LOWER AND UPPER BOUNDS FOR MONOTONIC INTEGRAL TRANSFORM OF SEPARATED POSITIVE OPERATORS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w,\mu)(T) := \int_0^\infty w(\lambda)\, T(\lambda + T)^{-1}\, d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

If the positive operators satisfy the separation condition

$$0 < \alpha \le A \le \beta < \gamma \le B \le \delta$$

for some positive constants α, β, γ, δ, then

$$0 \le (\gamma - \beta)\frac{\mathcal{M}(w,\mu)(\delta) - \mathcal{M}(w,\mu)(\beta)}{\delta - \beta} \le \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)$$
$$\le (\delta - \alpha)\frac{\mathcal{M}(w,\mu)(\gamma) - \mathcal{M}(w,\mu)(\alpha)}{\gamma - \alpha}.$$

If $f : (0,\infty) \to \mathbb{R}$ is operator monotone on $(0,\infty)$, then

$$0 \le (\gamma - \beta)\frac{f(\delta) - f(\beta)}{\delta - \beta} \le f(B) - f(A) \le (\delta - \alpha)\frac{f(\gamma) - f(\alpha)}{\gamma - \alpha}.$$

In particular, we have the power inequalities

$$0 < (\gamma - \beta)\frac{\delta^r - \beta^r}{\delta - \beta} \le B^r - A^r \le (\delta - \alpha)\frac{\gamma^r - \alpha^r}{\gamma - \alpha}$$

and the logarithmic inequalities

$$0 < (\gamma - \beta)\frac{\ln\delta - \ln\beta}{\delta - \beta} \le \ln B - \ln A \le (\delta - \alpha)\frac{\ln\gamma - \ln\alpha}{\gamma - \alpha}.$$

Some examples for operator convex functions as well as for integral transforms $\mathcal{M}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x\rangle \ge 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \ge f(B)$ holds for any $A \ge B > 0$.

We have the following representation of operator monotone functions [10], see for instance [1, p. 144-145]:

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator convex functions, Operator inequalities, Löwner-Heinz inequality. Logarithmic operator inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 79, 15 pp. Received 09/07/20

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.1}$$

where $a\in R$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.2}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.1).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.3}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.1).

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}d\lambda.$$

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.4}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.4) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.5}$$

Now, assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.6}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.7}$$

for $T>0$.

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.8}$$

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t),\ t>0. \tag{1.9}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t)&=\int_0^\infty w(\lambda)\,t\,(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)(t+\lambda-\lambda)(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[1-\lambda(t+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned} \tag{1.10}$$

If $\int_0^\infty w(\lambda)\,d\mu(\lambda)<\infty$, then

$$\mathcal{M}(w,\mu)(t)=\int_0^\infty w(\lambda)\,d\mu(\lambda)-\mathcal{D}(\ell w,\mu)(t), \tag{1.11}$$

where $\ell(t)=t$, $t>0$.

Consider the kernel $e_{-a}(\lambda):=\exp(-a\lambda)$, $\lambda\geq 0$ and $a>0$. Then, after some calculations, we get

$$\mathcal{D}(e_{-a})(t)=\int_0^\infty\frac{\exp(-a\lambda)}{t+\lambda}d\lambda=E_1(at)\exp(at),\ t\geq 0$$

and

$$\int_0^\infty w(\lambda)\,d\lambda=\int_0^\infty\exp(-a\lambda)\,d\lambda=\frac{1}{a},$$

where the exponential integral is defined by

$$E_1(t):=\int_t^\infty\frac{e^{-u}}{u}du.$$

This gives that

$$\mathcal{M}(e_{-a})(t)=t\mathcal{D}(w,\mu)(t)=tE_1(at)\exp(at),\ t\geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t)=\int_0^\infty\frac{\lambda\exp(-a\lambda)}{t+\lambda}d\lambda=\frac{1}{a}-tE_1(at)\exp(at)$$

for $t>0$.

One observes that

$$\mathcal{M}(e_{-a})(t)=\int_0^\infty w(\lambda)\,d\lambda-\mathcal{D}(\ell e_{-a},\mu)(t),\ t>0$$

and the equality (1.11) is verified in this case.

If we take $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda=\infty$ and the equality (1.11) does not hold in this case.

For all $T > 0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T) = T\mathcal{D}(w,\mu)(T) = \int_0^\infty w(\lambda)\left[1-\lambda(T+\lambda)^{-1}\right]d\mu(\lambda). \tag{1.12}$$

This gives the representation

$$T^r = \frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T),$$

where $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$ and μ is the usual Lebesgue norm.

Assume that $0 < A < B$. We say that these operators are *separated* if there exists $0 < \beta < \gamma$ such that $0 < A \le \beta < \gamma \le B$.

For a positive operator $T > 0$, we have the operator inequalities $\left\|T^{-1}\right\|^{-1} \le T \le \|T\|$. Therefore, if A, $B > 0$ with $\|A\|\left\|B^{-1}\right\| < 1$, then

$$0 < \left\|A^{-1}\right\|^{-1} \le A \le \|A\| < \left\|B^{-1}\right\|^{-1} \le B \le \|B\|.$$

The class of two separated positive operators play an important role in establishing various refinements and reverses of operator Young inequalities as pointed out in numerous recent papers from which we only mention [3], [13] and the references therein.

If the positive operators satisfy the separation condition

$$0 < \alpha \le A \le \beta < \gamma \le B \le \delta$$

for some positive constants α, β, γ, δ, then we show among others that

$$\begin{aligned}0 &\le (\gamma-\beta)\frac{\mathcal{M}(w,\mu)(\delta)-\mathcal{M}(w,\mu)(\beta)}{\delta-\beta} \le \mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\ &\le (\delta-\alpha)\frac{\mathcal{M}(w,\mu)(\gamma)-\mathcal{M}(w,\mu)(\alpha)}{\gamma-\alpha}.\end{aligned}$$

If $f:(0,\infty)\to\mathbb{R}$ is operator monotone on $(0,\infty)$, then

$$0 \le (\gamma-\beta)\frac{f(\delta)-f(\beta)}{\delta-\beta} \le f(B)-f(A) \le (\delta-\alpha)\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}.$$

In particular, we have the power inequalities

$$0 < (\gamma-\beta)\frac{\delta^r-\beta^r}{\delta-\beta} \le B^r - A^r \le (\delta-\alpha)\frac{\gamma^r-\alpha^r}{\gamma-\alpha}$$

and the logarithmic inequalities

$$0 < (\gamma-\beta)\frac{\ln\delta-\ln\beta}{\delta-\beta} \le \ln B - \ln A \le (\delta-\alpha)\frac{\ln\gamma-\ln\alpha}{\gamma-\alpha}.$$

Some examples for operator convex functions as well as for integral transforms $\mathcal{M}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

We have the following equality that is of interest in itself:

Lemma 1. *For all A, $B>0$ we have the representation*

$$\begin{aligned}\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A) \\ =\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right) \\ \times\lambda w(\lambda)\,d\mu(\lambda).\end{aligned}\tag{2.1}$$

Proof. From (1.12) we have for all A, $B\geq 0$ that

$$\begin{aligned}\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A) \\ =\int_0^\infty w(\lambda)\left[1-\lambda(B+\lambda)^{-1}\right]d\mu(\lambda)-\int_0^\infty w(\lambda)\left[1-\lambda(A+\lambda)^{-1}\right]d\mu(\lambda) \\ =\int_0^\infty\lambda w(\lambda)\left[(A+\lambda)^{-1}-(B+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned}\tag{2.2}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1}\tag{2.3}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt.\tag{2.4}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt.\tag{2.5}$$

Now, if we take in (1.6) $C=\lambda+A$, $D=\lambda+B$, then

$$\begin{aligned}(\lambda+A)^{-1}-(\lambda+B)^{-1} \\ =\int_0^1((1-t)(\lambda+A)+t(\lambda+B))^{-1}(B-A) \\ \times((1-t)(\lambda+A)+t(\lambda+B))^{-1}dt \\ =\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt.\end{aligned}\tag{2.6}$$

By employing (2.2) and (2.6), we derive (2.1). □

Corollary 1. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and it has the representation (1.1), then for all A, $B>0$ we have the representation*

$$\begin{aligned} &f(B)-f(A)-b(B-A) \\ &=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right) \\ &\times\lambda^2 d\mu(\lambda). \end{aligned} \tag{2.7}$$

Proof. From (1.1) we have for $T>0$ that

$$f(T)-a-bT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$. Therefore

$$\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)=f(B)-f(A)-b(B-A)$$

and by (2.1) we get (2.7). □

Corollary 2. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and it has the representation (1.3), then for all A, $B>0$ we have the representation*

$$\begin{aligned} &f(B)B^{-1}-f(A)A^{-1}-a\left(B^{-1}-A^{-1}\right)-c(B-A) \\ &=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right) \\ &\times\lambda^2 d\mu(\lambda). \end{aligned} \tag{2.8}$$

Proof. From (1.3) we have for $T>0$ that

$$(f(T)-a)T^{-1}-b-cT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ. Therefore

$$\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)=(f(B)-a)B^{-1}-(f(A)-a)A^{-1}-c(B-A)$$

and by (2.1) we get (2.8). □

Remark 1. *If $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$, then we can take $a=f(0)$ and by (2.8) we get*

$$\begin{aligned} &f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A) \\ &=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right) \\ &\times\lambda^2 d\mu(\lambda). \end{aligned} \tag{2.9}$$

The following monotonicity result for separated operators holds:

Theorem 3. *If the positive operators satisfy the separation condition*

$$0<\alpha\leq A\leq\beta<\gamma\leq B\leq\delta \tag{2.10}$$

for some positive constants α, β, γ, δ, then

$$\begin{aligned} 0&\leq(\gamma-\beta)\frac{\mathcal{M}(w,\mu)(\delta)-\mathcal{M}(w,\mu)(\beta)}{\delta-\beta} \\ &\leq\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A) \\ &\leq(\delta-\alpha)\frac{\mathcal{M}(w,\mu)(\gamma)-\mathcal{M}(w,\mu)(\alpha)}{\gamma-\alpha}. \end{aligned} \tag{2.11}$$

Proof. From (2.10) we have

$$0<\gamma-\beta\leq B-A\leq\delta-\alpha,$$

which implies that

$$\begin{aligned}0 &\leq (\gamma-\beta)\left((1-t)A+tB+\lambda\right)^{-2}\\ &\leq \left((1-t)A+tB+\lambda\right)^{-1}(B-A)\left((1-t)A+tB+\lambda\right)^{-1}\\ &\leq (\delta-\alpha)\left((1-t)A+tB+\lambda\right)^{-2}\end{aligned}$$

for all $t\in[0,1]$ and $\lambda\geq 0$.

By integration over $t\in[0,1]$ we deduce

$$\begin{aligned}0 &\leq (\gamma-\beta)\int_0^1\left((1-t)A+tB+\lambda\right)^{-2}dt\\ &\leq \int_0^1\left((1-t)A+tB+\lambda\right)^{-1}(B-A)\left((1-t)A+tB+\lambda\right)^{-1}dt\\ &\leq (\delta-\alpha)\int_0^1\left((1-t)A+tB+\lambda\right)^{-2}dt\end{aligned}$$

for all $\lambda\geq 0$.

If we multiply this inequality by $\lambda w(\lambda)\geq 0$ and integrate over the measure $\mu(\lambda)$, we get

$$\begin{aligned}0 &\leq (\gamma-\beta)\int_0^\infty\lambda w(\lambda)\left(\int_0^1\left((1-t)A+tB+\lambda\right)^{-2}dt\right)d\mu(\lambda)\\ &\leq \int_0^\infty\left(\int_0^1\left((1-t)A+tB+\lambda\right)^{-1}(B-A)\left((1-t)A+tB+\lambda\right)^{-1}dt\right)\\ &\times\lambda w(\lambda)\,d\mu(\lambda)\\ &\leq (\delta-\alpha)\int_0^\infty\lambda w(\lambda)\left(\int_0^1\left((1-t)A+tB+\lambda\right)^{-2}dt\right)d\mu(\lambda),\end{aligned}$$

and, by (2.1) we derive the inequality of interest

$$\begin{aligned}0 &\leq (\gamma-\beta)\int_0^\infty\lambda w(\lambda)\left(\int_0^1\left((1-t)A+tB+\lambda\right)^{-2}dt\right)d\mu(\lambda)\\ &\leq \mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\ &\leq (\delta-\alpha)\int_0^\infty\lambda w(\lambda)\left(\int_0^1\left((1-t)A+tB+\lambda\right)^{-2}dt\right)d\mu(\lambda).\end{aligned}\tag{2.12}$$

From (2.10) we derive that

$$(1-t)A+tB+\lambda\leq(1-t)\beta+t\delta+\lambda,$$

which implies that

$$\left((1-t)A+tB+\lambda\right)^{-1}\geq\left((1-t)\beta+t\delta+\lambda\right)^{-1}$$

and

$$\left((1-t)A+tB+\lambda\right)^{-2}\geq\left((1-t)\beta+t\delta+\lambda\right)^{-2}$$

for all $t\in[0,1]$ and $\lambda\geq 0$.

Also

$$(1-t)A+tB+\lambda\geq(1-t)\alpha+t\gamma+\lambda,$$

which implies that

$$((1-t)A+tB+\lambda)^{-1} \leq ((1-t)\alpha+t\gamma+\lambda)^{-1}$$

and

$$((1-t)A+tB+\lambda)^{-2} \leq ((1-t)\alpha+t\gamma+\lambda)^{-2}$$

for all $t\in[0,1]$ and $\lambda\geq 0$.

Therefore

$$\begin{aligned}(2.13)\quad &(\gamma-\beta)\int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)\beta+t\delta+\lambda)^{-2}\,dt\right)d\mu(\lambda)\\ &\leq(\gamma-\beta)\int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)A+tB+\lambda)^{-2}\,dt\right)d\mu(\lambda)\end{aligned}$$

and

$$\begin{aligned}(2.14)\quad &(\delta-\alpha)\int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)A+tB+\lambda)^{-2}\,dt\right)d\mu(\lambda)\\ &\leq(\delta-\alpha)\int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)\alpha+t\gamma+\lambda)^{-2}\,dt\right)d\mu(\lambda).\end{aligned}$$

Since

$$\begin{aligned}&(\gamma-\beta)\int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)\beta+t\delta+\lambda)^{-2}\,dt\right)d\mu(\lambda)\\ &=\frac{\gamma-\beta}{\delta-\beta}\\ &\times\int_0^\infty\left(\int_0^1 ((1-t)\beta+t\delta+\lambda)^{-1}(\delta-\beta)((1-t)\beta+t\delta+\lambda)^{-1}\,dt\right)\\ &\times\lambda w(\lambda)\,d\mu(\lambda)\\ &=\frac{\gamma-\beta}{\delta-\beta}\left[\mathcal{M}(w,\mu)(\delta)-\mathcal{M}(w,\mu)(\beta)\right]\ \text{ (by (2.1))}\end{aligned}$$

and

$$\begin{aligned}&(\delta-\alpha)\int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)\alpha+t\gamma+\lambda)^{-2}\,dt\right)d\mu(\lambda)\\ &=\frac{\delta-\alpha}{\gamma-\alpha}\\ &\times\int_0^\infty\left(\int_0^1 ((1-t)\alpha+t\gamma+\lambda)^{-1}(\gamma-\alpha)((1-t)\alpha+t\gamma+\lambda)^{-1}\,dt\right)\\ &\times\lambda w(\lambda)\,d\mu(\lambda)\\ &=\frac{\delta-\alpha}{\gamma-\alpha}\left[\mathcal{M}(w,\mu)(\gamma)-\mathcal{D}(w,\mu)_1(\alpha)\right]\ \text{ (by (2.1))},\end{aligned}$$

then (2.13) and (2.14) become

$$\begin{aligned}(2.15)\quad &\frac{\gamma-\beta}{\delta-\beta}\left[\mathcal{M}(w,\mu)(\delta)-\mathcal{M}(w,\mu)(\beta)\right]\\ &\leq(\gamma-\beta)\int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)A+tB+\lambda)^{-2}\,dt\right)d\mu(\lambda)\end{aligned}$$

and

$$(\delta-\alpha)\int_0^{\infty}\lambda w(\lambda)\left(\int_0^1((1-t)A+tB+\lambda)^{-2}dt\right)d\mu(\lambda) \tag{2.16}$$
$$\leq\frac{\delta-\alpha}{\gamma-\alpha}\left[\mathcal{M}(w,\mu)(\gamma)-\mathcal{M}(w,\mu)(\alpha)\right].$$

Finally, on making use of (2.12), (2.15) and (2.16), we derive (2.11). □

Remark 2. *If A, $B>0$ with $\|A\|\left\|B^{-1}\right\|<1$, then*

$$0\leq\left(\frac{1-\|A\|\left\|B^{-1}\right\|}{\|B^{-1}\|}\right)\frac{\mathcal{M}(w,\mu)(\|B\|)-\mathcal{M}(w,\mu)(\|A\|)}{\|B\|-\|A\|} \tag{2.17}$$
$$\leq\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)$$
$$\leq\left(\frac{\|B\|\left\|A^{-1}\right\|-1}{\|A^{-1}\|}\right)\frac{\mathcal{M}(w,\mu)\left(\left\|B^{-1}\right\|^{-1}\right)-\mathcal{M}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right)}{\|B^{-1}\|^{-1}-\|A^{-1}\|^{-1}}.$$

The case of operator monotone function is as follows:

Corollary 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and it has the representation (1.1). If A and B satisfy the separation condition (2.10), then*

$$0\leq(\gamma-\beta)\frac{f(\delta)-f(\beta)}{\delta-\beta} \tag{2.18}$$
$$\leq\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)-f(\beta)-b(\delta-\beta)\right]+b(B-A)$$
$$\leq f(B)-f(A)$$
$$\leq\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)-f(\alpha)-b(\gamma-\alpha)\right]+b(B-A)$$
$$\leq(\delta-\alpha)\frac{f(\gamma)-f(\alpha)}{\gamma-\alpha}.$$

Proof. From (1.1) we have for $T>0$ that

$$f(T)-a-bT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda\geq0$. Therefore

$$\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)=f(B)-f(A)-b(B-A),$$
$$\mathcal{M}(w,\mu)(\delta)-\mathcal{M}(w,\mu)(\beta)=f(\delta)-f(\beta)-b(\delta-\beta)$$

and

$$\mathcal{M}(w,\mu)(\gamma)-\mathcal{M}(w,\mu)(\alpha)=f(\gamma)-f(\alpha)-b(\gamma-\alpha).$$

By (2.11) we get

$$0\leq\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)-f(\beta)-b(\delta-\beta)\right]$$
$$\leq f(B)-f(A)-b(B-A)$$
$$\leq\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)-f(\alpha)-b(\gamma-\alpha)\right],$$

namely

$$0 \le \frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)-f(\beta)-b(\delta-\beta)\right]+b(B-A) \le f(B)-f(A) \le \frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)-f(\alpha)-b(\gamma-\alpha)\right]+b(B-A). \tag{2.19}$$

Observe that

$$\begin{aligned} &\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)-f(\beta)-b(\delta-\beta)\right]+b(B-A) \\ &= \frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)-f(\beta)\right]-b(\gamma-\beta)+b(B-A) \\ &= \frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)-f(\beta)\right]+b\left[(B-A)-(\gamma-\beta)\right] \\ &\ge \frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)-f(\beta)\right]\ge 0 \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)-f(\alpha)-b(\gamma-\alpha)\right]+b(B-A) \\ &= \frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)-f(\alpha)\right]-b(\delta-\alpha)+b(B-A) \\ &= \frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)-f(\alpha)\right]-b\left[(\delta-\alpha)-(B-A)\right] \\ &\le \frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)-f(\alpha)\right]. \end{aligned} \tag{2.21}$$

By (2.19)-(2.21) we derive the desired result (2.18). □

Remark 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If A, $B>0$ with $\|A\|\left\|B^{-1}\right\|<1$ then*

$$\begin{aligned} 0 &\le \left(\frac{1-\|A\|\left\|B^{-1}\right\|}{\|B^{-1}\|}\right)\frac{f(\|B\|)-f(\|A\|)}{\|B\|-\|A\|} \\ &\le f(B)-f(A) \\ &\le \left(\frac{\|B\|\left\|A^{-1}\right\|-1}{\|A^{-1}\|}\right)\frac{f\left(\left\|B^{-1}\right\|^{-1}\right)-f\left(\left\|A^{-1}\right\|^{-1}\right)}{\left\|B^{-1}\right\|^{-1}-\left\|A^{-1}\right\|^{-1}}. \end{aligned} \tag{2.22}$$

Remark 4. *Assume that A and B satisfy the separation condition (2.10). Since $f(t)=t^r$, $r\in(0,1]$ is operator monotone, then by (2.18) we get*

$$0<(\gamma-\beta)\frac{\delta^r-\beta^r}{\delta-\beta}\le B^r-A^r\le(\delta-\alpha)\frac{\gamma^r-\alpha^r}{\gamma-\alpha}. \tag{2.23}$$

Since $f(t)=\ln t$ is operator monotone, then by (2.18) we get

$$0<(\gamma-\beta)\frac{\ln\delta-\ln\beta}{\delta-\beta}\le \ln B-\ln A\le(\delta-\alpha)\frac{\ln\gamma-\ln\alpha}{\gamma-\alpha}. \tag{2.24}$$

If A, $B>0$ with $\|A\|\left\|B^{-1}\right\|<1$, then

$$0\le\left(\frac{1-\|A\|\left\|B^{-1}\right\|}{\|B^{-1}\|}\right)\frac{\|B\|^{r}-\|A\|^{r}}{\|B\|-\|A\|}\le B^{r}-A^{r}\le\left(\frac{\|B\|\left\|A^{-1}\right\|-1}{\|A^{-1}\|}\right)\frac{\left\|B^{-1}\right\|^{-r}-\left\|A^{-1}\right\|^{-r}}{\|B^{-1}\|^{-1}-\|A^{-1}\|^{-1}} \tag{2.25}$$

and

$$0\le\left(\frac{1-\|A\|\left\|B^{-1}\right\|}{\|B^{-1}\|}\right)\frac{\ln\|B\|-\ln\|A\|}{\|B\|-\|A\|}\le\ln B-\ln A\le\left(\frac{\|B\|\left\|A^{-1}\right\|-1}{\|A^{-1}\|}\right)\frac{\ln\left(\left\|B^{-1}\right\|^{-1}\right)-\ln\left(\left\|A^{-1}\right\|^{-1}\right)}{\|B^{-1}\|^{-1}-\|A^{-1}\|^{-1}}. \tag{2.26}$$

The case of operator convex functions is as follows:

Corollary 4. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and it has the representation (1.3). If A and B satisfy the separation condition (2.10), then*

$$\begin{aligned}0&\le\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)\delta^{-1}-f(\beta)\beta^{-1}-f(0)\left(\delta^{-1}-\beta^{-1}\right)\right]\\&\le\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)\delta^{-1}-f(\beta)\beta^{-1}-f(0)\left(\delta^{-1}-\beta^{-1}\right)\right]\\&\quad-c(\gamma-\beta)+c(B-A)\\&\le f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\\&\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}-f(0)\left(\gamma^{-1}-\alpha^{-1}\right)\right]\\&\quad-c(\delta-\alpha)+c(B-A)\\&\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}-f(0)\left(\gamma^{-1}-\alpha^{-1}\right)\right].\end{aligned}\tag{2.27}$$

In particular, if $f(0)=0$, then

$$\begin{aligned}0&\le(\gamma-\beta)\frac{f(\delta)\delta^{-1}-f(\beta)\beta^{-1}}{\delta-\beta}\\&\le\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)\delta^{-1}-f(\beta)\beta^{-1}\right]-c(\gamma-\beta)+c(B-A)\\&\le f(B)B^{-1}-f(A)A^{-1}\\&\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}\right]-c(\delta-\alpha)+c(B-A)\\&\le(\delta-\alpha)\frac{f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}}{\gamma-\alpha}.\end{aligned}\tag{2.28}$$

Proof. From (1.3) we have for $T>0$ that

$$(f(T)-f(0))T^{-1}-b-cT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ. Therefore

$$\begin{aligned}&\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\&=f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A),\end{aligned}$$

$$\begin{aligned}&\mathcal{M}(w,\mu)(\delta)-\mathcal{M}(w,\mu)(\beta)\\&=f(\delta)\delta^{-1}-f(\beta)\beta^{-1}-f(0)\left(\delta^{-1}-\beta^{-1}\right)-c(\delta-\beta)\end{aligned}$$

and

$$\begin{aligned}&\mathcal{M}(w,\mu)(\gamma)-\mathcal{M}(w,\mu)(\alpha)\\&=f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}-f(0)\left(\gamma^{-1}-\alpha^{-1}\right)-c(\gamma-\alpha).\end{aligned}$$

From (2.11) we get

$$\begin{aligned}0&\le\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)\delta^{-1}-f(\beta)\beta^{-1}-f(0)\left(\delta^{-1}-\beta^{-1}\right)\right]-c(\gamma-\beta)\\&\le f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\\&\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}-f(0)\left(\gamma^{-1}-\alpha^{-1}\right)\right]-c(\delta-\alpha).\end{aligned}$$

This is equivalent to

$$\begin{aligned}&\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)\delta^{-1}-f(\beta)\beta^{-1}-f(0)\left(\delta^{-1}-\beta^{-1}\right)\right]\\&-c(\gamma-\beta)+c(B-A)\\&\le f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\\&\le\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}-f(0)\left(\gamma^{-1}-\alpha^{-1}\right)\right]\\&-c(\delta-\alpha)+c(B-A).\end{aligned}$$

Since

$$\begin{aligned}&\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)\delta^{-1}-f(\beta)\beta^{-1}-f(0)\left(\delta^{-1}-\beta^{-1}\right)\right]\\&-c(\gamma-\beta)+c(B-A)\\&=\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)\delta^{-1}-f(\beta)\beta^{-1}-f(0)\left(\delta^{-1}-\beta^{-1}\right)\right]\\&+c\left[(B-A)-(\gamma-\beta)\right]\\&\ge\frac{\gamma-\beta}{\delta-\beta}\left[f(\delta)\delta^{-1}-f(\beta)\beta^{-1}-f(0)\left(\delta^{-1}-\beta^{-1}\right)\right]\end{aligned}$$

and

$$\begin{aligned}
&\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}-f(0)\left(\gamma^{-1}-\alpha^{-1}\right)\right]\\
&-c(\delta-\alpha)+c(B-A)\\
&=\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}-f(0)\left(\gamma^{-1}-\alpha^{-1}\right)\right]\\
&-c[(\delta-\alpha)-(B-A)]\\
&\leq\frac{\delta-\alpha}{\gamma-\alpha}\left[f(\gamma)\gamma^{-1}-f(\alpha)\alpha^{-1}-f(0)\left(\gamma^{-1}-\alpha^{-1}\right)\right]
\end{aligned}$$

and the inequality (2.27) is thus proved. □

Remark 5. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator convex in* $[0,\infty)$ *and* $f(0)=0$. *If* $A,\ B>0$ *with* $\|A\|\left\|B^{-1}\right\|<1$, *then*

$$\begin{aligned}
0&\leq\left(\frac{1-\|A\|\left\|B^{-1}\right\|}{\left\|B^{-1}\right\|}\right)\frac{f(\|B\|)\|B\|^{-1}-f(\|A\|)\|A\|^{-1}}{\|B\|-\|A\|}\\
&\leq f(B)B^{-1}-f(A)A^{-1}\\
&\leq\left(\frac{\|B\|\left\|A^{-1}\right\|-1}{\left\|A^{-1}\right\|}\right)\frac{f\left(\left\|B^{-1}\right\|^{-1}\right)\left\|B^{-1}\right\|-f\left(\left\|A^{-1}\right\|^{-1}\right)\left\|A^{-1}\right\|}{\left\|B^{-1}\right\|^{-1}-\left\|A^{-1}\right\|^{-1}}.
\end{aligned}\tag{2.29}$$

Remark 6. *Consider the operator convex function* $f(t)=-\ln(t+1)$, $t\in[0,\infty)$. *Then by (2.28) we have*

$$\begin{aligned}
0&\leq(\gamma-\beta)\frac{\beta^{-1}\ln(\beta+1)-\delta^{-1}\ln(\delta+1)}{\delta-\beta}\\
&\leq A^{-1}\ln(A+1)-B^{-1}\ln(B+1)\\
&\leq(\delta-\alpha)\frac{\alpha^{-1}\ln(\alpha+1)-\gamma^{-1}\ln(\gamma+1)}{\gamma-\alpha},
\end{aligned}\tag{2.30}$$

provided $0<\alpha\leq A\leq\beta<\gamma\leq B\leq\delta$.

If $A,\ B>0$ *with* $\|A\|\left\|B^{-1}\right\|<1$, *then*

$$\begin{aligned}
0&\leq\left(\frac{1-\|A\|\left\|B^{-1}\right\|}{\left\|B^{-1}\right\|}\right)\\
&\times\frac{\|A\|^{-1}\ln(\|A\|+1)-\|B\|^{-1}\ln(\|B\|+1)}{\|B\|-\|A\|}\\
&\leq A^{-1}\ln(A+1)-B^{-1}\ln(B+1)\\
&\leq\left(\frac{\|B\|\left\|A^{-1}\right\|-1}{\left\|A^{-1}\right\|}\right)\\
&\times\frac{\left\|A^{-1}\right\|\ln\left(\left\|A^{-1}\right\|^{-1}+1\right)-\left\|B^{-1}\right\|\ln\left(\left\|B^{-1}\right\|^{-1}+1\right)}{\left\|B^{-1}\right\|^{-1}-\left\|A^{-1}\right\|^{-1}}.
\end{aligned}\tag{2.31}$$

3. More Examples

We define the *upper incomplete Gamma function* as [8]

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [9]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{3.1}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (3.1) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{3.2}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (3.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^t E_1(t) \tag{3.3}$$

for $t > 0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du.$$

We then have

$$\mathcal{M}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T) = \Gamma(1-a)T^{1-a}\exp(T)\Gamma(a,T) \tag{3.4}$$

for $a < 1$ and

$$\mathcal{M}\left(w_{e^{-\cdot}}\right)(T) = T\exp(T)E_1(T) \tag{3.5}$$

for $T > 0$.

If the positive operators satisfy the separation condition $0 < \alpha \le A \le \beta < \gamma \le B \le \delta$ for some positive constants α, β, γ, δ, then

$$\begin{aligned} 0 &\le (\gamma-\beta)\frac{\delta^{1-a}\exp(\delta)\Gamma(a,\delta) - \beta^{1-a}\exp(\beta)\Gamma(a,\beta)}{\delta-\beta} \\ &\le B^{1-a}\exp(B)\Gamma(a,B) - A^{1-a}\exp(A)\Gamma(a,A) \\ &\le (\delta-\alpha)\frac{\gamma^{1-a}\exp(\gamma)\Gamma(a,\gamma) - \alpha^{1-a}\exp(\alpha)\Gamma(a,\alpha)}{\gamma-\alpha} \end{aligned} \tag{3.6}$$

for $a < 1$, and, in particular, for $a = 0$,

$$\begin{aligned} 0 &\le (\gamma-\beta)\frac{\delta\exp(\delta)E_1(\delta) - \beta\exp(\beta)E_1(\beta)}{\delta-\beta} \\ &\le B\exp(B)E_1(B) - A\exp(A)E_1(A) \\ &\le (\delta-\alpha)\frac{\gamma\exp(\gamma)E_1(\gamma) - \alpha\exp(\alpha)E_1(\alpha)}{\gamma-\alpha}. \end{aligned} \tag{3.7}$$

If A, $B > 0$ with $\|A\| \left\|B^{-1}\right\| < 1$, then

$$
\begin{aligned}
(3.8) \quad 0 &\leq \left(\frac{1 - \|A\| \left\|B^{-1}\right\|}{\|B^{-1}\|}\right) \\
&\times \frac{\|B\| \exp\left(\|B\|\right) E_1(\|B\|) - \|A\| \exp\left(\|A\|\right) E_1(\|A\|)}{\|B\| - \|A\|} \\
&\leq B \exp\left(B\right) E_1(B) - A \exp\left(A\right) E_1(A) \\
&\leq \left(\frac{\|B\| \left\|A^{-1}\right\| - 1}{\|A^{-1}\|}\right) \\
&\times \frac{\left\|B^{-1}\right\|^{-1} \exp\left(\left\|B^{-1}\right\|^{-1}\right) E_1(\left\|B^{-1}\right\|^{-1}) - \left\|A^{-1}\right\|^{-1} \exp\left(\left\|A^{-1}\right\|^{-1}\right) E_1(\left\|A^{-1}\right\|^{-1})}{\left\|B^{-1}\right\|^{-1} - \left\|A^{-1}\right\|^{-1}}.
\end{aligned}
$$

The interested reader may state other similar inequalities by using the examples of operator monotone functions from [2], [4] and the references therein.

References

[1] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[3] S. Furuichi, Refined Young inequalities with Specht's ratio. *J. Egypt. Math. Soc.* 2011;20(1):46–49.

[4] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.

[5] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[6] Generalized Exponential Integral, *Digital Library of Mathematical Functions,* NIST. [`Online https://dlmf.nist.gov/8.19#E1`].

[7] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[8] Incomplete Gamma and Related Functions, Definitions, *Digital Library of Mathematical Functions*, NIST. [`Online https://dlmf.nist.gov/8.2].`

[9] Incomplete Gamma and Related Functions, Integral Representations, *Digital Library of Mathematical Functions,* NIST. [`Online https://dlmf.nist.gov/8.6`].

[10] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[11] M. S. Moslehian, H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.

[12] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

[13] H. Zuo, G. Shi and M. Fujii Refined Young inequality with Kantorovich constant. *J. Math. Inequal.* 2011;5(4):551–556.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.