BOUNDS IN TERMS OF DERIVATIVE FOR MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M. In this paper we show that

$$0 \leq m\mathcal{M}'(w, \mu)(\delta) \leq \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \leq M\mathcal{M}'(w, \mu)(\alpha),$$

where $\mathcal{M}'(w, \mu)(t)$ is the derivative of $\mathcal{M}(w, \mu)(t)$ as a function of $t > 0$.

If $f : (0, \infty) \to \mathbb{R}$ is operator monotone on $(0, \infty)$, then

$$0 \leq mf'(\delta) \leq f(B) - f(A) \leq Mf'(\alpha).$$

In particular, we have the power inequality

$$0 < rm\delta^{r-1} \leq B^r - A^r \leq M\alpha^{r-1}$$

and the logarithmic inequality

$$0 < \frac{m}{\delta} \leq \ln B - \ln A \leq \frac{M}{\alpha}.$$

Some examples for operator convex functions as well as for integral transforms $\mathcal{M}(\cdot, \cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

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By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.2}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.3}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.3) exists for all $t > 0$.

For μ, the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \ t > 0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1) \mathcal{D}(w_{\ln})(t), \ t > 0. \tag{1.6}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.8}$$

for $T > 0$.

A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.9}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty. \tag{1.10}$$

If f is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.9).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left(\left(1-\lambda\right)A+\lambda B\right)\leq\left(\geq\right)\left(1-\lambda\right)f\left(A\right)+\lambda f\left(B\right) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f\left(t\right)=a+bt+ct^{2}+\int_{0}^{\infty}\frac{t^{2}\lambda}{t+\lambda}d\mu\left(\lambda\right), \tag{1.11}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f\left(0\right)$ and $b=f_{+}^{\prime}\left(0\right)$, the right derivative, in (1.11).

For a continuous and positive function $w\left(\lambda\right)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w,\mu)\left(t\right):=t\mathcal{D}(w,\mu)\left(t\right),\ t>0. \tag{1.12}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{M}(w,\mu)\left(t\right):=t\mathcal{D}(w,\mu)\left(t\right)&=\int_{0}^{\infty}w\left(\lambda\right)t\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right)\\&=\int_{0}^{\infty}w\left(\lambda\right)\left(t+\lambda-\lambda\right)\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right)\\&=\int_{0}^{\infty}w\left(\lambda\right)\left[1-\lambda\left(t+\lambda\right)^{-1}\right]d\mu\left(\lambda\right).\end{aligned} \tag{1.13}$$

If $\int_{0}^{\infty}w\left(\lambda\right)d\mu\left(\lambda\right)<\infty$, then

$$\mathcal{M}(w,\mu)\left(t\right)=\int_{0}^{\infty}w\left(\lambda\right)d\mu\left(\lambda\right)-\mathcal{D}(\ell w,\mu)\left(t\right), \tag{1.14}$$

where $\ell\left(t\right)=t$, $t>0$.

Consider the kernel $e_{-a}\left(\lambda\right):=\exp\left(-a\lambda\right)$, $\lambda\geq 0$ and $a>0$. Then after some calculations, we get

$$\mathcal{D}(e_{-a})\left(t\right)=\int_{0}^{\infty}\frac{\exp\left(-a\lambda\right)}{t+\lambda}d\lambda=E_{1}\left(at\right)\exp\left(at\right),\ t\geq 0$$

and

$$\int_{0}^{\infty}w\left(\lambda\right)d\lambda=\int_{0}^{\infty}\exp\left(-a\lambda\right)d\lambda=\frac{1}{a},$$

where

$$E_{1}\left(t\right):=\int_{t}^{\infty}\frac{e^{-u}}{u}du.$$

This gives that

$$\mathcal{M}(e_{-a})\left(t\right)=t\mathcal{D}(w,\mu)\left(t\right)=tE_{1}\left(at\right)\exp\left(at\right),\ t\geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t)=\int_0^\infty \frac{\lambda\exp(-a\lambda)}{t+\lambda}d\lambda=\frac{1}{a}-tE_1(at)\exp(at)$$

for $t>0$.

One observes that

$$\mathcal{M}(e_{-a})(t)=\int_0^\infty w(\lambda)\,d\lambda-\mathcal{D}(\ell e_{-a},\mu)(t),\ t>0$$

and the equality (1.14) is verified in this case.

If we take $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda=\infty$ and the equality (1.14) does not hold in this case.

For all $T>0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T)=T\mathcal{D}(w,\mu)(T)=\int_0^\infty w(\lambda)\left[1-\lambda(T+\lambda)^{-1}\right]d\mu(\lambda). \tag{1.15}$$

This gives the representation

$$T^r=\frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T),$$

where $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$ and μ is the usual Lebesgue norm. Also, from (1.6) we have the representation

$$T\ln T=(T-1)\mathcal{M}(w_{\ln})(T),\ T>0, \tag{1.16}$$

where $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$.

Assume that $A\geq\alpha>0$, $\delta\geq B>0$ and $0<m\leq B-A\leq M$ for some constants α, δ, m, M. In this paper we show that

$$0\leq m\mathcal{M}'(w,\mu)(\delta)\leq\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\leq M\mathcal{M}'(w,\mu)(\alpha),$$

where $\mathcal{M}'(w,\mu)(t)$ is the derivative of $\mathcal{M}(w,\mu)(t)$ as a function of $t>0$.

If $f:(0,\infty)\to\mathbb{R}$ is operator monotone on $(0,\infty)$, then

$$0\leq mf'(\delta)\leq f(B)-f(A)\leq Mf'(\alpha).$$

In particular, we have the power inequality

$$0<rm\delta^{r-1}\leq B^r-A^r\leq M\alpha^{r-1}$$

and the logarithmic inequality

$$0<\frac{m}{\delta}\leq\ln B-\ln A\leq\frac{M}{\alpha}.$$

Some examples for operator convex functions as well as for integral transforms $\mathcal{D}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

Let f be an operator convex function on the interval of real numbers I. For A, $B\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t):=f((1-t)A+tB). \tag{2.1}$$

For $x \in H$ we can also consider the auxiliary function $\varphi_{(A,B);x} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t) := \left\langle \varphi_{(A,B)}(t)\, x, x \right\rangle = \left\langle f\left((1-t)A + tB\right) x, x \right\rangle . \tag{2.2}$$

We have the following basic fact [2]:

Lemma 1. *Let f be an operator convex function on I. For any A, $B \in \mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $A, B \in \mathcal{SA}_I(H)$ and $x \in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

A continuous function $g : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$ along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s\to 0} \frac{g(A+sB) - g(A)}{s} \in \mathcal{B}(H). \tag{2.3}$$

If the limit (2.3) exists for all $B \in \mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators A, $B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that A, $B \in [A,B]$ and $[A,B] \subset \mathcal{SA}_I(H)$.

We also have [2]:

Lemma 2. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB}(B-A). \tag{2.4}$$

In particular,

$$\varphi'_{(A,B)}(0+) = \nabla f_A(B-A) \tag{2.5}$$

and

$$\varphi'_{(A,B)}(1-) = \nabla f_B(B-A). \tag{2.6}$$

and, see [2],

Lemma 3. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A,B])$, then for $0 < t_1 < t_2 < 1$*

$$\nabla f_{(1-t_1)A+t_1B}(B-A) \le \nabla f_{(1-t_2)A+t_2B}(B-A) \tag{2.7}$$

in the operator order.

In particular,

$$\nabla f_A(B-A) \le \nabla f_{(1-t_1)A+t_1B}(B-A) \tag{2.8}$$

and

$$\nabla f_{(1-t_2)A+t_2B}(B-A) \le \nabla f_B(B-A). \tag{2.9}$$

Also, we have

$$\nabla f_A(B-A) \le \nabla f_{(1-t)A+tB}(B-A) \le \nabla f_B(B-A) \tag{2.10}$$

for all $t \in (0,1)$.

We have the following gradient inequalities:

Lemma 4. *Let* f *be an operator convex function on* I *and* A, $B \in \mathcal{SA}_I(H)$, *with* $A \ne B$. *If* $f \in \mathcal{G}([A,B])$, *then*

$$\nabla_B f(B-A) \ge f(B) - f(A) \ge \nabla_A f(B-A). \tag{2.11}$$

Proof. By the properties of Bochner integral, we have

$$\begin{aligned} f(B) - f(A) &= \varphi_{(A,B)}(1) - \varphi_{(A,B)}(0) = \int_0^1 \varphi'_{(A,B)}(t)\,dt \\ &= \int_0^1 \nabla f_{(1-t)A+tB}(B-A)\,dt. \end{aligned}$$

From (2.10) we have, by integration, that

$$\nabla f_A(B-A) \le \int_0^1 \nabla f_{(1-t)A+tB}(B-A)\,dt \le \nabla f_B(B-A),$$

and the inequality (2.11) is proved. □

Let T, $S > 0$. The function $f(t) = t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \to 0}\left[\frac{f(T+tS) - f(T)}{t}\right] = -T^{-1}ST^{-1} \tag{2.12}$$

for T, $S > 0$.

Using (2.12) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1} \ge D^{-1} - C^{-1} \ge -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$D^{-1}(D-C)D^{-1} \le C^{-1} - D^{-1} \le C^{-1}(D-C)C^{-1} \tag{2.13}$$

for all C, $D > 0$.

Theorem 3. *Assume that* $A \ge \alpha > 0$, $\delta \ge B > 0$ *and* $0 < m \le B - A \le M$ *for some constants* α, δ, m, M. *Then*

$$0 \le m\mathcal{M}'(w,\mu)(\delta) \le \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) \le M\mathcal{M}'(w,\mu)(\alpha). \tag{2.14}$$

Proof. From (1.15) we have for all A, $B \ge 0$ that

$$\begin{aligned} &\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) \\ &= \int_0^\infty w(\lambda)\left[1 - \lambda(B+\lambda)^{-1}\right]d\mu(\lambda) \\ &- \int_0^\infty w(\lambda)\left[1 - \lambda(A+\lambda)^{-1}\right]d\mu(\lambda) \\ &= \int_0^\infty \lambda w(\lambda)\left[(A+\lambda)^{-1} - (B+\lambda)^{-1}\right]d\mu(\lambda). \end{aligned} \tag{2.15}$$

From (2.13) we get for $C=\lambda+A$ and $D=\lambda+B$ that

$$(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1} \leq (\lambda+A)^{-1}-(\lambda+B)^{-1} \leq (\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} \tag{2.16}$$

for all $\lambda \geq 0$.

If we multiply (2.16) by $\lambda w(\lambda) \geq 0$ and integrate over $\mu(\lambda)$ we get

$$\begin{aligned} &\int_0^\infty \lambda w(\lambda)(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1} d\mu(\lambda) \\ &\leq \mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A) \\ &\leq \int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} d\mu(\lambda). \end{aligned} \tag{2.17}$$

Since $m \leq B-A \leq M$, hence

$$m(\lambda+B)^{-2} \leq (\lambda+B)^{-1}(B-A)(\lambda+B)^{-1},$$

which implies, by integration that

$$\begin{aligned} &m\int_0^\infty \lambda w(\lambda)(\lambda+B)^{-2} d\mu(\lambda) \\ &\leq \int_0^\infty \lambda w(\lambda)(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1} d\mu(\lambda). \end{aligned} \tag{2.18}$$

Also

$$(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} \leq M(\lambda+A)^{-2},$$

which implies, by integration, that

$$\begin{aligned} &\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} d\mu(\lambda) \\ &\leq M\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-2} d\mu(\lambda). \end{aligned} \tag{2.19}$$

Since $B \leq \delta$, then $\lambda+B \leq \lambda+\delta$ for all $\lambda \geq 0$, which implies that $(\lambda+B)^{-1} \geq (\lambda+\delta)^{-1}$ and therefore $(\lambda+B)^{-2} \geq (\lambda+\delta)^{-2}$. Consequently

$$m\int_0^\infty \lambda w(\lambda)(\lambda+B)^{-2} d\mu(\lambda) \geq m\int_0^\infty \lambda w(\lambda)(\lambda+\delta)^{-2} d\mu(\lambda). \tag{2.20}$$

Also, since $A \geq \alpha > 0$, then $\lambda+A \geq \lambda+\alpha > 0$, which implies that $(\lambda+A)^{-1} \leq (\lambda+\alpha)^{-1}$, therefore $(\lambda+A)^{-2} \leq (\lambda+\alpha)^{-2}$ and

$$M\int_0^\infty \lambda w(\lambda)(\lambda+A)^{-2} d\mu(\lambda) \leq M\int_0^\infty \lambda w(\lambda)(\lambda+\alpha)^{-2} d\mu(\lambda). \tag{2.21}$$

From (2.17)-(2.21) we get

$$\begin{aligned} m\int_0^\infty \lambda w(\lambda)(\lambda+\delta)^{-2} d\mu(\lambda) &\leq \mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A) \\ &\leq M\int_0^\infty \lambda w(\lambda)(\lambda+\alpha)^{-2} d\mu(\lambda). \end{aligned} \tag{2.22}$$

For $h \neq 0$ small,

$$\frac{\mathcal{M}(w,\mu)(t+h)-\mathcal{M}(w,\mu)(t)}{h} = \frac{1}{h}\int_0^\infty \left(\frac{\lambda w(\lambda)}{t+\lambda}-\frac{\lambda w(\lambda)}{t+h+\lambda}\right) d\mu(\lambda) = \int_0^\infty \frac{\lambda w(\lambda)}{(t+h+\lambda)(t+\lambda)} d\mu(\lambda).$$

By taking the limit over $h \to 0$ and using the properties of limits and integrals, we get the derivative of $\mathcal{M}(w,\mu)$ as

$$\mathcal{M}'(w,\mu)(t) = \int_0^\infty \frac{\lambda w(\lambda)}{(t+\lambda)^2} d\mu(\lambda) \leq 0,\ t>0. \tag{2.23}$$

From (2.22) and (2.23) we derive (2.14). □

We know that for $T>0$, we have the operator inequalities

$$0 < \left\|T^{-1}\right\|^{-1} \leq T \leq \|T\|. \tag{2.24}$$

Indeed, it is well known that, if $P \geq 0$, then

$$|\langle Px,y\rangle|^2 \leq \langle Px,x\rangle \langle Py,y\rangle$$

for all x, $y \in H$.

Therefore, if $T>0$, then

$$\begin{aligned} 0 &\leq \langle x,x\rangle^2 = \left\langle T^{-1}Tx,x\right\rangle^2 = \left\langle Tx,T^{-1}x\right\rangle^2 \\ &\leq \langle Tx,x\rangle \left\langle TT^{-1}x,T^{-1}x\right\rangle = \langle Tx,x\rangle \left\langle x,T^{-1}x\right\rangle \end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\|=1$, then

$$1 \leq \langle Tx,x\rangle \left\langle x,T^{-1}x\right\rangle \leq \langle Tx,x\rangle \sup_{\|x\|=1} \left\langle x,T^{-1}x\right\rangle = \langle Tx,x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} 1_H \leq T. \tag{2.25}$$

The second inequality in (2.25) is obvious.

Corollary 1. *If A, $B>0$ and $B-A>0$, then*

$$\begin{aligned} 0 &\leq \left\|(B-A)^{-1}\right\|^{-1} \mathcal{M}'(w,\mu)(\|B\|) \leq \mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A) \\ &\leq \|B-A\| \mathcal{M}'(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right). \end{aligned} \tag{2.26}$$

Proof. Since $A \geq \left\|A^{-1}\right\|^{-1} = \alpha > 0$, $\delta = \|B\| \geq B > 0$ and

$$0 < m = \left\|(B-A)^{-1}\right\|^{-1} \leq B-A \leq \|B-A\| = M,$$

then by (2.14) we get (2.25). □

Proposition 1. *Let $f:(0,\infty)\to\mathbb{R}$ be operator monotone in $(0,\infty)$ that has the representation (1.9). If $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B-A \leq M$ for some constants α, δ, m, M, then*

$$\begin{aligned} 0 &\leq mf'(\delta) \leq m[f'(\delta)-b]+b(B-A) \\ &\leq f(B)-f(A) \leq M[f'(\alpha)-b]+b(B-A) \leq Mf'(\alpha). \end{aligned} \tag{2.27}$$

Proof. From (1.9) we have

$$\mathcal{M}(\ell,\mu)(t) = f(t) - a - bt,\ t > 0 \tag{2.28}$$

where $a \in R$, $b \geq 0$ and μ is a positive measure on $(0,\infty)$.

Then

$$\mathcal{M}'(\ell,\mu)(t) = f'(t) - b,\ t > 0$$

and by (2.14) we get

$$m[f'(\delta) - b] \leq f(B) - f(A) - b(B-A) \leq M[f'(\alpha) - b],$$

namely

$$m[f'(\delta) - b] + b(B-A) \leq f(B) - f(A) \leq M[f'(\alpha) - b] + b(B-A). \tag{2.29}$$

Since

$$m[f'(\delta) - b] + b(B-A) = mf'(\delta) + b(B-A-m) \geq mf'(\delta) \tag{2.30}$$

and

$$M[f'(\alpha) - b] + b(B-A) \leq Mf'(\alpha) + b(B-A-M) \leq Mf'(\alpha), \tag{2.31}$$

hence by (2.29)-(2.31) we get (2.27). □

Corollary 2. *Let $f : (0,\infty) \to \mathbb{R}$ be operator monotone in $(0,\infty)$ that has the representation (1.9). If A, $B > 0$ and $B - A > 0$, then*

$$\begin{aligned} 0 &\leq \left\|(B-A)^{-1}\right\|^{-1} f'(\|B\|) \\ &\leq \left\|(B-A)^{-1}\right\|^{-1} [f'(\|B\|) - b] + b(B-A) \\ &\leq f(B) - f(A) \leq \|B-A\| \left[f'\left(\left\|A^{-1}\right\|^{-1}\right) - b\right] + b(B-A) \\ &\leq \|B-A\| f'\left(\left\|A^{-1}\right\|^{-1}\right). \end{aligned} \tag{2.32}$$

Remark 1. *If $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M, then by (2.27) for the operator monotone function $f(t) = t^r$, $r \in (0,1]$ we have*

$$0 < rm\delta^{r-1} \leq B^r - A^r \leq M\alpha^{r-1}. \tag{2.33}$$

If A, $B > 0$ and $B - A > 0$, then

$$0 < \frac{r\|B\|^{r-1}}{\left\|(B-A)^{-1}\right\|} \leq B^r - A^r \leq r\|B-A\| \left\|A^{-1}\right\|^{1-r}. \tag{2.34}$$

If we consider now the operator monotone function $f(t) = \ln t$ and assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$, then

$$0 < \frac{m}{\delta} \leq \ln B - \ln A \leq \frac{M}{\alpha}. \tag{2.35}$$

If A, $B > 0$ and $B - A > 0$, then

$$0 < \frac{1}{\left\|(B-A)^{-1}\right\| \|B\|} \leq \ln B - \ln A \leq \|B-A\| \left\|A^{-1}\right\|. \tag{2.36}$$

The case of operator convex functions is as follows:

Proposition 2. *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex in $[0,\infty)$ that has the representation (1.11). If $A\geq\alpha>0$, $\delta\geq B>0$ and $0<m\leq B-A\leq M$ for some constants α, δ, m, M, then*

$$
\begin{aligned}
0 &\leq m\left(\frac{f'(\delta)\delta-f(\delta)+f(0)}{\delta^2}\right) \\
&\leq m\left[\frac{f'(\delta)\delta-f(\delta)+f(0)}{\delta^2}-c\right]+c(B-A) \\
&\leq f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right) \\
&\leq M\left[\frac{f'(\alpha)\alpha-f(\alpha)+f(0)}{\alpha^2}-c\right]+c(B-A) \\
&\leq M\left(\frac{f'(\alpha)\alpha-f(\alpha)+f(0)}{\alpha^2}\right).
\end{aligned}
\tag{2.37}
$$

Proof. From (1.11) we have

$$\mathcal{M}(\ell,\mu)(t)=\frac{f(t)-f(0)}{t}-f'_+(0)-ct,\ t>0,$$

where $c\geq0$ and μ is a positive measure on $(0,\infty)$.

Then

$$\mathcal{M}'(\ell,\mu)(t)=\frac{f'(t)t-f(t)+f(0)}{t^2}-c,\ t>0$$

and by (2.14) we get

$$
\begin{aligned}
&m\left[\frac{f'(\delta)\delta-f(\delta)+f(0)}{\delta^2}-c\right] \\
&\leq f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A) \\
&\leq M\left[\frac{f'(\alpha)\alpha-f(\alpha)+f(0)}{\alpha^2}-c\right],
\end{aligned}
$$

namely

$$
\begin{aligned}
&m\left[\frac{f'(\delta)\delta-f(\delta)+f(0)}{\delta^2}-c\right]+c(B-A) \\
&\leq f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right) \\
&\leq M\left[\frac{f'(\alpha)\alpha-f(\alpha)+f(0)}{\alpha^2}-c\right]+c(B-A).
\end{aligned}
\tag{2.38}
$$

Since

$$
\begin{aligned}
&m\left[\frac{f'(\delta)\delta-f(\delta)+f(0)}{\delta^2}-c\right]+c(B-A) \\
&=m\left(\frac{f'(\delta)\delta-f(\delta)+f(0)}{\delta^2}\right)+c(B-A-m) \\
&\geq m\left(\frac{f'(\delta)\delta-f(\delta)+f(0)}{\delta^2}\right)
\end{aligned}
\tag{2.39}
$$

and

$$M\left[\frac{f'(\alpha)\alpha-f(\alpha)+f(0)}{\alpha^2}-c\right]+c(B-A) \tag{2.40}$$
$$=M\left(\frac{f'(\alpha)\alpha-f(\alpha)+f(0)}{\alpha^2}\right)+c(B-A-M)$$
$$\leq M\left(\frac{f'(\alpha)\alpha-f(\alpha)+f(0)}{\alpha^2}\right),$$

hence by (2.38)-(2.40) we derive (2.37). □

Corollary 3. *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex in $[0,\infty)$ that has the representation (1.11). If A, $B>0$ and $B-A>0$, then*

$$0\leq\frac{f'(\|B\|)\|B\|-f(\|B\|)+f(0)}{\|B\|^2\left\|(B-A)^{-1}\right\|} \tag{2.41}$$
$$\leq\left\|(B-A)^{-1}\right\|^{-1}\left[\frac{f'(\|B\|)\|B\|-f(\|B\|)+f(0)}{\|B\|^2}-c\right]+c(B-A)$$
$$\leq f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)$$
$$\leq\|B-A\|\left[\frac{f'\left(\|A^{-1}\|^{-1}\right)\|A^{-1}\|^{-1}-f\left(\|A^{-1}\|^{-1}\right)+f(0)}{\|A^{-1}\|^{-2}}-c\right]$$
$$+c(B-A)$$
$$\leq\|A^{-1}\|^2\|B-A\|\left(\frac{f'\left(\|A^{-1}\|^{-1}\right)}{\|A^{-1}\|}-f\left(\|A^{-1}\|^{-1}\right)+f(0)\right).$$

Remark 2. *The function $f(t)=-\ln(t+1)$ is operator convex with $f(0)=0$. If $A\geq\alpha>0$, $\delta\geq B>0$ and $0<m\leq B-A\leq M$, then*

$$0\leq m\left(\frac{\ln(\delta+1)-\delta\ln(\delta+1)}{\delta^2}\right)\leq A^{-1}\ln(A+1)-B^{-1}\ln(B+1) \tag{2.42}$$
$$\leq M\left(\frac{\ln(\alpha+1)-\alpha\ln(\alpha+1)}{\alpha^2}\right).$$

If A, $B>0$ and $B-A>0$, then

$$0\leq\frac{\ln(\|B\|+1)-\|B\|(\|B\|+1)^{-1}}{\|B\|^2\left\|(B-A)^{-1}\right\|} \tag{2.43}$$
$$\leq A^{-1}\ln(A+1)-B^{-1}\ln(B+1)$$
$$\leq\|A^{-1}\|^2\|B-A\|\left[\ln\left(\|A^{-1}\|^{-1}+1\right)-\frac{1}{\|A^{-1}\|(1+\|A^{-1}\|)}\right].$$

3. More Examples

We define the *upper incomplete Gamma function* as [10]

$$\Gamma(a,z):=\int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a):=\int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a>0.$$

We have the integral representation [11]

$$\Gamma(a,z)=\frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{3.1}$$

for $\operatorname{Re} a<1$ and $|\operatorname{ph} z|<\pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda):=\lambda^{-a}e^{-\lambda}$ for $\lambda>0$. Then by (3.1) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=\int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{3.2}$$

for $a<1$ and $t>0$.

For $a=0$ in (3.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t)=\int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1)e^t\Gamma(0,t)=e^tE_1(t) \tag{3.3}$$

for $t>0$, where

$$E_1(t):=\int_t^\infty \frac{e^{-u}}{u}du.$$

We then have

$$\mathcal{M}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T)=\Gamma(1-a)T^{1-a}\exp(T)\Gamma(a,T) \tag{3.4}$$

for $a<1$ and

$$\mathcal{M}\left(w_{e^{-\cdot}}\right)(T)=T\exp(T)E_1(T) \tag{3.5}$$

for $T>0$.

Consider $\mathcal{M}\left(w_{e^{-\cdot}}\right)(t)=t\exp(t)E_1(t)$, $t>0$. Since $E_1'(t)=-\frac{e^{-t}}{t}$, $t>0$, then

$$\begin{aligned}\mathcal{M}'\left(w_{e^{-\cdot}}\right)(t) &= (t\exp(t)E_1(t))'=(t\exp(t))'E_1(t)+t\exp(t)E_1'(t)\\ &= (1+t)\exp(t)-1,\ t>0.\end{aligned}$$

By Theorem 3 we get

$$\begin{aligned}0&\le m\left[(1+\delta)\exp(\delta)-1\right]\le B\exp(B)E_1(B)-A\exp(A)E_1(A)\\ &\le M\left[(1+\alpha)\exp(\alpha)-1\right]\end{aligned} \tag{3.6}$$

if $A\ge\alpha>0$, $\delta\ge B>0$ and $0<m\le B-A\le M$ for some constants α, δ, m, M.

If A, $B>0$ and $B-A>0$, then

$$\begin{aligned}0&\le\frac{(1+\|B\|)\exp(\|B\|)-1}{\left\|(B-A)^{-1}\right\|}\le B\exp(B)E_1(B)-A\exp(A)E_1(A)\\ &\le\frac{\|B-A\|}{\|A^{-1}\|}\left[\left(1+\left\|A^{-1}\right\|\right)\exp\left(\left\|A^{-1}\right\|^{-1}\right)-1\right].\end{aligned} \tag{3.7}$$

More examples of such transforms are

$$\mathcal{M}(w_{1/(\ell^2+a^2)})(t):=\int_0^\infty \frac{t}{(t+\lambda)\left(\lambda^2+a^2\right)}d\lambda=\frac{\pi t^2-2at\ln(t/a)}{2a\left(t^2+a^2\right)},\ t\ge 0$$

and

$$\mathcal{M}(w_{\ell/(\ell^2+a^2)})(t) := \int_0^\infty \frac{t\lambda}{(t+\lambda)(\lambda^2+a^2)} d\lambda = \frac{\pi a t + 2t^2 \ln(t/a)}{2a(t^2+a^2)},\ t \geq 0$$

for $a > 0$.

The interested reader may state other similar results by employing the examples of monotone operator functions provided in [3], [4], [5], [8] and [9].

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