INEQUALITIES FOR THE MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following mapping that we call the *monotonic integral transform*

$$\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M. We prove among others that

$$0 \leq m\mathcal{M}'(w, \mu)(\delta) \leq \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \leq M\mathcal{M}'(w, \mu)(\alpha),$$

where $\mathcal{D}_1'(w, \mu)$ is the derivative of $\mathcal{D}_1(w, \mu)(t)$ as a function of $t > 0$.

As a consequence, if $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$, then

$$0 \leq mf'(\delta) \leq f(B) - f(A) \leq Mf'(\alpha).$$

Some examples for operator convex functions as well as for integral transforms $\mathcal{M}(\cdot, \cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator convex functions, Operator inequalities, Löwner-Heinz inequality. Logarithmic operator inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 77, 16 pp. Received 09/07/20

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.9}$$

where $a\in R$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.9).

In the recent paper [2] we obtained the following result:

Theorem 2. *For all A, $B>0$ we have the representation*

$$\begin{aligned}(1.11)\quad & \mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)\\ &=-\int_0^\infty\left(\int_0^1(\lambda+(1-t)B+tA)^{-1}(B-A)(\lambda+(1-t)B+tA)^{-1}dt\right)\\ &\times w(\lambda)\,d\mu(\lambda).\end{aligned}$$

If $B\geq A>0$, then

$$\mathcal{D}(w,\mu)(B)\leq\mathcal{D}(w,\mu)(A), \tag{1.12}$$

namely, the function $\mathcal{D}(w,\mu)(\cdot)$ is operator monotone decreasing on $(0,\infty)$.

As a consequence we also obtained the following result [2]:

Corollary 1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$. Then $[f(0)-f(t)]\,t^{-1}$ is operator monotone on $(0,\infty)$.*

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 3. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty\frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.13}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.13).

In [2] we also obtained the following result for operator convex functions:

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$. Then $\left[f(0)+f'_+(0)\,t-f(t)\right]t^{-2}$ is operator monotone on $(0,\infty)$*

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t),\ t>0. \tag{1.14}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t)&=\int_0^\infty w(\lambda)\,t\,(t+\lambda)^{-1}d\mu(\lambda)\\ &=\int_0^\infty w(\lambda)(t+\lambda-\lambda)(t+\lambda)^{-1}d\mu(\lambda)\\ &=\int_0^\infty w(\lambda)\left[1-\lambda(t+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned} \tag{1.15}$$

If $\int_0^\infty w(\lambda)\,d\mu(\lambda) < \infty$, then

$$\mathcal{M}(w,\mu)(t) = \int_0^\infty w(\lambda)\,d\mu(\lambda) - \mathcal{D}(\ell w,\mu)(t), \tag{1.16}$$

where $\ell(t) = t$, $t > 0$.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then after some calculations, we get

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda}d\lambda = E_1(at)\exp(at),\ t \geq 0$$

and

$$\int_0^\infty w(\lambda)\,d\lambda = \int_0^\infty \exp(-a\lambda)\,d\lambda = \frac{1}{a},$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du.$$

This gives that

$$\mathcal{M}(e_{-a})(t) = t\mathcal{D}(w,\mu)(t) = tE_1(at)\exp(at),\ t \geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda\exp(-a\lambda)}{t+\lambda}d\lambda = \frac{1}{a} - tE_1(at)\exp(at)$$

for $t > 0$.

One observes that

$$\mathcal{M}(e_{-a})(t) = \int_0^\infty w(\lambda)\,d\lambda - \mathcal{D}(\ell e_{-a},\mu)(t),\ t > 0$$

and the equality (1.16) is verified in this case.

If we take $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda = \infty$ and the equality (1.16) does not hold in this case.

For all $T > 0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T) = T\mathcal{D}(w,\mu)(T) = \int_0^\infty w(\lambda)\left[1 - \lambda(T+\lambda)^{-1}\right]d\mu(\lambda). \tag{1.17}$$

This gives the representation

$$T^r = \frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T),$$

where $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$ and μ is the usual Lebesgue norm. Also, from (1.6) we have the representation

$$\ln T = (T-1)\,\mathcal{D}(w_{\ln})(T),\ T > 0, \tag{1.18}$$

where $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$.

Motivated by the above results, in this paper we show among others that

$$0 \leq m\mathcal{M}'(w,\mu)(\delta) \leq \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) \leq M\mathcal{M}'(w,\mu)(\alpha),$$

where $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M and $\mathcal{D}_1'(w,\mu)$ is the derivative of $\mathcal{M}(w,\mu)(t)$ as a function of $t > 0$.

As a consequence, if $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$, then

$$0 \leq mf'(\delta) \leq f(B) - f(A) \leq Mf'(\alpha).$$

Some examples for operator convex functions as well as for integral transforms $\mathcal{M}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Monotonicity Properties

We have the following monotonicity result:

Theorem 4. *For all A, $B>0$ we have the representation*

$$\begin{aligned}(2.1)\quad &\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\ &=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right)\\ &\times\lambda w(\lambda)\,d\mu(\lambda).\end{aligned}$$

If $B\geq A>0$, then

$$\mathcal{M}(w,\mu)(B)\geq\mathcal{M}(w,\mu)(A),\tag{2.2}$$

namely $\mathcal{M}(w,\mu)$ is operator monotone on $(0,\infty)$.

Proof. From (1.17) we have for all A, $B\geq 0$ that

$$\begin{aligned}(2.3)\quad &\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\ &=\int_0^\infty w(\lambda)\left[1-\lambda(B+\lambda)^{-1}\right]d\mu(\lambda)-\int_0^\infty w(\lambda)\left[1-\lambda(A+\lambda)^{-1}\right]d\mu(\lambda)\\ &=\int_0^\infty\lambda w(\lambda)\left[(A+\lambda)^{-1}-(B+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1}\tag{2.4}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt.\tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt.\tag{2.6}$$

Now, if we take in (2.6) $C=\lambda+A$, $D=\lambda+B$, then

$$\begin{aligned}(2.7)\qquad &(\lambda+A)^{-1}-(\lambda+B)^{-1}\\&=\int_0^1((1-t)(\lambda+A)+t(\lambda+B))^{-1}(B-A)\\&\times((1-t)(\lambda+A)+t(\lambda+B))^{-1}dt\\&=\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt.\end{aligned}$$

By employing (2.3) and (2.7), we derive (2.1). □

Corollary 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and it has the representation (1.9), then for all A, $B>0$ we have the representation*

$$\begin{aligned}(2.8)\qquad &f(B)-f(A)-b(B-A)\\&=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right)\\&\times\lambda^2d\mu(\lambda).\end{aligned}$$

If $B\geq A$, then

$$(2.9)\qquad f(B)-f(A)\geq b(B-A)\geq 0$$

Proof. From (1.9) we have for $T>0$ that

$$f(T)-a-bT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$. Therefore

$$\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)=f(B)-f(A)-b(B-A)$$

and by (2.1) we get (2.8). □

Corollary 4. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and it has the representation (1.13), then for all A, $B>0$ we have the representation*

$$\begin{aligned}(2.10)\qquad &f(B)B^{-1}-f(A)A^{-1}-a\left(B^{-1}-A^{-1}\right)-c(B-A)\\&=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right)\\&\times\lambda^2d\mu(\lambda)\end{aligned}$$

If $B\geq A$, then

$$(2.11)\qquad f(B)B^{-1}-f(A)A^{-1}-a\left(B^{-1}-A^{-1}\right)\geq c(B-A)\geq 0.$$

Proof. From (1.13) we have for $T>0$ that

$$(f(T)-a)T^{-1}-b-cT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ. Therefore

$$\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)=(f(B)-a)B^{-1}-(f(A)-a)A^{-1}-c(B-A)$$

and by (2.1) we get (2.10). □

Remark 1. *If $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$, then we can take $a=f(0)$ and by (2.10) we get*

$$\begin{aligned}(2.12)\quad & f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\\ &=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right)\\ &\times\lambda^2d\mu(\lambda).\end{aligned}$$

If $B\geq A$, then

$$f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\geq c(B-A)\geq 0. \tag{2.13}$$

3. Lower and Upper Bounds

Let f be an operator convex function on I. For A, $B\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t):=f((1-t)A+tB). \tag{3.1}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)x,x\right\rangle=\langle f((1-t)A+tB)x,x\rangle. \tag{3.2}$$

We have the following basic fact [3]:

Lemma 1. *Let f be an operator convex function on I. For any A, $B\in\mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $A,B\in\mathcal{SA}_I(H)$ and $x\in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

A continuous function $g:\mathcal{SA}_I(H)\to\mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A\in\mathcal{SA}_I(H)$ along the direction $B\in\mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B):=\lim_{s\to 0}\frac{g(A+sB)-g(A)}{s}\in\mathcal{B}(H). \tag{3.3}$$

If the limit (3.3) exists for all $B\in\mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g\in\mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g\in\mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators $A,B\in\mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B]:=\{(1-t)A+tB\mid t\in[0,1]\}.$$

We observe that A, $B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_I(H)$.

We also have [3]:

Lemma 2. *Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t)=\nabla f_{(1-t)A+tB}(B-A). \tag{3.4}$$

In particular,

$$\varphi'_{(A,B)}(0+) = \nabla f_A (B - A) \tag{3.5}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B (B - A) . \tag{3.6}$$

and, see [3],

Lemma 3. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for $0 < t_1 < t_2 < 1$*

$$\nabla f_{(1-t_1)A+t_1B}(B - A) \leq \nabla f_{(1-t_2)A+t_2B}(B - A) \tag{3.7}$$

in the operator order.

In particular,

$$\nabla f_A (B - A) \leq \nabla f_{(1-t_1)A+t_1B}(B - A) \tag{3.8}$$

and

$$\nabla f_{(1-t_2)A+t_2B}(B - A) \leq \nabla f_B (B - A) . \tag{3.9}$$

Also, we have

$$\nabla f_A (B - A) \leq \nabla f_{(1-t)A+tB}(B - A) \leq \nabla f_B (B - A) \tag{3.10}$$

for all $t \in (0, 1)$.

We have the following gradient inequalities:

Lemma 4. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then*

$$\nabla_B f (B - A) \geq f(B) - f(A) \geq \nabla_A f (B - A) . \tag{3.11}$$

Proof. By the properties of Bochner integral, we have

$$\begin{aligned} f(B) - f(A) &= \varphi_{(A,B)}(1) - \varphi_{(A,B)}(0) = \int_0^1 \varphi'_{(A,B)}(t)\,dt \\ &= \int_0^1 \nabla f_{(1-t)A+tB}(B - A)\,dt. \end{aligned}$$

From (3.10) we have, by integration, that

$$\nabla f_A (B - A) \leq \int_0^1 \nabla f_{(1-t)A+tB}(B - A)\,dt \leq \nabla f_B (B - A) ,$$

and the inequality (3.11) is proved. □

Let T, $S > 0$. The function $f(t) = t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T (S) := \lim_{t \to 0} \left[\frac{f(T + tS) - f(T)}{t} \right] = -T^{-1} S T^{-1} \tag{3.12}$$

for T, $S > 0$.

Using (3.11) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D - C) D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1}(D - C) C^{-1}$$

that is equivalent to

$$D^{-1}(D - C) D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1}(D - C) C^{-1} \tag{3.13}$$

for all C, $D > 0$.

When more assumptions are made about the operators A and B, then we have the following lower and upper bounds for the difference $\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)$.

Theorem 5. *Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M. Then*

$$0 \leq m\mathcal{M}'(w,\mu)(\delta) \leq \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) \leq M\mathcal{M}'(w,\mu)(\alpha), \tag{3.14}$$

where $\mathcal{D}_1'(w,\mu)$ is the derivative of $\mathcal{M}(w,\mu)(t)$ as a function of $t > 0$.

Proof. We have for A, $B > 0$ that

$$\begin{aligned} &\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) \\ &= \int_0^\infty \lambda w(\lambda)\left[(A+\lambda)^{-1} - (B+\lambda)^{-1}\right] d\mu(\lambda). \end{aligned} \tag{3.15}$$

From (3.13) we get for $C = \lambda + A$ and $D = \lambda + B$ that

$$\begin{aligned} (\lambda+B)^{-1}(B-A)(\lambda+B)^{-1} &\leq (\lambda+A)^{-1} - (\lambda+B)^{-1} \\ &\leq (\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} \end{aligned} \tag{3.16}$$

for all $\lambda \geq 0$.

If we multiply (3.16) by $\lambda w(\lambda) \geq 0$ and integrate over $\mu(\lambda)$ we get

$$\begin{aligned} &\int_0^\infty \lambda w(\lambda)(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1} d\mu(\lambda) \\ &\leq \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) \\ &\leq \int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} d\mu(\lambda). \end{aligned} \tag{3.17}$$

Since $m \leq B - A \leq M$ hence

$$m(\lambda+B)^{-2} \leq (\lambda+B)^{-1}(B-A)(\lambda+B)^{-1},$$

which implies, by integration that

$$\begin{aligned} &m\int_0^\infty \lambda w(\lambda)(\lambda+B)^{-2} d\mu(\lambda) \\ &\leq \int_0^\infty \lambda w(\lambda)(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1} d\mu(\lambda). \end{aligned} \tag{3.18}$$

Also

$$(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} \leq M(\lambda+A)^{-2},$$

which implies, by integration, that

$$\begin{aligned} &\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} d\mu(\lambda) \\ &\leq M\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-2} d\mu(\lambda). \end{aligned} \tag{3.19}$$

Since $B \leq \delta$, then $\lambda + B \leq \lambda + \delta$ for all $\lambda \geq 0$ which implies that $(\lambda+B)^{-1} \geq (\lambda+\delta)^{-1}$ and therefore $(\lambda+B)^{-2} \geq (\lambda+\delta)^{-2}$. Consequently

$$m\int_0^\infty \lambda w(\lambda)(\lambda+B)^{-2} d\mu(\lambda) \geq m\int_0^\infty \lambda w(\lambda)(\lambda+\delta)^{-2} d\mu(\lambda). \tag{3.20}$$

Also, since $A \geq \alpha > 0$, then $\lambda + A \geq \lambda + \alpha > 0$, which implies that $(\lambda + A)^{-1} \leq (\lambda + \alpha)^{-1}$, therefore $(\lambda + A)^{-2} \leq (\lambda + \alpha)^{-2}$ and

$$M \int_0^\infty \lambda w(\lambda)(\lambda + A)^{-2} d\mu(\lambda) \leq M \int_0^\infty \lambda w(\lambda)(\lambda + \alpha)^{-2} d\mu(\lambda). \tag{3.21}$$

From (3.17)-(3.21) we get

$$\begin{aligned} m \int_0^\infty \lambda w(\lambda)(\lambda + \delta)^{-2} d\mu(\lambda) &\leq \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \\ &\leq M \int_0^\infty \lambda w(\lambda)(\lambda + \alpha)^{-2} d\mu(\lambda). \end{aligned} \tag{3.22}$$

For $h \neq 0$ small,

$$\begin{aligned} &\frac{\mathcal{M}(w, \mu)(t + h) - \mathcal{M}(w, \mu)(t)}{h} \\ &= \frac{1}{h} \int_0^\infty \lambda w(\lambda) \left(\frac{1}{t + \lambda} - \frac{1}{t + h + \lambda} \right) d\mu(\lambda) \\ &= \int_0^\infty \frac{\lambda w(\lambda)}{(t + h + \lambda)(t + \lambda)} d\mu(\lambda). \end{aligned}$$

By taking the limit over $h \to 0$ and using the properties of limits and integrals, we get the derivative of $\mathcal{M}(w, \mu)$ as

$$\mathcal{M}'(w, \mu)(t) = \int_0^\infty \frac{\lambda w(\lambda)}{(t + \lambda)^2} d\mu(\lambda) \geq 0, \ t > 0. \tag{3.23}$$

From (3.22) and (3.23) we derive (3.14). □

The case of operator monotone functions is as follows:

Corollary 5. *Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and it has the representation (1.9).If $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M, then*

$$\begin{aligned} 0 \leq m f'(\delta) &\leq m[f'(\delta) - b] + b(B - A) \leq f(B) - f(A) \\ &\leq M[f'(\alpha) - b] + b(B - A) \leq M f'(\alpha). \end{aligned} \tag{3.24}$$

Proof. From (1.9) we have

$$\mathcal{M}(w, \mu)(t) = f(t) - a - bt, \ t > 0.$$

By taking the derivative, we get

$$\mathcal{M}'(w, \mu)(t) = f'(t) - b, \ t > 0.$$

From (3.14) we get

$$0 \leq m[f'(\delta) - b] \leq f(B) - f(A) - b(B - A) \leq M[f'(\alpha) - b],$$

which is equivalent to

$$m[f'(\delta) - b] + b(B - A) \leq f(B) - f(A) \leq M[f'(\alpha) - b] + b(B - A).$$

Since

$$bm \leq b(B - A) \leq Mb,$$

hence

$$mf'(\delta) \leq m[f'(\delta) - b] + b(B - A)$$

and

$$M\left[f'(\alpha)-b\right]+b(B-A)\leq Mf'(\alpha)$$

and the inequalities in (3.24) are proved. □

The case of operator convex functions is as follows:

Corollary 6. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and it has the representation (1.13). If $A\geq\alpha>0$, $\delta\geq B>0$ and $0<m\leq B-A\leq M$ for some constants α, δ, m, M, then*

$$\begin{aligned}
0&\leq m\left(\frac{f'(\delta)\delta-f(\delta)+a}{\delta^2}\right)\\
&\leq m\left[\frac{f'(\delta)\delta-f(\delta)+a}{\delta^2}-c\right]+c(B-A)\\
&\leq f(B)B^{-1}-f(A)A^{-1}-a\left(B^{-1}-A^{-1}\right)\\
&\leq M\left[\frac{f'(\alpha)\alpha-f(\alpha)+a}{\alpha^2}-c\right]+c(B-A)\\
&\leq M\left(\frac{f'(\alpha)\alpha-f(\alpha)+a}{\alpha^2}\right).
\end{aligned}\tag{3.25}$$

Proof. From (1.13) we get

$$\mathcal{M}(w,\mu)(t)=\frac{f(t)-a}{t}-b-ct,\ t>0.$$

If we take the derivative in this equality, then we get

$$\mathcal{M}'(w,\mu)(t)=\frac{f'(t)t-f(t)+a}{t^2}-c.$$

By (3.14) we get

$$\begin{aligned}
0&\leq m\left[\frac{f'(\delta)\delta-f(\delta)+a}{\delta^2}-c\right]\\
&\leq f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\\
&\leq M\left[\frac{f'(\alpha)\alpha-f(\alpha)+a}{\alpha^2}-c\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&m\left[\frac{f'(\delta)\delta-f(\delta)+a}{\delta^2}-c\right]+c(B-A)\\
&\leq f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\\
&\leq M\left[\frac{f'(\alpha)\alpha-f(\alpha)+a}{\alpha^2}-c\right]+c(B-A).
\end{aligned}$$

Since

$$\begin{aligned}
&m\left[\frac{f'(\delta)\delta-f(\delta)+a}{\delta^2}-c\right]+c(B-A)\\
&\geq m\left[\frac{f'(\delta)\delta-f(\delta)+a}{\delta^2}-c\right]+cm=m\left(\frac{f'(\delta)\delta-f(\delta)+a}{\delta^2}\right)\geq 0
\end{aligned}$$

and

$$M\left[\frac{f'(\alpha)\alpha - f(\alpha) + a}{\alpha^2} - c\right] + c(B-A)$$
$$\le M\left[\frac{f'(\alpha)\alpha - f(\alpha) + a}{\alpha^2} - c\right] + cM = M\left(\frac{f'(\alpha)\alpha - f(\alpha) + a}{\alpha^2}\right),$$

the proof of (3.25) is thus completed. □

Remark 2. *If $f : [0,\infty) \to \mathbb{R}$ is operator convex in $[0,\infty)$ then we can take $a = f(0)$ and by (3.25) we get*

$$(3.26)\qquad 0 \le m\left(\frac{f'(\delta)\delta - f(\delta) + f(0)}{\delta^2}\right)$$
$$\le m\left[\frac{f'(\delta)\delta - f(\delta) + f(0)}{\delta^2} - c\right] + c(B-A)$$
$$\le f(B)B^{-1} - f(A)A^{-1} - f(0)\left(B^{-1} - A^{-1}\right)$$
$$\le M\left[\frac{f'(\alpha)\alpha - f(\alpha) + f(0)}{\alpha^2} - c\right] + c(B-A)$$
$$\le M\left(\frac{f'(\alpha)\alpha - f(\alpha) + f(0)}{\alpha^2}\right).$$

Its is well known that, if $P \ge 0$, then

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle \langle Py, y\rangle$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$0 \le \langle x, x\rangle^2 = \langle T^{-1}Tx, x\rangle^2 = \langle Tx, T^{-1}x\rangle^2$$
$$\le \langle Tx, x\rangle \langle TT^{-1}x, T^{-1}x\rangle = \langle Tx, x\rangle \langle x, T^{-1}x\rangle$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$1 \le \langle Tx, x\rangle \langle x, T^{-1}x\rangle \le \langle Tx, x\rangle \sup_{\|x\|=1} \langle x, T^{-1}x\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the first inequality below

$$(3.27)\qquad \left\|T^{-1}\right\|^{-1} \le T \le \|T\|.$$

The second inequality is obvious.

Proposition 1. *Let $B > A > 0$. Then*

$$(3.28)\qquad 0 \le \left\|(B-A)^{-1}\right\|^{-1} \mathcal{M}'(w,\mu)(\|B\|) \le \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)$$
$$\le \|B-A\| \mathcal{M}'(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right).$$

Proof. Since, by (3.27), $A \ge \left\|A^{-1}\right\|^{-1}$, $\|B\| \ge B$ and $\|B-A\| \ge B - A \ge \left\|(B-A)^{-1}\right\|^{-1}$ then by (3.14) for $\alpha = \left\|A^{-1}\right\|^{-1}$, $\delta = \|B\|$, $m = \left\|(B-A)^{-1}\right\|^{-1}$ and $M = \|B-A\|$ we get (3.28). □

In the case of operator monotone functions we have:

Corollary 7. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and $B>A>0$. Then*

$$0\le \frac{f'(\|B\|)}{\left\|(B-A)^{-1}\right\|}\le f(B)-f(A)\le \|B-A\|\, f'\left(\left\|A^{-1}\right\|^{-1}\right). \tag{3.29}$$

We also have:

Corollary 8. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and $B>A>0$. Then*

$$\begin{aligned}0&\le \frac{1}{\|B\|^2\left\|(B-A)^{-1}\right\|}\left(f'(\|B\|)\|B\|-f(\|B\|)+f(0)\right)\\&\le f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\\&\le \|B-A\|\left\|A^{-1}\right\|^2\left(f'\left(\left\|A^{-1}\right\|^{-1}\right)\left\|A^{-1}\right\|^{-1}-f\left(\left\|A^{-1}\right\|^{-1}\right)+f(0)\right).\end{aligned} \tag{3.30}$$

4. Some Examples

We consider the operator monotone function $f(t)=t^r$, $r\in(0,1]$, then for all A, $B\ge 0$ we have by (2.1) the representation

$$\begin{aligned}B^r-A^r&=\frac{\sin(r\pi)}{\pi}\\&\times\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right)\lambda^r d\lambda,\end{aligned} \tag{4.1}$$

which proves in one line the Löwner-Heinz inequality $B^r\ge A^r$ if $B\ge A$.

For logarithmic function we have the following representation for the difference:

Proposition 2. *For all A, $B>0$ we have*

$$\begin{aligned}&\ln B-\ln A\\&=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right)d\lambda.\end{aligned} \tag{4.2}$$

Proof. We have from (1.18) for A, $B>0$ that

$$\ln B-\ln A=\int_0^\infty\frac{1}{\lambda+1}\left[(B-1)(\lambda+B)^{-1}-(A-1)(\lambda+A)^{-1}\right]d\lambda. \tag{4.3}$$

Since

$$\begin{aligned}&(B-1)(\lambda+B)^{-1}-(A-1)(\lambda+A)^{-1}\\&=B(\lambda+B)^{-1}-A(\lambda+A)^{-1}-\left((\lambda+B)^{-1}-(\lambda+A)^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}&B(\lambda+B)^{-1}-A(\lambda+A)^{-1}\\&=(B+\lambda-\lambda)(\lambda+B)^{-1}-(A+\lambda-\lambda)(\lambda+A)^{-1}\\&=1-\lambda(\lambda+B)^{-1}-1+\lambda(\lambda+A)^{-1}=\lambda(\lambda+A)^{-1}-\lambda(\lambda+B)^{-1},\end{aligned}$$

hence

$$\begin{aligned}&(B-1)(\lambda+B)^{-1}-(A-1)(\lambda+A)^{-1}\\&=\lambda(\lambda+A)^{-1}-\lambda(\lambda+B)^{-1}-\left((\lambda+B)^{-1}-(\lambda+A)^{-1}\right)\\&=(\lambda+1)\left[(\lambda+A)^{-1}-(\lambda+B)^{-1}\right]\end{aligned}$$

and by (4.3) we get

$$\ln B-\ln A=\int_0^\infty\left[(\lambda+A)^{-1}-(\lambda+B)^{-1}\right]d\lambda. \tag{4.4}$$

Since, by (2.7) we have

$$\begin{aligned}&(\lambda+A)^{-1}-(\lambda+B)^{-1}\\&=\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt,\end{aligned} \tag{4.5}$$

for all $\lambda\geq 0$, hence by (4.4) and (4.5) we get (4.2). □

Assume that $A\geq\alpha>0$, $\delta\geq B>0$ and $0<m\leq B-A\leq M$ for some constants α, δ, m, M. Then by (3.24) we get

$$rm\delta^{r-1}\leq B^r-A^r\leq rM\alpha^{r-1}. \tag{4.6}$$

If $B>A>0$, then

$$\frac{r}{\left\|(B-A)^{-1}\right\|\|B\|^{1-r}}\leq B^r-A^r\leq r\|B-A\|\left\|A^{-1}\right\|^{1-r}. \tag{4.7}$$

The function $f(t)=\ln t$, $t>0$ is operator monotone on $(0,\infty)$ and if we assume that $A\geq\alpha>0$, $\delta\geq B>0$ and $0<m\leq B-A\leq M$ for some constants α, δ, m, M, then by (3.24) we get

$$\frac{m}{\delta}\leq\ln B-\ln A\leq\frac{M}{\alpha}. \tag{4.8}$$

If $B>A>0$, then

$$\frac{1}{\left\|(B-A)^{-1}\right\|\|B\|}\leq\ln B-\ln A\leq\|B-A\|\left\|A^{-1}\right\|. \tag{4.9}$$

We consider the operator convex function $f(t)=-\ln(t+1)$ on $[0,\infty)$. Then by (3.26),

$$\begin{aligned}0&\leq m\left(\frac{(\delta+1)\ln(\delta+1)-\delta}{\delta^2(\delta+1)}\right)\leq B^{-1}\ln(B+1)-A^{-1}\ln(A+1)\\&\leq M\left(\frac{(\alpha+1)\ln(\alpha+1)-\alpha}{\alpha^2(\alpha+1)}\right),\end{aligned} \tag{4.10}$$

where $A\geq\alpha>0$, $\delta\geq B>0$ and $0<m\leq B-A\leq M$ for some constants α, δ, m, M.

We define the *upper incomplete Gamma function* as [11]

$$\Gamma(a,z):=\int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [12]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{4.11}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (4.11) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{4.12}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (4.12) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^t E_1(t) \tag{4.13}$$

for $t > 0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du.$$

We then have

$$\mathcal{M}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T) = \Gamma(1-a)T^{1-a}\exp(T)\,\Gamma(a,T) \tag{4.14}$$

for $a < 1$ and

$$\mathcal{M}\left(w_{e^{-\cdot}}\right)(T) = T\exp(T)\,E_1(T) \tag{4.15}$$

for $T > 0$.

For all $A,\ B > 0$ we have the representation

$$\begin{aligned} &\mathcal{M}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(B) - \mathcal{M}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(A) \\ &= \int_0^\infty \left(\int_0^1 (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1}dt\right) \\ &\times \lambda^{1-a}e^{-\lambda}d\mu(\lambda). \end{aligned} \tag{4.16}$$

If $B \geq A > 0$, then

$$B^{1-a}\exp(B)\,\Gamma(a,B) \geq A^{1-a}\exp(A)\,\Gamma(a,A),$$

namely the function $g_a(t) := t^{1-a}\exp(t)\,\Gamma(a,t)$ is operator monotone on $(0,\infty)$.

Since $E_1'(t) = -\frac{e^{-t}}{t}$, $t > 0$, then

$$\begin{aligned} \mathcal{M}'\left(w_{e^{-\cdot}}\right)(t) &= (t\exp(t))' E_1(t) + t\exp(t)\,E_1'(t) \\ &= (\exp t + t\exp t)\,E_1(t) - t\exp(t)\left(\frac{e^{-t}}{t}\right) \\ &= (1+t)\exp t E_1(t) - 1 \end{aligned}$$

for $t > 0$.

From (3.24) we get

$$0 \leq m\left[(1+\delta)\exp\delta E_1(\delta)-1\right] \leq B\exp(B)E_1(B)-A\exp(A)E_1(A) \leq M\left[(1+\alpha)\exp\alpha E_1(\alpha)-1\right], \tag{4.17}$$

when $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M.

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