (real or complex) normed space, and let $\mathfrak{B} = \mathfrak{B}(E)$ denote the normed algebra of all bounded linear operators acting on E.

For $A = (A_1, ..., A_n)$, $B = (B_1, ..., B_n)$ be two n-tuples of elements in $\mathfrak{B}$, we define the elementary operator (induced by A, B) $R_{A,B}$ on $\mathfrak{B}$ by:

$$\forall X \in \mathfrak{B},\ R_{A,B}(X) = \sum_{i=1}^{n} A_i X B_i.$$

We denote by $\mathcal{R}(\mathfrak{B})$, the vector space of all elementary operators on $\mathfrak{B}$. We define the map $d(.) : \mathcal{R}(\mathfrak{B}) \longrightarrow \mathbb{R}$ by:

$$\forall R \in \mathcal{R}(\mathfrak{B}),\ d(R) = \sup_{\|X\|=1=rank X} \|R(X)\|.$$

We consider the tensor product space

$$\mathfrak{B} \otimes \mathfrak{B} = \left\{\sum_{i=1}^{n} A_i \otimes B_i : n \geq 1,\ A_i,\ B_i \in \mathfrak{B},\ i = 1, ..., n\right\}.$$

We denote by $\|.\|_\lambda$ be the injective norm on $\mathfrak{B} \otimes \mathfrak{B}$ given by:

$$\left\|\sum_{i=1}^{n} A_i \otimes B_i\right\|_\lambda = \sup_{f,g \in (\mathfrak{B}')_1} \left|\sum_{i=1}^{n} f(A_i) g(B_i)\right|.$$

For S be an invertible operator in $\mathfrak{B}(H)$, we define the two particular elementary operators $\varphi_S,\ \psi_S$ on $\mathfrak{B}(H)$ by

$$\begin{cases} \forall X \in \mathfrak{B}(H),\ \varphi_S(X) = SXS^{-1} + S^{-1}XS, \\ \forall X \in \mathfrak{B}(H),\ \psi_S(X) = S^*XS^{-1} + S^{-1}XS^*. \end{cases}$$

Notation 1. *For $A = (A_1, ..., A_n)$ be an n-tuple of commuting operators in $\mathfrak{B}(H)$, we denote by:*

1. Γ_A, the set of all multiplicative functional acting on the maximal commutative Banach algebra that contains the operators $A_1, ..., A_n$,

2. $\sigma(A) = \{(\varphi(A_1), ..., \varphi(A_n)) : \varphi \in \Gamma_A\}$, the joint spectrum of A.

In this section, we shall prove that $\|\sum_{i=1}^n A_i \otimes B_i\|_\lambda = d\left(R_{A,B}\right)$, and where $A = (A_1, ..., A_n),\ B = (B_1, ..., B_n) \in \mathfrak{B}^n$, and then the map $d(.)$ is a norm on the vector space $\mathcal{R}(\mathfrak{B})$, and the two normed spaces $(\mathcal{R}(\mathfrak{B}),\ d(.))$ and $(\mathfrak{B} \otimes \mathfrak{B},\ \|.\|_\lambda)$ are isometrically isomorph.

Concerning the characterizations given in section 4 for the invertible case, they are generated by φ_S, and by φ_S, ψ_S (where S is an invertible operator in $\mathfrak{B}(H)$). It was proved that the class of all invertible selfadjoint operators in $\mathfrak{B}(H)$ multiplied by nonzero scalars is the class of all invertible operators $S \in \mathfrak{B}(H)$ satisfying each of the three following inequalities:

$$\begin{array}{rrcl} (\mathbf{i})\ \forall X \in \mathfrak{B}(H), & \left\|\varphi_S(X)\right\| & \geq & 2\left\|X\right\|, \\ (\mathbf{ii})\ \forall X \in \mathfrak{B}(H), & \left\|\varphi_S(X)\right\| & = & \left\|\psi_S(X)\right\|, \\ (\mathbf{iii})\ \forall X \in \mathfrak{B}(H), & \left\|\varphi_S(X)\right\| & \geq & \left\|\psi_S(X)\right\|. \end{array}$$

In this section, we consider an invertible operator S in $\mathfrak{B}(H)$.

It is clear that:

$$\inf_{\|X\|=1} \left\|\varphi_S(X)\right\| \leq 2 \leq \sup_{\|X\|=1} \left\|\varphi_S(X)\right\|.$$

Then, the above infimum gets its maximal value 2 if and only if S satisfies the condition $(\mathbf{i})$. So, $\inf\limits_{\|X\|=1} \left\|\varphi_S(X)\right\| = 2$ if and only if S is an invertible selfadjoint operator multiplied by a nonzero scalar.

In this section, we aboard the problem when the above supremum gets its minimal value 2. We shall show that:

$(\mathbf{iv})$ this supremum (that is $\left\|\varphi_S\right\|$) gets it minimal value 2 if and only if S is a unitary operator multiplied by a nonzero scalar,

$(\mathbf{v})$ $\sup\limits_{\|X\|=1=rankX} \left\|\varphi_S(X)\right\| = \left\|S \otimes S^{-1} + S^{-1} \otimes S\right\|_\lambda \geq 2,$

$(\mathbf{vi})$ this last supremum (that is the injective norm of φ_S) gets its minimal value 2 if and only if S is normal and $\sup\limits_{\lambda,\mu\in\sigma(S)} \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = 2.$

From above, the following conditions are equivalent:

$(\cdot)$ $\inf\limits_{\|X\|=1} \left\|\varphi_S(X)\right\| = 2 = \sup\limits_{\|X\|=1} \left\|\varphi_S(X)\right\|,$

$(\cdot)$ $\forall X \in \mathfrak{B}(H),\ \left\|SXS^{-1} + S^{-1}XS\right\| = 2\left\|X\right\|,$

$(\cdot)$ S is a unitary reflection operator multiplied by a nonzero scalar.

Proposition 9. [14]. *For* $A = (A_1, ..., A_n)$, $B = (B_1, ..., B_n)$ *be two* n*-tuples of elements in* $\mathfrak{B}$*, the following equalities hold*

$$\begin{aligned} d\left(R_{A,B}\right) &= \sup_{f,g\in(\mathfrak{B}')_1} \left|\sum_{i=1}^{n} f(A_i)g\left(B_i\right)\right|, \\ &= \sup_{f\in(\mathfrak{B}')_1} \left\|\sum_{i=1}^{n} f(B_i)A_i\right\|, \\ &= \sup_{f\in(\mathfrak{B}')_1} \left\|\sum_{i=1}^{n} f(A_i)B_i\right\|. \end{aligned}$$

Proof. We denote by k_1, k_2 and k_3 be the supremum cited in the theorem in the same order. Let $x, y \in (E)_1$, $h \in \left(E'\right)_1$, and let $f, g \in (\mathfrak{B}')_1$. So, we have

$$\begin{aligned} d\left(R_{A,B}\right) &\geq \left\|\sum_{i=1}^{n} A_i\left(x\otimes h\right)B_i y\right\|, \\ &= \left\|\left(\sum_{i=1}^{n} h\left(B_i y\right)A_i\right)x\right\|. \end{aligned}$$

By taking the supremum over $x \in (E)_1$, we have, $d\left(R_{A,B}\right) \geq \|\sum_{i=1}^{n} h\left(B_i y\right)A_i\|$. Thus,

$$\begin{aligned} d\left(R_{A,B}\right) &\geq \left|\sum_{i=1}^{n} f\left(A_i\right)h\left(B_i y\right)\right|, \\ &= \left|h\left(\sum_{i=1}^{n} f\left(A_i\right)B_i y\right)\right|. \end{aligned}$$

By taking the supremum over $h \in \left(E'\right)_1$ and over $y \in (E)_1$, we obtain

$$d\left(R_{A,B}\right) \geq \left\|\sum_{i=1}^{n} f\left(A_i\right)B_i\right\|.$$

Then,

$$d\left(R_{A,B}\right) \geq \left|\sum_{i=1}^{n} f\left(A_i\right)g\left(B_i\right)\right|.$$

So, we have $d\left(R_{A,B}\right) \geq k_1$. Since, $k_1 \geq |g\left(\sum_{i=1}^{n} f\left(B_i\right)A_i\right)|$, then $k_1 \geq \|\sum_{i=1}^{n} f\left(B_i\right)A_i\|$. This gives us that $k_1, \geq k_2$. It is clear that $k_2 \geq |g\left(\sum_{i=1}^{n} f\left(A_i\right)B_i\right)|$, then $k_2 \geq k_3$. Since, $k_3 \geq |\sum_{i=1}^{n} f(A_i)h\left(B_i y\right)| = |f\left(\sum_{i=1}^{n} h\left(B_i y\right)A_i\right)|$, so we have,

$$\begin{aligned} k_3 &\geq \left\|\sum_{i=1}^{n} h\left(B_i y\right)A_i\right\|, \\ &\geq \left|\sum_{i=1}^{n} h\left(B_i y\right)A_i x\right|, \\ &= \left\|\left(\sum_{i=1}^{n} A_i\left(x\otimes h\right)B_i\right)y\right\|. \end{aligned}$$

Thus, $k_3 \geq \|\sum_{i=1}^{n} A_i (x \otimes h) B_i\|$. Therefore, $k_3 \geq d(R_{A,B})$. This complete the proof. ☐

Proposition 10. *The map* $d(.) : \mathcal{R}(\mathfrak{B}) \to \mathbb{R},\ R \mapsto d(R)$ *is a norm on* $\mathcal{R}(\mathfrak{B})$.

Proof. It is clear that

$$d(R) \geq 0,\ d(\lambda R) = |\lambda|\, d(R),\ d(R+S) \leq d(R) + d(S),$$

for every scalar λ, and for every $R, S \in \mathcal{R}(\mathfrak{B})$. So, it remains to prove that if $d(R) = 0$, then $R = 0$, for every $R \in \mathcal{R}(\mathfrak{B})$.

Now, let $A = (A_1, ..., A_n),\ B = (B_1, ..., B_n)$ be two n-tuples of elements in $\mathfrak{B}$ such that $d(R_{A,B}) = 0$.

We may assume that $B_1, ..., B_m$ (where $m \leq n$) form a maximal linearly independent subset of $B_1, ..., B_n$. There exist m operators $C_1, ..., C_m \in sp\{A_1, ..., A_n\}$ such that $R_{A,B} = R_{C,D}$, where $C = (C_1, ..., C_m),\ D = (B_1, ..., B_m)$. So, using the above proposition, we obtain $\sum_{i=1}^{m} f(C_i)B_i = 0$, for every $f \in \mathfrak{B}'$. Since $B_1, ..., B_n$ are linearly independent, then $f(C_i) = 0$, for $i = 1, ..., m$, and for every $f \in \mathfrak{B}'$. This proves, $C_i = 0$, for $i = 1, ..., m$. Hence, $R_{A,B} = R_{C,D} = 0$. ☐

Corollary 6. *The two normed spaces* $(\mathcal{R}(\mathfrak{B}), d(.))$ *and* $(\mathfrak{B} \otimes \mathfrak{B}, \|.\|_\lambda)$ *are isometrically isomorph.*

Proof. Let the map

$$\begin{array}{rccc} \Gamma : & (\mathfrak{B} \otimes \mathfrak{B}, \|.\|_\lambda) & \to & (\mathcal{R}(\mathfrak{B}), d(.)), \\ & \sum_{i=1}^{n} A_i \otimes B_i & \longmapsto & R_{A,B}, \end{array}$$

where $A = (A_1, ..., A_n),\ B = (B_1, ..., B_n) \in \mathfrak{B}^n$.

From *Proposition* 9, the map Γ is well-defined and injective. It is clear that Γ is linear and surjective. Using again *Proposition* 9, we deduce that Γ is an isometry. ☐

Notation 2. *According to the above identification, and for* $R \in \mathcal{R}(\mathfrak{B})$, *we use the notation* $\|R\|_\lambda$ *instead of* $d(R)$, *and we say it is the injective norm of* R

Corollary 7. [15]. *Let* $A = (A_1, ..., A_n),\ B = (B_1, ..., B_n)$ *be two n-tuples of commuting operators in* $\mathfrak{B}(H)$. *Then* $\|R_{A,B}\|_\lambda \geq |\sigma(A) \circ \sigma(B)|$; *and this inequality becomes an equality, if all* A_i *and* B_i *are normal operators.*

Proof. Let (φ, ψ) be an arbitrary pair in $\Gamma_A \times \Gamma_B$. Using the Hahn-Banach theorem, we may extend φ and ψ to unit functional f and g on $\mathfrak{B}(H)$, respectively. So from *Proposition* 9, it follows that $\|R_{A,B}\|_\lambda \geq |\sum_{i=1}^{n} f(A_i)g(B_i)| = |\sum_{i=1}^{n} \varphi(A_i)\psi(B_i)|$. Therefore $\|R_{A,B}\|_\lambda \geq |\sigma(A) \circ \sigma(B)|$.

Now, suppose all A_i and B_i are normal operators. It suffice to prove $\|R_{A,B}\|_\lambda \leq |\sigma(A) \circ \sigma(B)|$. Let $f,\ g$ be two arbitrary unit functional on $\mathfrak{B}(H)$, and let (φ, ψ) be an arbitrary pair in $\Gamma_A \times \Gamma_B$. Since $|\sigma(A) \circ \sigma(B)| \geq |\psi(\sum_{i=1}^{n} \varphi(A_i)B_i)|$, and $\sum_{i=1}^{n} \varphi(A_i)B_i$ is normal (from Putnam-Fuglede), then $|\sigma(A) \circ \sigma(B)| \geq \|\sum_{i=1}^{n} \varphi(A_i)B_i\|$. So that $|\sigma(A) \circ \sigma(B)| \geq \|\sum_{i=1}^{n} \varphi(A_i)g(B_i)\| = \|\varphi(\sum_{i=1}^{n} g(B_i)A_i)\|$. Using the same argument as used with B_i, we deduce that $|\sigma(A) \circ \sigma(B)| \geq \|\sum_{i=1}^{n} g(B_i)A_i\|$. From *Proposition* 9, it follows that $|\sigma(A) \circ \sigma(B)| \geq \|R_{A,B}\|_\lambda$. ☐

Lemma 7. *We have* $\|\psi_S\|_\lambda = \|\varphi_P\|_\lambda$, *where* $P = |S|$.

Proof. Let $S = UP$, be the polar decomposition of S. From the fact that

$$\{X \in \mathfrak{B}(H) : \|X\| = 1 = rankX\} = \{U^* X : X \in \mathfrak{B}(H),\ \|X\| = 1 = rankX\},$$

it follows that

$$\begin{aligned} \left\|\psi_{S}\right\|_{\lambda} &= \sup_{\|X\|=1=rankX} \left\|S^{*} X S^{-1} + S^{-1} X S^{*}\right\| \\ &= \sup_{\|X\|=1=rankX} \left\|P U^{*} X P^{-1} U^{*} + P^{-1} U^{*} X P U^{*}\right\| \\ &= \sup_{\|X\|=1=rankX} \left\|P \left(U^{*} X\right) P^{-1} + P^{-1} \left(U^{*} X\right) P\right\| \\ &= \left\|\varphi_{P}\right\|_{\lambda}. \end{aligned}$$

□

Proposition 11. [15]. *The following properties hold*

$$(i).\ \left\|\varphi_{S}\right\|_{\lambda} \geq \sup_{\lambda,\mu \in \sigma(S)} \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right|,$$

(ii). *if S is normal, the above inequality becomes an equality,*
(iii). *if S is normal,, the following holds*

$$\left\|\psi_{S}\right\|_{\lambda} = \sup_{\lambda,\mu \in \sigma(S)} \left(\left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|\right).$$

Proof. (i) and (ii) follows immediately from $Corollary$ 7.
(iii). Assume S normal, and let UP be its polar decomposition.
Since S is invertible and normal , then $\sigma(P) = \{|\lambda| : \lambda \in \sigma(S)\}$. So from the above lemma and (ii), we obtain $\left\|\psi_{S}\right\|_{\lambda} = \left\|\varphi_{P}\right\|_{\lambda} = \sup_{\lambda,\mu \in \sigma(S)} \left(\left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|\right)$. □

Corollary 8. (i). *We have* $\left\|\varphi_{S}\right\|_{\lambda} \geq 2$,
(ii). *if S is normal, then the injective norm of φ_{S} gets its minimal value 2, if and only if the following spectral condition holds*

$$\forall \in \lambda, \mu \in \sigma(S),\ \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \leq 2.$$

(iii).*if* $\left\|\varphi_{S}\right\|_{\lambda} = 2$, *then the interior of $\sigma(S)$ is empty.*

Proof. (i) and (ii) follows immediately from the above proposition.
(iii) Assume $\left\|\varphi_{S}\right\|_{\lambda} = 2$. Thus, $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \leq 2$, for every $\lambda, \mu \in \sigma(S)$. Hence, every straight line passing through the origin intercept $\sigma(S)$ in at most two points. This proves that the interior of $\sigma(S)$ is empty. □

Proposition 12. [15]. *Let P be a positive and invertible operator in $\mathfrak{B}(H)$. Then we have,*

$$\left\|\varphi_{P}\right\|_{\lambda} = \|P\| \left\|P^{-1}\right\| + \frac{1}{\|P\| \left\|P^{-1}\right\|}.$$

Proof. Let the operator M_P defined on $\mathfrak{B}(H)$ by

$$\forall X \in \mathfrak{B}(H),\ M_P(X) = PXP^{-1}.$$

Since $\sigma(M_P) = \sigma(P)\sigma(P^{-1})$, $\sigma(\varphi_P) = \left\{ f(M_P) + \frac{1}{f(M_P)} : f \in \Gamma \right\}$ (where Γ is the set of all multiplicative functional on the maximal commutative Banach algebra in $\mathfrak{B}(\mathfrak{B}(H))$ that contains M_P), and from the above proposition, it is easy to see that, $\left\|\varphi_P\right\|_\lambda = \sup_{\lambda,\mu\in\sigma(P)} \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = \sup_{z\in\sigma(M_P)} \left|z + \frac{1}{z}\right|$. So, using the fact that $\min\sigma(P) = \frac{1}{\|P^{-1}\|}$ and $\max\sigma(P) = \|P\|$, then $\min\sigma(M_P) = \frac{1}{\|P\|\|P^{-1}\|} = p$, and $\max\sigma(M_P) = \|P\|\left\|P^{-1}\right\| = \frac{1}{p}$. On the other hand, since $\max_{p\le t\le \frac{1}{p}} \left(t + \frac{1}{t}\right) = p + \frac{1}{p}$, this maximum is attainable at p and $\frac{1}{p}$. Thus, the result follows immediately from the fact that $p \in \sigma(M_P)$. □

Proposition 13. [15] *The following properties hold:*

(*i*) $\left\|\psi_S\right\|_\lambda = \|S\|\left\|S^{-1}\right\| + \frac{1}{\|S\|\|S^{-1}\|}$,

(*ii*) *If S is selfadjoint, then* $\left\|\varphi_S\right\|_\lambda = \|S\|\left\|S^{-1}\right\| + \frac{1}{\|S\|\|S^{-1}\|}$,

(*iii*) *if S normal, then* $\left\|\varphi_S\right\|_\lambda \le \|S\|\left\|S^{-1}\right\| + \frac{1}{\|S\|\|S^{-1}\|}$.

Proof. Let $S = UP$ be the polar decomposition of S.

(*i*). From *Lemma* 7 and *Proposition* 12 and since, $\|S\| = \|P\|$, $\left\|S^{-1}\right\| = \left\|P^{-1}\right\|$, it follows that:

$$\left\|\psi_S\right\|_\lambda = \left\|\varphi_P\right\|_\lambda = \|P\|\left\|P^{-1}\right\| + \frac{1}{\|P\|\left\|P^{-1}\right\|} = \|S\|\left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}.$$

(*ii*). This implication follows immediately from (*i*).

(*iii*). Assume S normal. Then, using *Proposition* 11, and (*i*), we obtain

$$\begin{aligned}\left\|\varphi_S\right\|_\lambda &= \sup_{\lambda,\mu\in\sigma(S)} \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| \\ &\le \sup_{\lambda,\mu\in\sigma(S)} \left(\left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|\right) \\ &= \left\|\psi_S\right\|_\lambda \\ &= \|S\|\left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}.\end{aligned}$$

□

Remark 8. *In the condition* (*iii*) *of this last proposition, the inequality may be strict. Indeed, in dimension two, we choose the invertible normal operator* $S = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1+i}{2} \end{bmatrix}$. *By a simple computation, we find that* $2 = \left\|\varphi_S\right\|_\lambda < \|S\|\left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|} = \frac{3\sqrt{2}}{2}$.

Notation 3. *We denote by*

$$\mathcal{E}(H) = \left\{ T \in \mathfrak{B}(H) : T \text{ normal and invertible, } \left\|\varphi_T\right\|_\lambda = \|T\|\left\|T^{-1}\right\| + \frac{1}{\|T\|\left\|T^{-1}\right\|} \right\}.$$

From above, $\mathcal{E}(H)$ contains every invertible selfadjoint (resp. every unitary) operators in $\mathfrak{B}(H)$, but $\mathcal{E}(H)$ does not contain every invertible normal operators in $\mathfrak{B}(H)$ (see the example in the above remark). In the next proposition, we give a characterization of this class $\mathcal{E}(H)$, where we use the following notations:

- $\sigma_1(S) = \left\{ \lambda \in \sigma(S) : |\lambda| = \min_{\mu \in \sigma(S)} |\mu| \right\}$,
- $\sigma_2(S) = \{\lambda \in \sigma(S) : |\lambda| = r(S)\}$,
- D_θ (where $\theta \in [0, \pi)$) is the straight line through the origin with slope $\tan\theta$.

Proposition 14. [15] *The two following properties are equivalent:*
(*i*) $S \in \mathcal{E}(H)$,
(*ii*) *S is normal, and there exists $\theta \in [0, \pi[$ such that*

$$D_\theta \cap \sigma_1(S) \neq \emptyset,\ D_\theta \cap \sigma_2(S) \neq \emptyset.$$

Proof. $(i) \Rightarrow (ii)$. Assume (i) holds. Using *Proposition* 11 and from the compactness of $\sigma(S)$, we may choose $\lambda, \mu \in \sigma(S)$ such that

$$\|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\| \left\|S^{-1}\right\|} = \left\|\varphi_S\right\|_\lambda = \left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right|.$$

Hence,

$$\begin{aligned} \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\| \left\|S^{-1}\right\|} &\leq \left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|, \\ &\leq \left\|\psi_S\right\|_\lambda, \\ &= \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\| \left\|S^{-1}\right\|}. \end{aligned}$$

Thus, $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = \left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right| = \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}$. Put $p = \frac{1}{\|S\|\left\|S^{-1}\right\|}$. Since S is normal, then $\min_{\lambda,\mu\in\sigma(S)} \left|\frac{\lambda}{\mu}\right| = p$, and $\max_{\lambda,\mu\in\sigma(S)} \left|\frac{\lambda}{\mu}\right| = \frac{1}{p}$. The positive function $f(t) = t + \frac{1}{t},\ p \leq t \leq \frac{1}{p}$, is bounded and attain its maximum $p + \frac{1}{p} = \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}$ only at $t = p$ and in $t = \frac{1}{p}$. So, we may choose λ in $\sigma_1(S)$ and μ in $\sigma_2(S)$. Since, $\left|\frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right| = \left|\frac{\lambda}{\mu}\right| + \left|\frac{\mu}{\lambda}\right|$, then, λ and μ must be belong to a straight line passing through the origin. This proves (ii).

$(ii) \Rightarrow (i)$. Assume (ii) holds. Let $\alpha \in D_\theta \cap \sigma_1(S)$ and $\beta \in D_\theta \cap \sigma_2(S)$. Since, S is normal, then $\alpha = \frac{e^{i\theta}}{\left\|S^{-1}\right\|}$ and $\beta = e^{i(\theta + k\pi)} \|S\|$, for some $k \in \{0, 1\}$. Thus, $\left\|\varphi_S\right\|_\lambda \geq \left|\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right| = \|S\| \left\|S^{-1}\right\| + \frac{1}{\|S\|\left\|S^{-1}\right\|}$. Then, using *Proposition* 13.(iii), (i) holds. □

In the next proposition, we shall give two necessary and sufficient conditions for which $\left\|\varphi_S\right\|_\lambda$ gets its minimal value 2.

We need the two following lemmas:

Lemma 8. [21]. *If $|\langle Sx, x\rangle| \leq 1$ and $\left|\left\langle S^{-1}x, x\right\rangle\right| \leq 1$, for every unit vector x in H, then S is unitary.*

Lemma 9. *The operator S is normal if and only if $S^* S^{-1}$ is unitary*

Proof. The proof is trivial. □

Proposition 15. [16] *The following properties are equivalent*

(*i*) $\left\|\varphi_{S}\right\|_{\lambda}$ *gets its minimal value* 2,

(*ii*) $\forall X\in\mathcal{F}_{1}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\|\leq 2\left\|X\right\|$,

(*iii*) S *is normal and* $\sup\limits_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right|=2.$

Proof. (*i*) ⇔ (*ii*). This equivalence follows immediately from *Proposition* 9 and *Corollary* 8.(*i*).

(*i*) ⇒ (*iii*). Assume (*i*) holds.

From *Proposition* 11.(*i*), we deduce that $\sup\limits_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right|=2.$

So, it remains to prove that S is normal. By using the same argument as used in [13, *Lemma* 1], we deduce the following inequality

$$\forall x,y\in(H)_{1},\ \left\|\varphi_{S}\right\|_{\lambda}\geq 2\left|\langle Sx,y\rangle\left\langle S^{-1}x,y\right\rangle\right|.$$

Hence, the inequality $\left|\langle Sx,y\rangle\left\langle S^{-1}x,y\right\rangle\right|\leq\|x\|\,\|y\|$ holds for every $x,\ y$ in H. So we obtain $\left|\left\langle S^{*}S^{-1}x,x\right\rangle\right|\leq 1$ and $\left|\left\langle\left(S^{*}S^{-1}\right)^{-1}x,x\right\rangle\right|\leq 1$, for every $x,\ y$ in $(H)_{1}$. Then, using the two above lemmas, we deduce that S is normal. Thus, (*iii*) holds.

(*iii*) ⇒ (*i*). This follows immediately from *Proposition* 11.(*ii*). □

Remark 9. *The class of all operators S for which $\left\|\varphi_{S}\right\|_{\lambda}$ is minimal contains strictly the class of all unitary operators, and contained strictly in the class of all invertible normal operators. Indeed, it is easy to see that $\left\|\varphi_{S}\right\|_{\lambda}=2$, if S is unitary, and for an operator $I_{1}\oplus\left(\frac{1}{2}iI_{2}\right)$ with respect to a some orthogonal direct sum $H=H_{1}\oplus H_{2}$ (where I_{i} is the identity on H_{i}, for $i=1,2$) belongs to this class, but it is not unitary; for the second inclusion is trivial.*

In the above proposition, we have given two characterizations for which the injective norm of φ_{S} gets its minimal value 2. In the next proposition, we shall present some characterizations for which the norm of φ_{S} gets its minimal value 2.

Proposition 16. [16] *The following properties are equivalent*

(*i*) $\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|=2\left\|X\right\|$,

(*ii*) $\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}\right\|+\left\|S^{-1}XS\right\|\leq 2\left\|X\right\|$,

(*iii*) $\forall X\in\mathfrak{B}(H),\ \left\|SXS^{-1}+S^{-1}XS\right\|\leq 2\left\|X\right\|$,

(*iv*) $\left\|\varphi_{S}\right\|$ *gets its minimal value* 2,

(*v*) $\frac{1}{\|S\|}S$ *is unitary.*

Proof. The two implications (*i*) ⟹ (*ii*), (*ii*) ⟹ (*iii*), and the equivalence (*iii*) ⟺ (*iv*) are trivial.

(*iii*) ⟹ (*v*). Assume (*iii*) holds.

So, it follows that, $\left\|\varphi_{S}\right\|_{\lambda}=2$. Using *Proposition* 15, then S is normal and $\sup\limits_{\lambda,\mu\in\sigma(S)}\left|\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right|=2.$

Using the spectral measure of S, there exists a sequence (S_n) of invertible normal operators in $\mathfrak{B}(H)$ with finite spectrum such that:

(a) $S_n \longrightarrow S$ uniformly,

(b) for every $\lambda \in \sigma(S)$, there exists a sequence (λ_n) such that $\lambda_n \in \sigma(S_n)$, for every n, and $\lambda_n \longrightarrow \lambda$.

Let $\lambda,\ \mu \in \sigma(S)$. Then from (b), there exist two sequences $(\lambda_n),\ (\mu_n)$ such that $\lambda_n,\ \mu_n \in \sigma(S_n)$, for every n, and $\lambda_n \longrightarrow \lambda,\ \mu_n \longrightarrow \mu$.

Let $\epsilon > 0$. Then, there exists an integer $N \geq 1$ such that

$$(*) \quad \forall n > N,\ \forall X \in \mathfrak{B}(H),\ \left\| S_n X S_n^{-1} + S_n^{-1} X S_n \right\| \leq (2+\epsilon)\left\| X \right\|.$$

Let $n > N$. Since S_n is normal with finite spectrum, there exist p orthogonal projections $E_1, ..., E_p$ in $\mathfrak{B}(H)$ such that $E_i E_j = 0$, if $i \neq j$, $\sum_{i=1}^{p} E_i = I$, $S_n = \sum_{i=}^{p} \alpha_i E_i$, where $\sigma(S_n) = \left\{ \alpha_1, ..., \alpha_p \right\}$, $\alpha_1 = \lambda_n$, $\mu_n = \alpha_2$.

Then, using $(*)$ and putting $A = \begin{bmatrix} 2 & \gamma_n \\ \gamma_n & 2 \end{bmatrix}$, where $\gamma_n = \frac{\lambda_n}{\mu_n} + \frac{\mu_n}{\lambda_n}$, we obtain

$$\forall X \in \mathfrak{B}(\mathbb{C}^2),\ \left\| A \circ X \right\| \leq (2+\epsilon)\left\| X \right\|.$$

Put $X = \begin{bmatrix} t \operatorname{Im} \gamma_n & i \\ i & t \operatorname{Im} \gamma_n \end{bmatrix}$ (where $t > 0$) in this last inequality, we obtain

$$(2t \operatorname{Im} \gamma_n)^2 + |\gamma_n|^2 + 4t\,(\operatorname{Im} \gamma_n)^2 \leq (2t \operatorname{Im} \gamma_n)^2 + 4 + \left(4\epsilon + \epsilon^2\right)\left((t \operatorname{Im} \gamma_n)^2 + 1\right).$$

Put $\gamma = \lim \gamma_n = \frac{\lambda}{\mu} + \frac{\mu}{\lambda}$, and letting $n \longrightarrow \infty$ in this last inequality, it follows that

$$|\gamma|^2 + 4t\,(\operatorname{Im} \gamma)^2 \leq 4 + \left(4\epsilon + \epsilon^2\right)\left((t \operatorname{Im} \gamma)^2 + 1\right).$$

Now, letting $\epsilon \longrightarrow 0$, we deduce that $4t\,(\operatorname{Im} \gamma)^2 \leq 4 - |\gamma|^2$, for every $t > 0$. Hence, $\operatorname{Im} \gamma = 0$, and $|\gamma| \leq 2$. Then, by a simple computation, we find that $|\lambda| = |\mu|$. Then $\sigma(S)$ is included in the circle centred at the origin and of radius $\|S\|$. Since S is normal, this proves (v).

$(v) \Longrightarrow (i)$. This implication is trivial. □

Corollary 9. *Then the following properties are equivalent:*

(i) *S is a unitary reflection operator multiplied by a nonzero scalar*

(ii) $\forall X \in \mathfrak{B}(H),\ \left\| SXS^{-1} + S^{-1}XS \right\| = 2\left\| X \right\|,$

(iii) $\inf\limits_{\|X\|=1} \left\| \varphi_S(X) \right\| = 2 = \sup\limits_{\|X\|=1} \left\| \varphi_S(X) \right\|.$

Proof. This corollary follows immediately from *Proposition* 7 and *Proposition* 16. □

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