INEQUALITIES FOR THE MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function \( w(\lambda) \), \( \lambda > 0 \) and \( \mu \) a positive measure on \( (0, \infty) \) we consider the following mapping that we call the monotonic integral transform
\[
\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T (\lambda + T)^{-1} \, d\mu(\lambda),
\]
where the integral is assumed to exist for \( T \) a positive operator on a complex Hilbert space \( H \).

Assume that \( A > 0 \), \( B > 0 \) and \( 0 < m \leq B - A \leq M \) for some constants \( a, \delta, m, M \). We prove among others that
\[
0 < m \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \leq M M\mathcal{M}'(w, \mu)(\delta),
\]
where \( \mathcal{D}'(w, \mu) \) is the derivative of \( \mathcal{D}(w, \mu)(t) \) as a function of \( t > 0 \).

As a consequence, if \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \( (0, \infty) \), then
\[
0 < m f'(\delta) \leq f(B) - f(A) \leq M f'(\alpha).
\]

Some examples for operator convex functions as well as for integral transforms \( \mathcal{M}(\cdot, \cdot) \) related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space \( (H, \langle \cdot, \cdot \rangle) \). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( \langle Tx, x \rangle \geq 0 \) for all \( x \in H \) and also an operator \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible.

We have the following integral representation for the power function when \( t > 0, r \in (0, 1] \), see for instance [1, p. 145]
\[
t^{-1} = \frac{\sin \left( r \pi \right)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} \, d\lambda. \tag{1.1}
\]

Observe that for \( t > 0, t \neq 1 \), we have
\[
\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln \left( \frac{u+t}{u+1} \right)
\]
for all \( u > 0 \).

By taking the limit over \( u \to \infty \) in this equality, we derive
\[
\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)}.
\]
which gives the representation for the logarithm

\[ \ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \]

for all \( t > 0 \).

Motivated by these representations, we introduce, for a continuous and positive function \( w(\lambda), \lambda > 0 \), the following integral transform

\[ \mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \quad t > 0, \]

where \( \mu \) is a positive measure on \((0, \infty)\) and the integral (1.3) exists for all \( t > 0 \).

For \( \mu \) the Lebesgue usual measure, we put

\[ \mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \quad t > 0, \]

If we take \( \mu \) to be the usual Lebesgue measure and the kernel \( w_r(\lambda) = \lambda^{r-1}, \quad r \in (0, 1] \), then

\[ t^{r-1} = \frac{\sin \left( r\pi \right)}{\pi} \mathcal{D}(w_r)(t), \quad t > 0. \]

For the same measure, if we take the kernel \( w_{\ln}(\lambda) = (\lambda + 1)^{-1}, \quad t > 0 \), we have the representation

\[ \ln t = (t - 1) \mathcal{D}(w_{\ln})(t), \quad t > 0. \]

Assume that \( T > 0 \), then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

\[ \mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \]

where \( w \) and \( \mu \) are as above. Also, when \( \mu \) is the usual Lebesgue measure, then

\[ \mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \]

for \( T > 0 \).

A real valued continuous function \( f \) on \((0, \infty)\) is said to be operator monotone if \( f(A) \geq f(B) \) holds for any \( A \geq B > 0 \).

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

**Theorem 1.** A function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) if and only if it has the representation

\[ f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \]

where \( a \in \mathbb{R}, b \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that

\[ \int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \]

If \( f \) is operator monotone in \([0, \infty)\), then \( a = f(0) \) in (1.9).

In the recent paper [2] we obtained the following result:
Theorem 2. For all $A, B > 0$ we have the representation
\begin{equation}
D(w; \mu)(B) - D(w; \mu)(A) = -\int_0^\infty \left( \int_0^1 (\lambda + (1 - t)B + tA)^{-1} (B - A)(\lambda + (1 - t)B + tA)^{-1} dt \right)
\times w(\lambda) \, d\mu(\lambda).
\end{equation}
If $B \geq A > 0$, then
\begin{equation}
D(w; \mu)(B) \leq D(w; \mu)(A),
\end{equation}
namely, the function $D(w; \mu)(\cdot)$ is operator monotone decreasing on $(0, \infty)$.

As a consequence we also obtained the following result [2]:

Corollary 1. Assume that $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function on $[0, \infty)$. Then $[f(0) - f(t)] \, t^{-1}$ is operator monotone on $(0, \infty)$.

A real valued continuous function $f$ on an interval $I$ is said to be operator convex (operator concave) on $I$ if
\begin{equation}
(f((1 - \lambda)A + \lambda B)) \preceq (\succeq) (1 - \lambda)f(A) + \lambda f(B)
\end{equation}
in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator $A$ and $B$ on a Hilbert space $H$ whose spectra are contained in $I$. Notice that a function $f$ is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 3. A function $f : (0, \infty) \to \mathbb{R}$ is operator convex in $(0, \infty)$ if and only if it has the representation
\begin{equation}
f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t + \lambda} \, d\mu(\lambda),
\end{equation}
where $a, b \in \mathbb{R}, c \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ such that (1.2) holds. If $f$ is operator convex in $[0, \infty)$, then $a = f(0)$ and $b = f_+(0)$, the right derivative, in (1.13).

In [2] we also obtained the following result for operator convex functions:

Corollary 2. Assume that $f : [0, \infty) \to \mathbb{R}$ is an operator convex function on $[0, \infty)$. Then $[f(0) + f_+(0) - f(t)] \, t^{-2}$ is operator monotone on $(0, \infty)$.

For a continuous and positive function $w(\lambda), \lambda > 0$ and a positive measure $\mu$ on $(0, \infty)$, we can define the following mapping, which we call monotonic integral transform, by
\begin{equation}
\mathcal{M}(w, \mu)(t) := tD(w, \mu)(t), \quad t > 0.
\end{equation}
For $t > 0$ we have
\begin{equation}
\mathcal{M}(w, \mu)(t) := tD(w, \mu)(t) = \int_0^\infty w(\lambda) \, t \, (t + \lambda)^{-1} \, d\mu(\lambda)
= \int_0^\infty w(\lambda) \, (t + \lambda - \lambda)(t + \lambda)^{-1} \, d\mu(\lambda)
= \int_0^\infty w(\lambda) \left[ 1 - \lambda (t + \lambda)^{-1} \right] \, d\mu(\lambda).
\end{equation}
If \( \int_{0}^{\infty} w(\lambda) \, d\mu(\lambda) < \infty \), then
\[
(1.16) \quad \mathcal{M}(w, \mu)(t) = \int_{0}^{\infty} w(\lambda) \, d\mu(\lambda) - D(\ell w, \mu)(t),
\]
where \( \ell(t) = t, \ t > 0 \).

Consider the kernel \( e^{-a}\lambda) := \exp(-a\lambda), \ \lambda \geq 0 \) and \( a > 0 \). Then after some calculations, we get
\[
D(e^{-a}) (t) = \int_{0}^{\infty} \frac{\exp(-a\lambda)}{t + \lambda} \, d\lambda = E_{1}(at) \exp(at), \ t \geq 0
\]
and
\[
\int_{0}^{\infty} w(\lambda) \, d\lambda = \int_{0}^{\infty} \exp(-a\lambda) \, d\lambda = \frac{1}{a},
\]
where
\[
E_{1}(t) := \int_{t}^{\infty} \frac{e^{-u}}{u} \, du.
\]
This gives that
\[
\mathcal{M}(e^{-a})(t) = tD(w, \mu)(t) = tE_{1}(at) \exp(at), \ t \geq 0.
\]

By integration we also have
\[
D(\ell e^{-a}, \mu)(t) = \int_{0}^{\infty} \frac{\lambda \exp(-a\lambda)}{t + \lambda} \, d\lambda = \frac{1}{a} - tE_{1}(at) \exp(at)
\]
for \( t > 0 \).

One observes that
\[
\mathcal{M}(e^{-a})(t) = \int_{0}^{\infty} w(\lambda) \, d\lambda - D(\ell e^{-a}, \mu)(t), \ t > 0
\]
and the equality (1.16) is verified in this case.

If we take \( w_r(\lambda) = \lambda^{r-1}, \ r \in (0, 1] \), then \( \int_{0}^{\infty} w_r(\lambda) \, d\lambda = \infty \) and the equality (1.16) does not hold in this case.

For all \( T > 0 \) we have, by the continuous functional calculus for selfadjoint operators, that
\[
(1.17) \quad \mathcal{M}(w, \mu)(T) = TD(w, \mu)(T) = \int_{0}^{\infty} w(\lambda) \left[ 1 - \lambda (T + \lambda)^{-1} \right] \, d\mu(\lambda).
\]
This gives the representation
\[
T^r = \frac{\sin(r\pi)}{\pi} \mathcal{M}(w_r, \mu)(T),
\]
where \( w_r(\lambda) = \lambda^{r-1}, \ r \in (0, 1] \) and \( \mu \) is the usual Lebesgue norm. Also, from (1.6) we have the representation
\[
\ln T = (T - 1) D(w_{in})(T), \ T > 0,
\]
where \( w_{in}(\lambda) = (\lambda + 1)^{-1}, \ t > 0 \).

Motivated by the above results, in this paper we show among others that
\[
0 \leq m \mathcal{M}'(w, \mu)(\delta) \leq \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \leq M \mathcal{M}'(w, \mu)(\alpha),
\]
where \( A \geq \alpha > 0, \ \delta \geq B > 0 \) and \( 0 < m \leq B - A \leq M \) for some constants \( \alpha, \ \delta, \ m, \ M \) and \( \mathcal{D}'(w, \mu) \) is the derivative of \( \mathcal{M}(w, \mu)(t) \) as a function of \( t > 0 \).

As a consequence, if \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty), \) then
\[
0 \leq mf'(\delta) \leq f(B) - f(A) \leq M f' (\alpha).
\]
Some examples for operator convex functions as well as for integral transforms \( \mathcal{M}(\cdot, \cdot) \) related to the exponential and logarithmic functions are also provided.

2. Monotonicity Properties

We have the following monotonicity result:

**Theorem 4.** For all \( A, B > 0 \) we have the representation

\[
\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) = \int_0^\infty \left( \int_0^1 (\lambda + (1-t)A + tB)^{-1} (B - A) (\lambda + (1-t)A + tB)^{-1} \, dt \right) \lambda w(\lambda) \, d\mu(\lambda).
\]

If \( B \geq A > 0 \), then

\[
\mathcal{M}(w, \mu)(B) \geq \mathcal{M}(w, \mu)(A),
\]

namely \( \mathcal{M}(w, \mu) \) is operator monotone on \((0, \infty)\).

**Proof.** From (1.17) we have for all \( A, B \geq 0 \) that

\[
\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) = \int_0^\infty w(\lambda) \left[ 1 - \lambda (B + \lambda)^{-1} \right] d\mu(\lambda) - \int_0^\infty w(\lambda) \left[ 1 - \lambda (A + \lambda)^{-1} \right] d\mu(\lambda)
\]

\[
= \int_0^\infty \lambda w(\lambda) \left[ (A + \lambda)^{-1} - (B + \lambda)^{-1} \right] d\mu(\lambda).
\]

Let \( T, S > 0 \). The function \( f(t) = -t^{-1} \) is operator monotone on \((0, \infty)\), operator Gâteaux differentiable and the Gâteaux derivative is given by

\[
\nabla f_T(S) := \lim_{t \to 0} \frac{f(T + tS) - f(T)}{t} = T^{-1}ST^{-1}
\]

for \( T, S > 0 \).

Consider the continuous function \( g \) defined on an interval \( I \) for which the corresponding operator function is Gâteaux differentiable on the segment \([C, D] : \{(1-t)C + tD, \ t \in [0,1]\}\) for \( C, D \) selfadjoint operators with spectra in \( I \). We consider the auxiliary function defined on \([0,1]\) by

\[
f_{C,D}(t) := f((1-t)C + tD), \ t \in [0,1].
\]

Then we have, by the properties of the Bochner integral, that

\[
f(D) - f(C) = \int_0^1 \frac{d}{dt} (f_{C,D}(t)) \, dt = \int_0^1 \nabla f_{(1-t)C+tD} (D-C) \, dt.
\]

If we write this equality for the function \( f(t) = -t^{-1} \) and \( C, D > 0 \), then we get the representation

\[
C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D - C) ((1-t)C + tD)^{-1} \, dt.
\]
Now, if we take in (2.6) \( C = \lambda + A, D = \lambda + B \), then

\[
(2.7) \quad (\lambda + A)^{-1} - (\lambda + B)^{-1} = \int_0^1 ((1 - t) (\lambda + A) + t(\lambda + B))^{-1} (B - A) \\
\times ((1 - t) (\lambda + A) + t(\lambda + B))^{-1} dt \\
= \int_0^1 (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} dt.
\]

By employing (2.3) and (2.7), we derive (2.1).

**Corollary 3.** Assume that the function \( f : (0, \infty) \rightarrow \mathbb{R} \) is operator monotone in \((0, \infty)\) and it has the representation (1.9), then for all \( A, B > 0 \) we have the representation

\[
(2.8) \quad f (B) - f (A) = b(B - A) \\
\text{and by (2.1) we get (2.8).}
\]

**Proof.** From (1.9) we have for \( T > 0 \) that

\[
f (T) - a - bT = M(\ell, \mu) (T),
\]

for some positive measure \( \mu \), where \( \ell (\lambda) = \lambda, \lambda \geq 0 \). Therefore

\[
M(w, \mu) (B) - M(w, \mu) (A) = f (B) - f (A) - b(B - A)
\]

and by (2.1) we get (2.8).

**Corollary 4.** Assume that the function \( f : (0, \infty) \rightarrow \mathbb{R} \) is operator convex in \((0, \infty)\) and it has the representation (1.13), then for all \( A, B > 0 \) we have the representation

\[
(2.10) \quad f (B) B^{-1} - f (A) A^{-1} - a (B^{-1} - A^{-1}) - c(B - A) \\
= \int_0^\infty \left( \int_0^1 (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} dt \right) \\
\times \lambda^2 d\mu (\lambda).
\]

If \( B \geq A \), then

\[
(2.11) \quad f (B) B^{-1} - f (A) A^{-1} - a (B^{-1} - A^{-1}) \geq c(B - A) \geq 0.
\]

**Proof.** From (1.13) we have for \( T > 0 \) that

\[
(f (T) - a) T^{-1} - b - cT = M(\ell, \mu) (T),
\]

for some positive measure \( \mu \). Therefore

\[
M(w, \mu) (B) - M(w, \mu) (A) = (f (B) - a) B^{-1} - (f (A) - a) A^{-1} - c(B - A)
\]

and by (2.1) we get (2.10).
Remark 1. If \( f : [0, \infty) \to \mathbb{R} \) is operator convex in \( [0, \infty) \), then we can take \( a = f(0) \) and by (2.10) we get
\[
(2.12) \quad f(B)B^{-1} - f(A)A^{-1} - f(0)B^{-1} - A^{-1}) - c(B - A)
= \int_0^\infty \left( \int_0^1 (\lambda + (1-t)A+A)^{-1}B - A) \right) (\lambda + (1-t)A+A)^{-1} dt \right) \times \lambda^2d\mu(\lambda).
\]
If \( B \geq A \), then
\[
(2.13) \quad f(B)B^{-1} - f(A)A^{-1} - f(0)B^{-1} - A^{-1}) \geq c(B - A) \geq 0.
\]

3. Lower and Upper Bounds

Let \( f \) be an operator convex function on \( I \). For \( A, B \in SA_I(H) \), the class of all selfadjoint operators with spectra in \( I \), we consider the auxiliary function \( \varphi_{(A,B)} : [0, 1] \to B(H) \) defined by
\[
(3.1) \quad \varphi_{(A,B)}(t) := f((1-t)A+tB).
\]
For \( x \in H \) we can also consider the auxiliary function \( \varphi_{(A,B);x} : [0, 1] \to \mathbb{R} \) defined by
\[
(3.2) \quad \varphi_{(A,B);x}(t) := \langle \varphi_{(A,B)}(t) x, x \rangle = \langle f((1-t)A+tB) x, x \rangle.
\]

We have the following basic fact [3]:

Lemma 1. Let \( f \) be an operator convex function on \( I \). For any \( A, B \in SA_I(H) \), \( \varphi_{(A,B)} \) is well defined and convex in the operator order. For any \( A, B \in SA_I(H) \) and \( x \in H \) the function \( \varphi_{(A,B);x} \) is convex in the usual sense on \([0, 1]\).

A continuous function \( g : SA_I(H) \to B(H) \) is said to be Gâteaux differentiable in \( A \in SA_I(H) \) along the direction \( B \in B(H) \) if the following limit exists in the strong topology of \( B(H) \)
\[
(3.3) \quad \nabla g_A(B) := \lim_{s \to 0} \frac{g(A+sB) - g(A)}{s} \in B(H).
\]

If the limit (3.3) exists for all \( B \in B(H) \), then we say that \( g \) is Gâteaux differentiable in \( A \) and we can write \( g \in G(A) \). If this is true for any \( A \) in an open set \( S \) from \( SA_I(H) \) we write that \( g \in G(S) \).

If \( g \) is a continuous function on \( I \), by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators \( A, B \in SA_I(H) \) we consider the segment of selfadjoint operators
\[
[A, B] := \{(1-t)A+tB \mid t \in [0, 1]\}.
\]
We observe that \( A, B \in [A, B] \) and \( [A, B] \subset SA_I(H) \).

We also have [3]:

Lemma 2. Let \( f \) be an operator convex function on \( I \) and \( A, B \in SA_I(H) \), with \( A \neq B \). If \( f \in G([A, B]) \), then the auxiliary function \( \varphi_{(A,B)} \) is differentiable on \((0, 1)\) and
\[
(3.4) \quad \varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A).
\]
In particular,
\[(3.5)\] 
\[\phi'_{(A,B)}(0^+) = \nabla f_A (B - A)\]
and
\[(3.6)\] 
\[\phi'_{(A,B)}(1^-) = \nabla f_B (B - A).\]

and, see [3],

**Lemma 3.** Let \( f \) be an operator convex function on \( I \) and \( A, B \in \mathcal{SA}_I (H) \), with \( A \neq B \). If \( f \in \mathcal{G} ([A, B]) \), then for \( 0 < t_1 < t_2 < 1 \)
\[(3.7)\] 
\[\nabla f_{(1-t_1)A+t_1B} (B - A) \leq \nabla f_{(1-t_2)A+t_2B} (B - A)\]
in the operator order.

In particular,
\[(3.8)\] 
\[\nabla f_A (B - A) \leq \nabla f_{(1-t_1)A+t_1B} (B - A)\]
and
\[(3.9)\] 
\[\nabla f_{(1-t_2)A+t_2B} (B - A) \leq \nabla f_B (B - A).\]

Also, we have
\[(3.10)\] 
\[\nabla f_A (B - A) \leq \nabla f_{(1-t)A+tB} (B - A) \leq \nabla f_B (B - A)\]
for all \( t \in (0, 1) \).

We have the following gradient inequalities:

**Lemma 4.** Let \( f \) be an operator convex function on \( I \) and \( A, B \in \mathcal{SA}_I (H) \), with \( A \neq B \). If \( f \in \mathcal{G} ([A, B]) \), then
\[(3.11)\] 
\[\nabla_B f (B - A) \geq f (B) - f (A) \geq \nabla_A f (B - A).\]

**Proof.** By the properties of Bochner integral, we have
\[ f (B) - f (A) = \varphi_{(A,B)} (1) - \varphi_{(A,B)} (0) = \int_0^1 \varphi'_{(A,B)} (t) dt \]
\[ = \int_0^1 \nabla f_{(1-t)A+tB} (B - A) dt.\]

From (3.10) we have, by integration, that
\[ \nabla f_A (B - A) \leq \int_0^1 \nabla f_{(1-t)A+tB} (B - A) dt \leq \nabla f_B (B - A),\]
and the inequality (3.11) is proved. \( \square \)

Let \( T, S > 0 \). The function \( f (t) = t^{-1} \) is operator Gâteaux differentiable and the Gâteaux derivative is given by
\[(3.12)\] 
\[\nabla f_T (S) := \lim_{t \to 0} \left[ \frac{f(T + tS) - f(T)}{t} \right] = -T^{-1}ST^{-1}\]
for \( T, S > 0 \).

Using (3.11) for the operator convex function \( f (t) = t^{-1} \), we get
\[-D^{-1} (D - C) D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1} (D - C) C^{-1}\]
that is equivalent to
\[(3.13)\] 
\[D^{-1} (D - C) D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1} (D - C) C^{-1}\]
for all \(C, D > 0\).

When more assumptions are made about the operators \(A\) and \(B\), then we have the following lower and upper bounds for the difference \(\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A)\).

**Theorem 5.** Assume that \(A \geq \alpha > 0, \delta \geq B > 0\) and \(0 < m \leq B - A \leq M\) for some constants \(\alpha, \delta, m, M\). Then

\[
0 \leq m \mathcal{M}'(w, \mu) (\delta) \leq \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \leq M \mathcal{M}'(w, \mu) (\alpha),
\]

where \(\mathcal{D}'(w, \mu)\) is the derivative of \(\mathcal{M}(w, \mu)(t)\) as a function of \(t > 0\).

**Proof.** We have for \(A, B > 0\) that

\[
\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) = \int_0^\infty \lambda w(\lambda) \left[(A + \lambda)^{-1} - (B + \lambda)^{-1}\right] d\mu(\lambda).
\]

From (3.13) we get for \(C = \lambda + A\) and \(D = \lambda + B\) that

\[
(\lambda + B)^{-1} (B - A) (\lambda + B)^{-1} \leq (\lambda + A)^{-1} - (\lambda + B)^{-1} \leq (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1}
\]

for all \(\lambda \geq 0\).

If we multiply (3.16) by \(\lambda w(\lambda) \geq 0\) and integrate over \(\mu(\lambda)\) we get

\[
\int_0^\infty \lambda w(\lambda) (\lambda + B)^{-1} (B - A) (\lambda + B)^{-1} d\mu(\lambda) \leq \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \leq \int_0^\infty \lambda w(\lambda) (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} d\mu(\lambda).
\]

Since \(m \leq B - A \leq M\) hence

\[
m (\lambda + B)^{-2} \leq (\lambda + B)^{-1} (B - A) (\lambda + B)^{-1},
\]

which implies, by integration that

\[
m \int_0^\infty \lambda w(\lambda) (\lambda + B)^{-2} d\mu(\lambda) \leq \int_0^\infty \lambda w(\lambda) (\lambda + B)^{-1} (B - A) (\lambda + B)^{-1} d\mu(\lambda).
\]

Also

\[
(\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} \leq M (\lambda + A)^{-2},
\]

which implies, by integration, that

\[
\int_0^\infty \lambda w(\lambda) (\lambda + A)^{-1} (B - A) (\lambda + A)^{-1} d\mu(\lambda) \leq M \int_0^\infty \lambda w(\lambda) (\lambda + A)^{-2} d\mu(\lambda).
\]

Since \(B \leq \delta\), then \(\lambda + B \leq \lambda + \delta\) for all \(\lambda \geq 0\) which implies that \((\lambda + B)^{-1} \geq (\lambda + \delta)^{-1}\) and therefore \((\lambda + B)^{-2} \geq (\lambda + \delta)^{-2}\). Consequently

\[
m \int_0^\infty \lambda w(\lambda) (\lambda + B)^{-2} d\mu(\lambda) \geq m \int_0^\infty \lambda w(\lambda) (\lambda + \delta)^{-2} d\mu(\lambda).
\]
Also, since $A \geq \alpha > 0$, then $\lambda + A \geq \lambda + \alpha > 0$, which implies that $(\lambda + A)^{-1} \leq (\lambda + \alpha)^{-1}$, therefore $(\lambda + A)^{-2} \leq (\lambda + \alpha)^{-2}$ and

\begin{equation}
M \int_0^\infty \lambda w (\lambda) (\lambda + A)^{-2} d\mu (\lambda) \leq M \int_0^\infty \lambda w (\lambda) (\lambda + \alpha)^{-2} d\mu (\lambda).
\end{equation}

From (3.17)-(3.21) we get

\begin{equation}
(3.22) \quad m \int_0^\infty \lambda w (\lambda) (\lambda + \delta)^{-2} d\mu (\lambda) \leq M(w, \mu) (A) - M(w, \mu) (B)
\end{equation}

\begin{equation*}
\leq M \int_0^\infty \lambda w (\lambda) (\lambda + \alpha)^{-2} d\mu (\lambda).
\end{equation*}

For $h \neq 0$ small,

\begin{align*}
\frac{M(w, \mu) (t + h) - M(w, \mu) (t)}{h} &= \frac{1}{h} \int_0^\infty \lambda w (\lambda) \left( \frac{1}{t + \lambda} - \frac{1}{t + h + \lambda} \right) d\mu (\lambda) \\
&= \int_0^\infty \frac{\lambda w (\lambda)}{(t + h + \lambda) (t + \lambda)} d\mu (\lambda).
\end{align*}

By taking the limit over $h \to 0$ and using the properties of limits and integrals, we get the derivative of $M(w, \mu)$ as

\begin{equation}
(3.23) \quad M'(w, \mu) (t) = \int_0^\infty \frac{\lambda w (\lambda)}{(t + \lambda)^2} d\mu (\lambda) \geq 0, \quad t > 0.
\end{equation}

From (3.22) and (3.23) we derive (3.14). \hfill \Box

The case of operator monotone functions is as follows:

**Corollary 5.** Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and it has the representation (1.9). If $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants $\alpha, \delta, m, M$, then

\begin{equation}
0 \leq mf' (\delta) - b \leq f (B) - f (A) \leq M f' (\alpha) - b.
\end{equation}

**Proof.** From (1.9) we have

\begin{equation*}
M(w, \mu) (t) = f (t) - at - bt, \quad t > 0.
\end{equation*}

By taking the derivative, we get

\begin{equation*}
M'(w, \mu) (t) = f' (t) - b, \quad t > 0.
\end{equation*}

From (3.14) we get

\begin{equation*}
0 \leq mf' (\delta) - b \leq f (B) - f (A) - b (B - A) \leq M [f' (\alpha) - b],
\end{equation*}

which is equivalent to

\begin{equation*}
m [f' (\delta) - b] + b (B - A) \leq f (B) - f (A) \leq M [f' (\alpha) - b] + b (B - A).
\end{equation*}

Since

\begin{equation*}
bm \leq b (B - A) \leq Mb,
\end{equation*}

hence

\begin{equation*}
mf' (\delta) \leq m [f' (\delta) - b] + b (B - A)
\end{equation*}
and

\[ M \left[ f' (\alpha) - b \right] + b (B - A) \leq M f' (\alpha) \]

and the inequalities in (3.24) are proved. \( \Box \)

The case of operator convex functions is as follows:

**Corollary 6.** Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is operator convex in \((0, \infty)\) and it has the representation (1.13). If \( A \geq \alpha > 0, \delta \geq B > 0 \) and \( 0 < m \leq B - A \leq M \) for some constants \( \alpha, \delta, m, M, \) then

\[ 0 \leq m \left( \frac{f' (\delta) \delta - f (\delta) + a}{\delta^2} - c \right) + c (B - A) \]

\[ \leq f (B) B^{-1} - f (A) A^{-1} - a (B^{-1} - A^{-1}) \]

\[ \leq M \left( \frac{f' (\alpha) \alpha - f (\alpha) + a}{\alpha^2} - c \right) + c (B - A) \]

\[ \leq M \left( \frac{f' (\alpha) \alpha - f (\alpha) + a}{\alpha^2} \right). \]

**Proof.** From (1.13) we get

\[ \mathcal{M}(w, \mu) (t) = \frac{f (t) - a}{t} - b - ct, \ t > 0. \]

If we take the derivative in this equality, then we get

\[ \mathcal{M}'(w, \mu) (t) = \frac{f' (t) t - f (t) + a}{t^2} - c. \]

By (3.14) we get

\[ 0 \leq m \left( \frac{f' (\delta) \delta - f (\delta) + a}{\delta^2} - c \right) \]

\[ \leq f (B) B^{-1} - f (A) A^{-1} - f (0) (B^{-1} - A^{-1}) \]

\[ \leq M \left( \frac{f' (\alpha) \alpha - f (\alpha) + a}{\alpha^2} - c \right), \]

which is equivalent to

\[ m \left( \frac{f' (\delta) \delta - f (\delta) + a}{\delta^2} - c \right) + c (B - A) \]

\[ \leq f (B) B^{-1} - f (A) A^{-1} - f (0) (B^{-1} - A^{-1}) \]

\[ \leq M \left( \frac{f' (\alpha) \alpha - f (\alpha) + a}{\alpha^2} - c \right) + c (B - A). \]

Since

\[ m \left( \frac{f' (\delta) \delta - f (\delta) + a}{\delta^2} - c \right) + c (B - A) \]

\[ \geq m \left( \frac{f' (\delta) \delta - f (\delta) + a}{\delta^2} - c \right) + cm = m \left( \frac{f' (\delta) \delta - f (\delta) + a}{\delta^2} \right) \geq 0 \]
and
\[
M \left[ \frac{f'(\alpha) \alpha - f(\alpha) + a}{\alpha^2} - c \right] + c(B - A) \\
\leq M \left[ \frac{f'(\alpha) \alpha - f(\alpha) + a}{\alpha^2} - c \right] + cM = M \left( \frac{f'(\alpha) \alpha - f(\alpha) + a}{\alpha^2} \right),
\]
the proof of (3.25) is thus completed. \qed

Remark 2. If \( f : [0, \infty) \to \mathbb{R} \) is operator convex in \([0, \infty)\) then we can take \( a = f(0) \) and by (3.25) we get
\[
0 \leq m \left( \frac{f'(\delta) \delta - f(\delta) + f(0)}{\delta^2} \right) \\
\leq f(B) B^{-1} - f(A) A^{-1} - f(0) (B^{-1} - A^{-1}) \\
\leq M \left( \frac{f'(\alpha) \alpha - f(\alpha) + f(0)}{\alpha^2} - c \right) + c(B - A) \\
\leq M \left( \frac{f'(\alpha) \alpha - f(\alpha) + f(0)}{\alpha^2} \right).
\]

It is well known that, if \( P \succeq 0 \), then
\[
\|P_{x, y}\|^2 \leq \langle Px, x \rangle \langle Py, y \rangle
\]
for all \( x, y \in H \).

Therefore, if \( T > 0 \), then
\[
0 \leq \langle x, x \rangle^2 = \langle T^{-1}Tx, x \rangle^2 = \langle Tx, T^{-1}x \rangle^2 \\
\leq \langle Tx, x \rangle \langle TT^{-1}x, T^{-1}x \rangle = \langle Tx, x \rangle \langle x, T^{-1}x \rangle
\]
for all \( x \in H \).

If \( x \in H \), \( \|x\| = 1 \), then
\[
1 \leq \langle Tx, x \rangle \langle x, T^{-1}x \rangle \leq \langle Tx, x \rangle \sup_{\|x\|=1} \langle x, T^{-1}x \rangle = \langle Tx, x \rangle \|T^{-1}\|,
\]
which implies the first inequality below
\[
\|T^{-1}\|^{-1} \leq T \leq \|T\|.
\]
The second inequality is obvious.

Proposition 1. Let \( B > A > 0 \). Then
\[
0 \leq \left\| (B - A)^{-1} \right\|^{-1} M'(w, \mu)(\|B\|) \leq M(w, \mu)(B) - M(w, \mu)(A) \\
\leq \|B - A\| M'(w, \mu) \left( \|A^{-1}\|^{-1} \right).
\]

Proof. Since, by (3.27), \( A \geq \|A^{-1}\|^{-1}, \|B\| \geq B \) and \( \|B - A\| \geq B - A \geq \|B - A\|^{-1} \) then by (3.14) for \( \alpha = \|A^{-1}\|^{-1}, \delta = \|B\|, m = \left\| (B - A)^{-1} \right\|^{-1} \) and \( M = \|B - A\| \) we get (3.28). \qed

In the case of operator monotone functions we have:
Corollary 7. Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and $B > A > 0$. Then

$$0 \leq \frac{f'(\|B\|)}{(B-A)^{-1}} \leq f(B) - f(A) \leq \|B - A\| f'(\|A^{-1}\|^{-1}).$$

We also have:

Corollary 8. Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ and $B > A > 0$. Then

$$0 \leq \frac{1}{\|B\|^2 \|B - A\|^{-1}} (f'(\|B\|) \|B\| - f(\|B\|) + f(0))$$

$$\leq f(B) B^{-1} - f(A) A^{-1} - f(0) (B^{-1} - A^{-1})$$

$$\leq \|B - A\| \|A^{-1}\|^2 \left( f'(\|A^{-1}\|^{-1}) \|A^{-1}\|^{-1} - f(\|A^{-1}\|^{-1}) + f(0) \right).$$

4. Some Examples

We consider the operator monotone function $f(t) = t^r$, $r \in (0, 1]$, then for all $A$, $B \geq 0$ we have

$$B^r - A^r = \frac{\sin(r\pi)}{\pi}$$

$$\times \int_0^\infty \left( \int_0^1 (\lambda + (1-t) A + tB)^{-1} (B - A) (\lambda + (1-t) A + tB)^{-1} dt \right) \lambda^r d\lambda,$$

which proves in one line the Löwner-Heinz inequality $B^r \geq A^r$ if $B \geq A$.

For logarithmic function we have the following representation for the difference:

Proposition 2. For all $A$, $B > 0$ we have

$$\ln B - \ln A = \int_0^{\infty} \int_0^1 (B - A) (\lambda + (1-t) A + tB)^{-1} \lambda^r d\lambda.$$

Proof. We have from (1.18) for $A$, $B > 0$ that

$$\ln B - \ln A = \int_0^{\infty} \frac{1}{\lambda + 1} \left[ (B - 1) (\lambda + B)^{-1} - (A - 1) (\lambda + A)^{-1} \right] d\lambda.$$

Since

$$(B - 1) (\lambda + B)^{-1} - (A - 1) (\lambda + A)^{-1} = B (\lambda + B)^{-1} - A (\lambda + A)^{-1} - (\lambda + B)^{-1} - (\lambda + A)^{-1}$$

and

$$B (\lambda + B)^{-1} - A (\lambda + A)^{-1} = (B + \lambda - \lambda) (\lambda + B)^{-1} - (A + \lambda - \lambda) (\lambda + A)^{-1}$$

$$= 1 - \lambda (\lambda + B)^{-1} - 1 + \lambda (\lambda + A)^{-1} = \lambda (\lambda + A)^{-1} - \lambda (\lambda + B)^{-1},$$
hence
\[(B - 1) (\lambda + B)^{-1} - (A - 1) (\lambda + A)^{-1}
= \lambda (\lambda + A)^{-1} - (\lambda + B)^{-1} - (\lambda + B)^{-1} - (\lambda + A)^{-1}
= (\lambda + 1) \left[ (\lambda + A)^{-1} - (\lambda + B)^{-1} \right]
\]
and by (4.3) we get
(4.4) \[\ln B - \ln A = \int_0^\infty \left[ (\lambda + A)^{-1} - (\lambda + B)^{-1} \right] d\lambda.\]
Since, by (2.7) we have
(4.5) \[(\lambda + A)^{-1} - (\lambda + B)^{-1}
= \int_0^1 (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} dt,
\]
for all \(\lambda \geq 0\), hence by (4.4) and (4.5) we get (4.2).

\[\text{Assume that } A \geq \alpha > 0, \delta \geq B > 0 \text{ and } 0 < m \leq B - A \leq M \text{ for some constants } \alpha, \delta, m, M. \text{ Then by (3.24) we get}\]

(4.6) \[r m \delta^{-1} \leq B^r - A^r \leq r M \alpha^{r-1}.\]
If \(B > A > 0\), then
(4.7) \[\frac{r}{\| (B - A)^{-1} \|^{1-r} \| B \|^{1-r}} \leq B^r - A^r \leq r \| B - A \| \| A^{-1} \|^{1-r}.\]

The function \(f(t) = \ln t, t > 0\) is operator monotone on \((0, \infty)\) and if we assume that \(A \geq \alpha > 0, \delta \geq B > 0 \text{ and } 0 < m \leq B - A \leq M \text{ for some constants } \alpha, \delta, m, M, \text{ then by (3.24) we get}\]

(4.8) \[\frac{m}{\delta} \leq \ln B - \ln A \leq \frac{M}{\alpha}.\]
If \(B > A > 0\), then
(4.9) \[\frac{1}{\| (B - A)^{-1} \| \| B \|} \leq \ln B - \ln A \leq \| B - A \| \| A^{-1} \|.\]

We consider the operator convex function \(f(t) = -\ln (t + 1) \text{ on } [0, \infty).\) Then by (3.26),

(4.10) \[0 \leq m \left( \frac{(\delta + 1) \ln (\delta + 1) - \delta}{\delta^2 (\delta + 1)} \right) \leq B^{-1} \ln (B + 1) - A^{-1} \ln (A + 1)
\leq M \left( \frac{(\alpha + 1) \ln (\alpha + 1) - \alpha}{\alpha^2 (\alpha + 1)} \right),\]
where \(A \geq \alpha > 0, \delta \geq B > 0 \text{ and } 0 < m \leq B - A \leq M \text{ for some constants } \alpha, \delta, m, M.\)

We define the upper incomplete Gamma function as \([11]\]
\[\Gamma(a, z) := \int_z^\infty t^{a-1} e^{-t} dt,\]
which for \( z = 0 \) gives \textit{Gamma function}

\[
\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \quad \text{for} \quad \Re a > 0.
\]

We have the integral representation [12]

\[
(4.11) \quad \Gamma(a, z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt
\]

for \( \Re a < 1 \) and \( |\text{ph} z| < \pi \).

Now, we consider the weight \( w_{-a e^{-\cdot}} (\lambda) := \lambda^{-a} e^{-\lambda} \) for \( \lambda > 0 \). Then by (4.11) we have

\[
(4.12) \quad D(w_{-a e^{-\cdot}})(t) = \int_0^\infty \frac{\lambda^{-a} e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t \Gamma(a, t)
\]

for \( a < 1 \) and \( t > 0 \).

For \( a = 0 \) in (4.12) we get

\[
(4.13) \quad D\left( w_{e^{-\cdot}} \right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1) e^t \Gamma(0, t) = e^t E_1(t)
\]

for \( t > 0 \), where

\[
E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du.
\]

We then have

\[
(4.14) \quad M\left( w_{-a e^{-\cdot}} \right)(T) = \Gamma(1-a) T^{1-a} \exp(T) \Gamma(a, T)
\]

for \( a < 1 \) and

\[
(4.15) \quad M\left( w_{e^{-\cdot}} \right)(T) = T \exp(T) E_1(T)
\]

for \( T > 0 \).

For all \( A, B > 0 \) we have the representation

\[
(4.16) \quad M\left( w_{-a e^{-\cdot}} \right)(B) - M\left( w_{-a e^{-\cdot}} \right)(A)
= \int_0^\infty \left( \int_0^1 (\lambda + (1-t) A + tB)^{-1}(B-A)(\lambda + (1-t) A + tB)^{-1}dt \right) \times \lambda^{-a} e^{-\lambda}d\mu(\lambda).
\]

If \( B \geq A > 0 \), then

\[
B^{1-a} \exp(B) \Gamma(a, B) \geq A^{1-a} \exp(A) \Gamma(a, A),
\]

namely the function \( g_a(t) := t^{1-a} \exp(t) \Gamma(a, t) \) is operator monotone on \( (0, \infty) \).

Since \( E_1^t(t) = -\frac{e^{-t}}{t} \), \( t > 0 \), then

\[
M'(w_{e^{-\cdot}})(t) = (t \exp(t))' E_1(t) + t \exp(t) E_1'(t)
= (\exp t + t \exp t) E_1(t) - t \exp(t) \left( e^{-t} \frac{1}{t} \right)
= (1 + t) \exp t E_1(t) - 1
\]

for \( t > 0 \).
From (3.24) we get

\begin{equation}
0 \leq m [(1 + \delta) \exp \delta E_1 (\delta) - 1] \leq B \exp (B) E_1 (B) - A \exp (A) E_1 (A) \\
\leq M [(1 + \alpha) \exp \alpha E_1 (\alpha) - 1],
\end{equation}

when \( A \geq \alpha > 0, \delta \geq B > 0 \) and \( 0 < m \leq B - A \leq M \) for some constants \( \alpha, \delta, m, M \).

References


[6] T. Furuta, Precise lower bound of \( f(A) - f(B) \) for \( A > B > 0 \) and non-constant operator monotone function \( f \) on \( [0, \infty) \). *J. Math. Inequal.* 9 (2015), no. 1, 47–52.


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