SEVERAL BOUNDS FOR MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2}[\mathcal{M}(w, \mu)(M + \beta) - \mathcal{M}(w, \mu)(\beta)] \\ &\leq \frac{m^2}{M}[\mathcal{M}(w, \mu)(M + \beta) - \mathcal{M}(w, \mu)(\beta)](B - A)^{-1} \\ &\leq \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \\ &\leq \frac{M^2}{m}[\mathcal{M}(w, \mu)(m + \alpha) - \mathcal{M}(w, \mu)(\alpha)](B - A)^{-1} \\ &\leq \frac{M^2}{m^2}[\mathcal{M}(w, \mu)(m + \alpha) - \mathcal{M}(w, \mu)(\alpha)]. \end{aligned}$$

In particular, we derived that for $r \in (0, 1]$

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2}[(M + \beta)^r - \beta^r] \leq \frac{m^2}{M}[f(M + \beta)^r - \beta^r](B - A)^{-1} \\ &\leq B^r - A^r \\ &\leq \frac{M^2}{m}[(m + \alpha)^r - \alpha^r](B - A)^{-1} \leq \frac{M^2}{m^2}[(m + \alpha)^r - \alpha^r]. \end{aligned}$$

Some examples for operator monotone and operator convex functions as well as for monotonic integral transforms $\mathcal{M}(\cdot, \cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

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Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ, the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [6], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.9}$$

where $a \in R$, $b \geq 0$ *and a positive measure* μ *on* $(0, \infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.10}$$

If f *is operator monotone in* $[0, \infty)$, *then* $a = f(0)$ *in (1.9).*

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda) A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function* $f : (0, \infty) \to \mathbb{R}$ *is operator convex in* $(0, \infty)$ *if and only if it has the representation*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t+\lambda} d\mu(\lambda), \tag{1.11}$$

where $a, b \in \mathbb{R}$, $c \geq 0$ *and a positive measure* μ *on* $(0, \infty)$ *such that (1.2) holds. If* f *is operator convex in* $[0, \infty)$, *then* $a = f(0)$ *and* $b = f'_+(0)$, *the right derivative, in (1.11).*

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t), \ t > 0. \tag{1.12}$$

For $t > 0$ we have

$$\begin{aligned} \mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t) &= \int_0^\infty w(\lambda) t (t+\lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)(t+\lambda-\lambda)(t+\lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\left[1 - \lambda (t+\lambda)^{-1}\right] d\mu(\lambda). \end{aligned} \tag{1.13}$$

If $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$, then

$$\mathcal{M}(w, \mu)(t) = \int_0^\infty w(\lambda) d\mu(\lambda) - \mathcal{D}(\ell w, \mu)(t), \tag{1.14}$$

where $\ell(t) = t$, $t > 0$.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then after some calculations, we get

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda} d\lambda = E_1(at)\exp(at), \ t \geq 0$$

and

$$\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a},$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

This gives that

$$\mathcal{M}(e_{-a})(t) = t\mathcal{D}(w,\mu)(t) = tE_1(at)\exp(at), \ t \geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda \exp(-a\lambda)}{t+\lambda} d\lambda = \frac{1}{a} - tE_1(at)\exp(at)$$

for $t > 0$.

One observes that

$$\mathcal{M}(e_{-a})(t) = \int_0^\infty w(\lambda) d\lambda - \mathcal{D}(\ell e_{-a},\mu)(t), \ t > 0$$

and the equality (1.14) is verified in this case.

If we take $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then $\int_0^\infty w_r(\lambda) d\lambda = \infty$ and the equality (1.14) does not hold in this case.

For all $T > 0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T) = T\mathcal{D}(w,\mu)(T) = \int_0^\infty w(\lambda)\left[1 - \lambda(T+\lambda)^{-1}\right] d\mu(\lambda). \tag{1.15}$$

This gives the representation

$$T^r = \frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T), \tag{1.16}$$

where $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$ and μ is the usual Lebesgue norm. Also, from (1.6) we have the representation

$$T\ln T = (T-1)\mathcal{M}(w_{\ln})(T), \ T > 0, \tag{1.17}$$

where $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$.

In this paper, we show among others that, if $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\mathcal{M}(w,\mu)(M+\beta) - \mathcal{M}(w,\mu)(\beta)\right] \\
&\leq \frac{m^2}{M}\left[\mathcal{M}(w,\mu)(M+\beta) - \mathcal{M}(w,\mu)(\beta)\right](B-A)^{-1} \\
&\leq \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) \\
&\leq \frac{M^2}{m}\left[\mathcal{M}(w,\mu)(m+\alpha) - \mathcal{M}(w,\mu)(\alpha)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{M}(w,\mu)(m+\alpha) - \mathcal{M}(w,\mu)(\alpha)\right].
\end{aligned}$$

Some examples for operator monotone and operator convex functions as well as for monotonic integral transforms $\mathcal{M}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

We have the following identity of interest:

Theorem 3. *For all A, $B>0$ with $B\geq A$ we have the representation*

$$(B-A)^{1/2}\left[\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\right](B-A)^{1/2} \tag{2.1}$$
$$=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+(1-t)A+tB)^{-1}(B-A)^{1/2}\right]^2dt\right)\lambda w(\lambda)\,d\mu(\lambda)\geq 0.$$

Proof. From (1.15) we have for all A, $B\geq 0$ that

$$\begin{aligned}\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)&\\ =\int_0^\infty w(\lambda)\left[1-\lambda(B+\lambda)^{-1}\right]d\mu(\lambda)&-\int_0^\infty w(\lambda)\left[1-\lambda(A+\lambda)^{-1}\right]d\mu(\lambda)\\ =\int_0^\infty \lambda w(\lambda)\left[(A+\lambda)^{-1}-(B+\lambda)^{-1}\right]d\mu(\lambda)&.\end{aligned} \tag{2.2}$$

Let T, $S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.3}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.4}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt. \tag{2.5}$$

Now, if we take in (2.5) $C=\lambda+A$, $D=\lambda+B$, then

$$\begin{aligned}&(\lambda+A)^{-1}-(\lambda+B)^{-1}\\ &=\int_0^1((1-t)(\lambda+A)+t(\lambda+B))^{-1}(B-A)\\ &\times((1-t)(\lambda+A)+t(\lambda+B))^{-1}dt\\ &=\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt.\end{aligned} \tag{2.6}$$

By employing (2.2) and (2.6), we derive

$$\begin{aligned}(2.7)\quad & \mathcal{M}(w,\mu)\,(B)-\mathcal{M}(w,\mu)\,(A) \\ & =\int_0^\infty\left(\int_0^1 (\lambda+(1-t)\,A+tB)^{-1}\,(B-A)\,(\lambda+(1-t)\,A+tB)^{-1}\,dt\right) \\ & \times\lambda w\,(\lambda)\,d\mu\,(\lambda)\end{aligned}$$

for all A, $B>0$.

Now, if we multiply both sides of (2.7) by $(B-A)^{1/2}\geq 0$, then we get

$$\begin{aligned}(2.8)\quad & (B-A)^{1/2}\,[\mathcal{M}(w,\mu)\,(B)-\mathcal{M}(w,\mu)\,(A)]\,(B-A)^{1/2} \\ & =\int_0^\infty\left(\int_0^1 (B-A)^{1/2}\,(\lambda+(1-t)\,A+tB)^{-1}\,(B-A)\right. \\ & \left.\times(\lambda+(1-t)\,A+tB)^{-1}\,(B-A)^{1/2}\,dt\right)\lambda w\,(\lambda)\,d\mu\,(\lambda) \\ & =\int_0^\infty\int_0^1 (B-A)^{1/2}\,(\lambda+(1-t)\,A+tB)^{-1}\,(B-A)^{1/2} \\ & \times\,(B-A)^{1/2}\,(\lambda+(1-t)\,A+tB)^{-1}\,(B-A)^{1/2}\,dt\Big)\,\lambda w\,(\lambda)\,d\mu\,(\lambda) \\ & =\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}\,(\lambda+(1-t)\,A+tB)^{-1}\,(B-A)^{1/2}\right]^2 dt\right) \\ & \times\lambda w\,(\lambda)\,d\mu\,(\lambda) \\ & \geq 0\end{aligned}$$

and the identity and inequality in (2.1) are thus proved. □

The case of operator monotone functions is as follows:

Corollary 1. *Assume that $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). Then for all A, $B>0$ with $B\geq A$ we have the equality*

$$\begin{aligned}(2.9)\quad & (B-A)^{1/2}\,[f\,(B)-f\,(A)]\,(B-A)^{1/2}-b\,(B-A)^2 \\ & =\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}\,(\lambda+(1-t)\,A+tB)^{-1}\,(B-A)^{1/2}\right]^2 dt\right)\lambda^2 d\mu\,(\lambda) \\ & \geq 0.\end{aligned}$$

Proof. We have by (1.9) that

$$f\,(t)=a+bt+\mathcal{M}(\ell,\mu),$$

for some positive measure μ, where $\ell\,(t)=t$, $t>0$ and $b\geq 0$.

Therefore

$$\begin{aligned}& (B-A)^{1/2}\,[\mathcal{M}(\ell,\mu)\,(B)-\mathcal{M}(\ell,\mu)\,(A)]\,(B-A)^{1/2} \\ & =(B-A)^{1/2}\,[f\,(B)-f\,(A)-bB+bA]\,(B-A)^{1/2} \\ & =(B-A)^{1/2}\,[f\,(B)-f\,(A)]\,(B-A)^{1/2}-b\,(B-A)^{1/2}\,(B-A)\,(B-A)^{1/2} \\ & =(B-A)^{1/2}\,[f\,(B)-f\,(A)]\,(B-A)^{1/2}-b\,(B-A)^2\end{aligned}$$

and by (2.1) we derive (2.9). □

The case of operator convex functions is as follows:

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.11). Then for all A, $B>0$ with $B\geq A$ we have the equality*

$$\begin{aligned}(2.10)\quad & (B-A)^{1/2}\left[f(B)B^{-1}-f(A)A^{-1}\right](B-A)^{1/2}\\ &-f(0)(B-A)^{1/2}\left(B^{-1}-A^{-1}\right)(B-A)^{1/2}-c(B-A)^{2}\\ &=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+(1-t)A+tB)^{-1}(B-A)^{1/2}\right]^2dt\right)\lambda^2d\mu(\lambda)\geq 0.\end{aligned}$$

If $f(0)=0$, then we have the simpler equality

$$\begin{aligned}(2.11)\quad & (B-A)^{1/2}\left[f(B)B^{-1}-f(A)A^{-1}\right](B-A)^{1/2}-c(B-A)^{2}\\ &=\int_0^\infty\left(\int_0^1\left[(B-A)^{1/2}(\lambda+(1-t)A+tB)^{-1}(B-A)^{1/2}\right]^2dt\right)\lambda^2d\mu(\lambda)\geq 0.\end{aligned}$$

Proof. We have by (1.9) that

$$\frac{f(t)-f(0)}{t}-f'_{+}(0)-ct=\mathcal{M}(\ell,\mu)$$

for some positive measure μ, where $\ell(t)=t$, $t>0$ and $c\geq 0$.

Therefore

$$\begin{aligned}&(B-A)^{1/2}\left[\mathcal{M}(\ell,\mu)(B)-\mathcal{M}(\ell,\mu)(A)\right](B-A)^{1/2}\\ &=(B-A)^{1/2}\left([f(B)-f(0)]B^{-1}-cB-[f(A)-f(0)]A^{-1}+cA\right)(B-A)^{1/2}\\ &=(B-A)^{1/2}\left(f(B)B^{-1}-f(0)B^{-1}-cB-f(A)A^{-1}+f(0)A^{-1}+cA\right)(B-A)^{1/2}\\ &=(B-A)^{1/2}\left(f(B)B^{-1}-f(A)A^{-1}-f(0)B^{-1}+f(0)A^{-1}-cB+cA\right)(B-A)^{1/2}\\ &=(B-A)^{1/2}\left[f(B)B^{-1}-f(A)A^{-1}\right](B-A)^{1/2}\\ &-f(0)(B-A)^{1/2}\left(B^{-1}-A^{-1}\right)(B-A)^{1/2}-c(B-A)^{1/2}(B-A)(B-A)^{1/2}\\ &=(B-A)^{1/2}\left[f(B)B^{-1}-f(A)A^{-1}\right](B-A)^{1/2}\\ &-f(0)(B-A)^{1/2}\left(B^{-1}-A^{-1}\right)(B-A)^{1/2}-c(B-A)^{2}\end{aligned}$$

and by (2.1) we derive (2.11). □

If more conditions are imposed on the operators A and B then better inequalities may be stated as follows:

Theorem 4. *If $\beta\geq A\geq\alpha>0$, $B>0$ with $M\geq B-A\geq m>0$ for some constants α, β, m, M, then*

$$\begin{aligned}(2.12)\quad 0&\leq\frac{m^2}{M^2}\left[\mathcal{M}(w,\mu)(M+\beta)-\mathcal{M}(w,\mu)(\beta)\right]\\ &\leq\frac{m^2}{M}\left[\mathcal{M}(w,\mu)(M+\beta)-\mathcal{M}(w,\mu)(\beta)\right](B-A)^{-1}\\ &\leq\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\ &\leq\frac{M^2}{m}\left[\mathcal{M}(w,\mu)(m+\alpha)-\mathcal{M}(w,\mu)(\alpha)\right](B-A)^{-1}\\ &\leq\frac{M^2}{m^2}\left[\mathcal{M}(w,\mu)(m+\alpha)-\mathcal{M}(w,\mu)(\alpha)\right].\end{aligned}$$

Proof. For $t\in[0,1]$ we have

$$\lambda+tB+(1-t)A=\lambda+t(B-A)+A.$$

Also,

$$\lambda+t(B-A)+A\geq\lambda+tm+A\geq\lambda+tm+\alpha=\lambda+(1-t)\alpha+t(m+\alpha)$$

for $t\in[0,1]$ and $\lambda\geq0$, which implies that

$$(\lambda+tB+(1-t)A)^{-1}\leq[\lambda+(1-t)\alpha+t(m+\alpha)]^{-1}$$

and, by multiplying both sides by $(B-A)^{1/2}\geq0$,

$$\begin{aligned}&(B-A)^{1/2}(\lambda+tB+(1-t)A)^{-1}(B-A)^{1/2}\\&\leq[\lambda+(1-t)\alpha+t(m+\alpha)]^{-1}(B-A)\\&\leq M[\lambda+(1-t)\alpha+t(m+\alpha)]^{-1}.\end{aligned}$$

Furthermore, by taking the square,

$$\begin{aligned}&\left[(B-A)^{1/2}(\lambda+tB+(1-t)A)^{-1}(B-A)^{1/2}\right]^2\\&\leq M^2[\lambda+(1-t)\alpha+t(m+\alpha)]^{-2}\end{aligned}$$

for $t\in[0,1]$ and $\lambda\geq0$, which implies by integration that

$$\begin{aligned}&\int_0^\infty\lambda w(\lambda)\left(\int_0^1\left[(B-A)^{1/2}(\lambda+tB+(1-t)A)^{-1}(B-A)^{1/2}\right]^2dt\right)d\mu(\lambda)\\&\leq M^2\int_0^\infty\lambda w(\lambda)\left(\int_0^1[\lambda+(1-t)\alpha+t(m+\alpha)]^{-2}dt\right)d\mu(\lambda)\\&=\frac{M^2}{m}\int_0^\infty\lambda w(\lambda)\left(\int_0^1[\lambda+(1-t)\alpha+t(m+\alpha)]^{-1}(m+\alpha-\alpha)\right.\\&\left.\times[\lambda+(1-t)\alpha+t(m+\alpha)]^{-1}dt\right)d\mu(\lambda)\ \text{(and by (2.7))}\\&=\frac{M^2}{m}[\mathcal{M}(w,\mu)(m+\alpha)-\mathcal{M}(w,\mu)(\alpha)].\end{aligned}$$

Using (2.1) we get

$$\begin{aligned}&(B-A)^{1/2}[\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)](B-A)^{1/2}\\&\leq\frac{M^2}{m}[\mathcal{M}(w,\mu)(m+\alpha)-\mathcal{M}(w,\mu)(\alpha)].\end{aligned}$$

Multiplying both sides with $(B-A)^{-1/2}$ we deduce the fourth inequality in (2.12).

We also have

$$\lambda+t(B-A)+A\leq\lambda+tM+A\leq\lambda+tM+\beta=\lambda+(1-t)\beta+t(M+\beta),$$

which implies that

$$(\lambda+tB+(1-t)A)^{-1}\geq[\lambda+(1-t)\beta+t(M+\beta)]^{-1}$$

and, by multiplying both sides by $(B-A)^{1/2} \geq 0$,

$$\begin{aligned}
&(B-A)^{1/2}(\lambda+tB+(1-t)A)^{-1}(B-A)^{1/2} \\
&\geq [\lambda+(1-t)\beta+t(M+\beta)]^{-1}(B-A) \\
&\geq m[\lambda+(1-t)\beta+t(M+\beta)]^{-1}
\end{aligned}$$

for $t \in [0,1]$ and $\lambda \geq 0$.

By taking the square, we get

$$\begin{aligned}
&\left[(B-A)^{1/2}(\lambda+tB+(1-t)A)^{-1}(B-A)^{1/2}\right]^2 \\
&\geq m^2[\lambda+(1-t)\beta+t(M+\beta)]^{-2}
\end{aligned}$$

for $t \in [0,1]$ and $\lambda \geq 0$.

By taking the integrals in this inequality we obtain

$$\begin{aligned}
&\int_0^\infty \lambda w(\lambda)\left(\int_0^1 \left[(B-A)^{1/2}(\lambda+tB+(1-t)A)^{-1}(B-A)^{1/2}\right]^2 dt\right) d\mu(\lambda) \\
&\geq m^2 \int_0^\infty \lambda w(\lambda)\left(\int_0^1 [\lambda+(1-t)\beta+t(M+\beta)]^{-2} dt\right) d\mu(\lambda) \\
&= \frac{m^2}{M}\int_0^\infty \lambda w(\lambda)\left(\int_0^1 [\lambda+(1-t)\beta+t(M+\beta)]^{-1}(M+\beta-\beta)\right. \\
&\left.\times[\lambda+(1-t)\beta+t(M+\beta)]^{-1} dt\right) d\mu(\lambda) \ \text{(and by (2.7))} \\
&= \frac{m^2}{M}[\mathcal{M}(w,\mu)(M+\beta)-\mathcal{M}(w,\mu)(\beta)].
\end{aligned}$$

Using (2.1) we derive

$$\begin{aligned}
&(B-A)^{1/2}[\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)](B-A)^{1/2} \\
&\geq \frac{m^2}{M}[\mathcal{M}(w,\mu)(M+\beta)-\mathcal{M}(w,\mu)(\beta)].
\end{aligned}$$

Multiplying both sides with $(B-A)^{-1/2}$ we deduce the second inequality in (2.12).

The rest of the inequalities are obvious. □

Its is well known that, if $P \geq 0$, then

$$|\langle Px,y\rangle|^2 \leq \langle Px,x\rangle\langle Py,y\rangle$$

for all x, $y \in H$.

Therefore, if $T>0$, then

$$\begin{aligned}
0 &\leq \langle x,x\rangle^2 = \left\langle T^{-1}Tx,x\right\rangle^2 = \left\langle Tx,T^{-1}x\right\rangle^2 \\
&\leq \langle Tx,x\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle = \langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle
\end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\|=1$, then

$$1 \leq \langle Tx,x\rangle\left\langle x,T^{-1}x\right\rangle \leq \langle Tx,x\rangle \sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle = \langle Tx,x\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} \leq T. \tag{2.13}$$

We also have that $T \leq \|T\|$ for $T \geq 0$.

Remark 1. *If $A > 0$ and $B - A > 0$, then obviously $\|A\| \geq A \geq \left\|A^{-1}\right\|^{-1}$ and $\|B-A\| \geq B - A \geq \left\|(B-A)^{-1}\right\|^{-1}$. So, if we take $\beta = \|A\|$, $\alpha = \left\|A^{-1}\right\|^{-1}$, $M = \|B-A\|$ and $m = \left\|(B-A)^{-1}\right\|^{-1}$ in (2.12), then we get*

$$
\begin{aligned}
(2.14)\quad 0 &\leq \frac{\mathcal{M}(w,\mu)(\|B-A\|+\|A\|) - \mathcal{M}(w,\mu)(\|A\|)}{\|B-A\|^2 \left\|(B-A)^{-1}\right\|^2} \\
&\leq \frac{\mathcal{M}(w,\mu)(\|B-A\|+\|A\|) - \mathcal{M}(w,\mu)(\|A\|)}{\|B-A\| \left\|(B-A)^{-1}\right\|^2}(B-A)^{-1} \\
&\leq \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) \\
&\leq \|B-A\|^2 \left\|(B-A)^{-1}\right\| \\
&\times \left[\mathcal{M}(w,\mu)\left(\left\|(B-A)^{-1}\right\|^{-1} + \left\|A^{-1}\right\|^{-1}\right) - \mathcal{M}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right)\right] \\
&\times (B-A)^{-1} \\
&\leq \|B-A\|^2 \left\|(B-A)^{-1}\right\|^2 \\
&\times \left[\mathcal{M}(w,\mu)\left(\left\|(B-A)^{-1}\right\|^{-1} + \left\|A^{-1}\right\|^{-1}\right) - \mathcal{M}(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right)\right].
\end{aligned}
$$

The case of operator monotone functions is as follows:

Corollary 3. *Assume that $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). If $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then*

$$
\begin{aligned}
(2.15)\quad 0 &\leq \frac{m^2}{M^2}\left[f(M+\beta) - f(\beta) - bM\right] \\
&\leq \frac{m^2}{M}\left[f(M+\beta) - f(\beta) - bM\right](B-A)^{-1} \\
&\leq f(B) - f(A) - b(B-A) \\
&\leq \frac{M^2}{m}\left[f(m+\alpha) - f(\alpha) - bm\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[f(m+\alpha) - f(\alpha) - bm\right].
\end{aligned}
$$

Proof. From (1.9) we have

$$\mathcal{M}(\ell,\mu)(B) - \mathcal{M}(\ell,\mu)(A) = f(B) - f(A) - b(B-A),$$

$$\mathcal{M}(\ell,\mu)(M+\beta) - \mathcal{M}(\ell,\mu)(\beta) = f(M+\beta) - f(\beta) - bM$$

and

$$\mathcal{M}(\ell,\mu)(m+\alpha) - \mathcal{M}(\ell,\mu)(\alpha) = f(m+\alpha) - f(\alpha) - bm$$

and by (2.12) we obtain (2.15). □

Remark 2. *If we apply (2.15) for the power function $f(t)=t^r$, $r\in(0,1]$ via the representation (1.16), then we get*

$$0\leq\frac{m^2}{M^2}\left[(M+\beta)^r-\beta^r\right]\leq\frac{m^2}{M}\left[f(M+\beta)^r-\beta^r\right](B-A)^{-1}\leq B^r-A^r\leq\frac{M^2}{m}\left[(m+\alpha)^r-\alpha^r\right](B-A)^{-1}\leq\frac{M^2}{m^2}\left[(m+\alpha)^r-\alpha^r\right], \tag{2.16}$$

where $\beta\geq A\geq\alpha>0$, $B>0$ with $M\geq B-A\geq m>0$ for some constants α, β, m, M.

From (2.15) we have

$$\frac{m^2}{M}\left[f(M+\beta)-f(\beta)-bM\right](B-A)^{-1}+b(B-A)\leq f(B)-f(A)\leq\frac{M^2}{m}\left[f(m+\alpha)-f(\alpha)-bm\right](B-A)^{-1}+b(B-A). \tag{2.17}$$

Observe that

$$\frac{m^2}{M}\left[f(M+\beta)-f(\beta)-bM\right](B-A)^{-1}+b(B-A)=\frac{m^2}{M}\left[f(M+\beta)-f(\beta)\right](B-A)^{-1}-m^2b(B-A)^{-1}+b(B-A).$$

Since $B-A\geq m>0$, hence $(B-A)^2\geq m^2>0$ and by multiplying both sides with $(B-A)^{-1/2}$ we get $B-A\geq m^2(B-A)^{-1}$ which implies that $b(B-A)\geq m^2b(B-A)^{-1}$. From this we derive that

$$\frac{m^2}{M}\left[f(M+\beta)-f(\beta)\right](B-A)^{-1}-m^2b(B-A)^{-1}+b(B-A)\geq\frac{m^2}{M}\left[f(M+\beta)-f(\beta)\right](B-A)^{-1}. \tag{2.18}$$

Also

$$\frac{M^2}{m}\left[f(m+\alpha)-f(\alpha)-bm\right](B-A)^{-1}+b(B-A)=\frac{M^2}{m}\left[f(m+\alpha)-f(\alpha)\right](B-A)^{-1}-bM^2(B-A)^{-1}+b(B-A).$$

Since $B-A\leq M$, hence $(B-A)^2\leq M^2$ and by multiplying both sides with $(B-A)^{-1/2}$ we get $B-A\leq M^2(B-A)^{-1}$ which implies that $b(B-A)\leq M^2b(B-A)^{-1}$. From this we derive that

$$\frac{M^2}{m}\left[f(m+\alpha)-f(\alpha)\right](B-A)^{-1}-bM^2(B-A)^{-1}+b(B-A)\leq\frac{M^2}{m}\left[f(m+\alpha)-f(\alpha)\right](B-A)^{-1}. \tag{2.19}$$

Using (2.17)-(2.19) we can state the following lower and upper bounds for the difference $f(B)-f(A)$ which does not depend on the nonnegative parameter b.

Proposition 1. *Assume that $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $\beta\geq A\geq\alpha>0$, $B>0$ with $M\geq B-A\geq m>0$ for some constants α, β, m, M, then*

$$\begin{aligned}(2.20)\qquad 0 &\leq \frac{m^2}{M^2}\left[f\left(M+\beta\right)-f\left(\beta\right)\right]\leq\frac{m^2}{M}\left[f\left(M+\beta\right)-f\left(\beta\right)\right]\left(B-A\right)^{-1}\\ &\leq f\left(B\right)-f\left(A\right)\\ &\leq\frac{M^2}{m}\left[f\left(m+\alpha\right)-f\left(\alpha\right)\right]\left(B-A\right)^{-1}\leq\frac{M^2}{m^2}\left[f\left(m+\alpha\right)-f\left(\alpha\right)\right].\end{aligned}$$

Remark 3. . *If we write the inequality (2.20) for the operator monotone function $f(t)=\ln t$, $t\in(0,\infty)$, then we obtain the logarithmic inequalities*

$$\begin{aligned}(2.21)\qquad 0 &\leq \frac{m^2}{M^2}\ln\left(\frac{M+\beta}{\beta}\right)\leq\frac{m^2}{M}\ln\left(\frac{M+\beta}{\beta}\right)\left(B-A\right)^{-1}\\ &\leq \ln B-\ln A\\ &\leq\frac{M^2}{m}\ln\left(\frac{m+\alpha}{\alpha}\right)\left(B-A\right)^{-1}\leq\frac{M^2}{m^2}\ln\left(\frac{m+\alpha}{\alpha}\right).\end{aligned}$$

The case of operator convex functions is as follows:

Corollary 4. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.11) with $f(0)=0$. If $\beta\geq A\geq\alpha>0$, $B>0$ with $M\geq B-A\geq m>0$ for some constants α, β, m, M, then*

$$\begin{aligned}(2.22)\qquad 0 &\leq \frac{m^2}{M^2}\left[f\left(M+\beta\right)\left(M+\beta\right)^{-1}-f\left(\beta\right)\beta^{-1}-cM\right]\\ &\leq\frac{m^2}{M}\left[f\left(M+\beta\right)\left(M+\beta\right)^{-1}-f\left(\beta\right)\beta^{-1}-cM\right]\left(B-A\right)^{-1}\\ &\leq f\left(B\right)B^{-1}-f\left(A\right)A^{-1}-c\left(B-A\right)\\ &\leq\frac{M^2}{m}\left[f\left(m+\alpha\right)\left(m+\alpha\right)^{-1}-f\left(\alpha\right)\alpha^{-1}-cm\right]\left(B-A\right)^{-1}\\ &\leq\frac{M^2}{m^2}\left[f\left(m+\alpha\right)\left(m+\alpha\right)^{-1}-f\left(\alpha\right)\alpha^{-1}-cm\right].\end{aligned}$$

Proof. From (1.11) we derive that

$$\mathcal{M}\left(\ell,\mu\right)\left(B\right)-\mathcal{M}\left(\ell,\mu\right)\left(A\right)=\left[f\left(B\right)B^{-1}-f\left(A\right)A^{-1}\right]-c\left(B-A\right),$$

$$\begin{aligned}&\mathcal{M}\left(\ell,\mu\right)\left(M+\beta\right)-\mathcal{M}\left(\ell,\mu\right)\left(\beta\right)\\ &=f\left(M+\beta\right)\left(M+\beta\right)^{-1}-f\left(\beta\right)\beta^{-1}-cM\end{aligned}$$

and

$$\begin{aligned}&\mathcal{M}\left(\ell,\mu\right)\left(m+\alpha\right)-\mathcal{M}\left(\ell,\mu\right)\left(\alpha\right)\\ &=f\left(m+\alpha\right)\left(m+\alpha\right)^{-1}-f\left(\alpha\right)\alpha^{-1}-cm\end{aligned}$$

and by (2.12) we get (2.22). □

Utilising a similar argument as above, we can prove the following result in which the positive constant c is not involved:

Proposition 2. *Assume that* $f : [0, \infty) \to \mathbb{R}$ *is operator convex in* $[0, \infty)$ *with* $f(0) = 0$. *If* $\beta \geq A \geq \alpha > 0$, $B > 0$ *with* $M \geq B - A \geq m > 0$ *for some constants* α, β, m, M, *then*

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2}\left[f(M+\beta)(M+\beta)^{-1} - f(\beta)\beta^{-1}\right] \\ &\leq \frac{m^2}{M}\left[f(M+\beta)(M+\beta)^{-1} - f(\beta)\beta^{-1}\right](B-A)^{-1} \\ &\leq f(B)B^{-1} - f(A)A^{-1} \\ &\leq \frac{M^2}{m}\left[f(m+\alpha)(m+\alpha)^{-1} - f(\alpha)\alpha^{-1}\right](B-A)^{-1} \\ &\leq \frac{M^2}{m^2}\left[f(m+\alpha)(m+\alpha)^{-1} - f(\alpha)\alpha^{-1}\right]. \end{aligned} \tag{2.23}$$

If we consider the operator convex function $f(t) = -\ln(t+1)$, then we get

$$\begin{aligned} 0 &\leq \frac{m^2}{M^2}\left[\beta^{-1}\ln(\beta+1) - (M+\beta)^{-1}\ln(M+\beta+1)\right] \\ &\leq \frac{m^2}{M}\left[\beta^{-1}\ln(\beta+1) - (M+\beta)^{-1}\ln(M+\beta+1)\right](B-A)^{-1} \\ &\leq A^{-1}\ln(A+1) - B^{-1}\ln(B+1) \\ &\leq \frac{M^2}{m}\left[\alpha^{-1}\ln(\alpha+1) - (m+\alpha)^{-1}\ln(m+\alpha+1)\right](B-A)^{-1} \\ &\leq \frac{M^2}{m^2}\left[\alpha^{-1}\ln(\alpha+1) - (m+\alpha)^{-1}\ln(m+\alpha+1)\right], \end{aligned} \tag{2.24}$$

where $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$.

We define the *upper incomplete Gamma function* as [9]

$$\Gamma(a, z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z = 0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [10]

$$\Gamma(a, z) = \frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{2.25}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (2.25) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a, t) \tag{2.26}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (2.26) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0, t) = e^tE_1(t) \tag{2.27}$$

for $t>0$, where

$$E_1(t):=\int_t^\infty \frac{e^{-u}}{u}du.$$

We then have

$$\mathcal{M}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T)=\Gamma(1-a)T^{1-a}\exp(T)\,\Gamma(a,T) \tag{2.28}$$

for $a<1$ and

$$\mathcal{M}\left(w_{e^{-\cdot}}\right)(T)=T\exp(T)\,E_1(T) \tag{2.29}$$

for $T>0$.

Proposition 3. *If $\beta\geq A\geq\alpha>0$, $B>0$ with $M\geq B-A\geq m>0$ for some constants α, β, m, M, then*

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[(M+\beta)^{1-a}\,\Gamma(a,M+\beta)\exp(M+\beta)-\beta^{1-a}\Gamma(a,\beta)\exp\beta\right] \\
&\leq \frac{m^2}{M}\left[(M+\beta)^{1-a}\,\Gamma(a,M+\beta)\exp(M+\beta)-\beta^{1-a}\Gamma(a,\beta)\exp\beta\right] \\
&\times(B-A)^{-1} \\
&\leq B^{1-a}\exp(B)\,\Gamma(a,B)-A^{1-a}\exp(A)\,\Gamma(a,A) \\
&\leq \frac{M^2}{m}\left[(m+\alpha)^{1-a}\,\Gamma(a,m+\alpha)\exp(m+\alpha)-\alpha^{1-a}\Gamma(a,\alpha)\exp(\alpha)\right] \\
&\times(B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[(m+\alpha)^{1-a}\,\Gamma(a,m+\alpha)\exp(m+\alpha)-\alpha^{1-a}\Gamma(a,\alpha)\exp(\alpha)\right]
\end{aligned} \tag{2.30}$$

for $a<1$.

In particular, for $a=0$,

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[(M+\beta)\,E_1(M+\beta)\exp(M+\beta)-\beta E_1(\beta)\exp\beta\right] \\
&\leq \frac{m^2}{M}\left[(M+\beta)\,E_1(M+\beta)\exp(M+\beta)-\beta E_1(\beta)\exp\beta\right] \\
&\times(B-A)^{-1} \\
&\leq B\exp(B)\,E_1(B)-A\exp(A)\,E_1(A) \\
&\leq \frac{M^2}{m}\left[(m+\alpha)\,E_1(m+\alpha)\exp(m+\alpha)-\alpha E_1(\alpha)\exp(\alpha)\right] \\
&\times(B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[(m+\alpha)\,E_1(m+\alpha)\exp(m+\alpha)-\alpha E_1(\alpha)\exp(\alpha)\right].
\end{aligned} \tag{2.31}$$

The interested reader may state other similar results by employing the examples of monotone operator functions provided in [2], [3], [4], [7] and [8].

3. Integral Inequalities

We can state now the following integral inequalities:

Theorem 5. *If $\beta \geq A \geq \alpha > 0$, $B > 0$ with $M \geq B - A \geq m > 0$ for some constants α, β, m, M, then*

$$
\begin{aligned}
(3.1)\quad 0 &\leq \frac{m^2}{M^2}\left[\int_0^1 \mathcal{M}(w,\mu)(tM+\beta)\,tdt - \frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)(tM+\beta)\,dt\right] \\
&\leq \frac{m^2}{M}\left[\int_0^1 \mathcal{M}(w,\mu)(tM+\beta)\,tdt - \frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)(tM+\beta)\,dt\right] \\
&\times (B-A)^{-1} \\
&\leq \int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,tdt - \frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt \\
&\leq \frac{M^2}{m}\left[\int_0^1 \mathcal{M}(w,\mu)(tm+\alpha)\,tdt - \frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)(tm+\alpha)\,dt\right] \\
&\times (B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\int_0^1 \mathcal{M}(w,\mu)(tm+\alpha)\,tdt - \frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)(tm+\alpha)\,dt\right].
\end{aligned}
$$

Proof. Let $t > s$ with $t,\ s \in [0,1]$, then

$$(1-t)A + tB - ((1-s)A + sB) = (t-s)(B-A),$$

which gives

$$(t-s)M \geq (1-t)A + tB - ((1-s)A + sB) \geq (t-s)m.$$

We also have

$$\alpha + sm \leq ((1-s)A + sB) = A + s(B-A) \leq \beta + sM.$$

If we use the inequality (2.12), then we get

$$
\begin{aligned}
0 &\leq \frac{[(t-s)m]^2}{[(t-s)M]^2}\left[\mathcal{M}(w,\mu)((t-s)M+\beta+sM) - \mathcal{M}(w,\mu)(\beta+sM)\right] \\
&\leq \frac{[(t-s)m]^2}{[(t-s)M]}\left[\mathcal{M}(w,\mu)((t-s)M+\beta+sM) - \mathcal{M}(w,\mu)(\beta+sM)\right] \\
&\times ((t-s)(B-A))^{-1} \\
&\leq \mathcal{M}(w,\mu)((1-t)A+tB) - \mathcal{M}(w,\mu)((1-s)A+sB) \\
&\leq \frac{[(t-s)M]^2}{(t-s)m}\left[\mathcal{M}(w,\mu)((t-s)m+\alpha+sm) - \mathcal{M}(w,\mu)(\alpha+sm)\right] \\
&\times ((t-s)(B-A))^{-1} \\
&\leq \frac{[(t-s)M]^2}{[(t-s)m]^2}\left[\mathcal{M}(w,\mu)((t-s)m+\alpha+sm) - \mathcal{M}(w,\mu)(\alpha+sm)\right],
\end{aligned}
$$

namely

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\mathcal{M}(w,\mu)(tM+\beta)-\mathcal{M}(w,\mu)(sM+\beta)\right] \\
&\leq \frac{m^2}{M}\left[\mathcal{M}(w,\mu)(tM+\beta)-\mathcal{M}(w,\mu)(sM+\beta)\right](B-A)^{-1} \\
&\leq \mathcal{M}(w,\mu)((1-t)A+tB)-\mathcal{M}(w,\mu)((1-s)A+sB) \\
&\leq \frac{M^2}{m}\left[\mathcal{M}(w,\mu)(tm+\alpha)-\mathcal{M}(w,\mu)(sm+\alpha)\right](B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{M}(w,\mu)(tm+\alpha)-\mathcal{M}(w,\mu)(sm+\alpha)\right],
\end{aligned}$$

for $t>s$ with $t,\ s\in[0,1]$.

If we multiply this inequality by $(t-s)\geq 0$, then we get

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\left[\mathcal{M}(w,\mu)(tM+\beta)-\mathcal{M}(w,\mu)(sM+\beta)\right](t-s) \\
&\leq \frac{m^2}{M}\left[\mathcal{M}(w,\mu)(tM+\beta)-\mathcal{M}(w,\mu)(sM+\beta)\right](t-s)(B-A)^{-1} \\
&\leq \left[\mathcal{M}(w,\mu)((1-t)A+tB)-\mathcal{M}(w,\mu)((1-s)A+sB)\right](t-s) \\
&\leq \frac{M^2}{m}\left[\mathcal{M}(w,\mu)(tm+\alpha)-\mathcal{M}(w,\mu)(sm+\alpha)\right](t-s)(B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\left[\mathcal{M}(w,\mu)(tm+\alpha)-\mathcal{M}(w,\mu)(sm+\alpha)\right](t-s).
\end{aligned} \tag{3.2}$$

This inequality also holds for $t<s$ with $t,\ s\in[0,1]$, and therefore for all t, $s\in[0,1]$.

Furthermore, by integrating (3.2) on $[0,1]^2$, we obtain

$$\begin{aligned}
0 &\leq \frac{m^2}{M^2}\int_0^1\int_0^1\left[\mathcal{M}(w,\mu)(tM+\beta)-\mathcal{M}(w,\mu)(sM+\beta)\right](t-s)\,dtds \\
&\leq \frac{m^2}{M}\int_0^1\int_0^1\left[\mathcal{M}(w,\mu)(tM+\beta)-\mathcal{M}(w,\mu)(sM+\beta)\right](t-s)\,dtds \\
&\times(B-A)^{-1} \\
&\leq \int_0^1\int_0^1\left[\mathcal{M}(w,\mu)((1-t)A+tB)-\mathcal{M}(w,\mu)((1-s)A+sB)\right] \\
&\times(t-s)\,dtds \\
&\leq \frac{M^2}{m}\int_0^1\int_0^1\left[\mathcal{M}(w,\mu)(tm+\alpha)-\mathcal{M}(w,\mu)(sm+\alpha)\right](t-s)\,dtds \\
&\times(B-A)^{-1} \\
&\leq \frac{M^2}{m^2}\int_0^1\int_0^1\left[\mathcal{M}(w,\mu)(tm+\alpha)-\mathcal{M}(w,\mu)(sm+\alpha)\right](t-s)\,dtds.
\end{aligned} \tag{3.3}$$

Now, observe that

$$\begin{aligned}
&\int_0^1\int_0^1 \left[\mathcal{M}(w,\mu)(tM+\beta)-\mathcal{M}(w,\mu)(sM+\beta)\right](t-s)\,dtds\\
&=\int_0^1\int_0^1 \left[\mathcal{M}(w,\mu)(tM+\beta)\,t+\mathcal{M}(w,\mu)(sM+\beta)\,s\right]dtds\\
&-\int_0^1\int_0^1 \left[\mathcal{M}(w,\mu)(tM+\beta)\,s+\mathcal{M}(w,\mu)(sM+\beta)\,t\right]dtds\\
&=2\left[\int_0^1 \mathcal{M}(w,\mu)(tM+\beta)\,tdt-\frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)(tM+\beta)\,dt\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 \left[\mathcal{M}(w,\mu)((1-t)A+tB)-\mathcal{M}(w,\mu)((1-s)A+sB)\right](t-s)\,dtds\\
&=\int_0^1\int_0^1 \left[\mathcal{M}(w,\mu)((1-t)A+tB)\,t+\mathcal{M}(w,\mu)((1-s)A+sB)\,s\right]dtds\\
&-\int_0^1\int_0^1 \left[\mathcal{M}(w,\mu)((1-t)A+tB)\,s+\mathcal{M}(w,\mu)((1-s)A+sB)\,t\right]dtds\\
&=2\left[\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,tdt-\frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt\right].
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_0^1\int_0^1 \left[\mathcal{M}(w,\mu)(tm+\alpha)-\mathcal{M}(w,\mu)(sm+\alpha)\right](t-s)\,dtds\\
&=2\left[\int_0^1 \mathcal{M}(w,\mu)(tm+\alpha)\,tdt-\frac{1}{2}\int_0^1 \mathcal{M}(w,\mu)(tm+\alpha)\,dt\right]
\end{aligned}$$

and by (3.3) we get the desired result (3.1). □

The case of operator monotone functions is as follows:

Corollary 5. *Assume that $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $\beta\geq A\geq\alpha>0$, $B>0$ with $M\geq B-A\geq m>0$ for some constants α, β, m, M, then*

$$\begin{aligned}
0&\leq\frac{m^2}{M^2}\left[\int_0^1 f(tM+\beta)\,tdt-\frac{1}{2}\int_0^1 f(tM+\beta)\,dt\right] \qquad (3.4)\\
&\leq\frac{m^2}{M}\left[\int_0^1 f(tM+\beta)\,tdt-\frac{1}{2}\int_0^1 f(tM+\beta)\,dt\right](B-A)^{-1}\\
&\leq\int_0^1 f((1-t)A+tB)\,tdt-\frac{1}{2}\int_0^1 f((1-t)A+tB)\,dt\\
&\leq\frac{M^2}{m}\left[\int_0^1 f(tm+\alpha)\,tdt-\frac{1}{2}\int_0^1 f(tm+\alpha)\,dt\right](B-A)^{-1}\\
&\leq\frac{M^2}{m^2}\left[\int_0^1 f(tm+\alpha)\,tdt-\frac{1}{2}\int_0^1 f(tm+\alpha)\,dt\right].
\end{aligned}$$

Proof. The proof follows by Proposition 1 and Theorem 5 for $\mathcal{M}(w,\mu)(t)=f(t)-a$. We omit the details. □

For a, $b>0$ and $t\in(0,1]$ we define

$$P(a,b,r):=\int_0^1 (at+b)^r\left(t-\frac{1}{2}\right)dt>0.$$

This can be calculated as follows

$$\begin{aligned}P(a,b,r)&=\frac{1}{a(r+1)}\int_0^1\left(t-\frac{1}{2}\right)d\left((at+b)^{r+1}\right)\\&=\frac{1}{a(r+1)}\left[\left(t-\frac{1}{2}\right)(at+b)^{r+1}\Big|_0^1-\int_0^1(at+b)^{r+1}dt\right]\\&=\frac{1}{a(r+1)}\left[\frac{(a+b)^{r+1}-b^{r+1}}{2}-\frac{(a+b)^{r+2}-b^{r+2}}{a(r+2)}\right].\end{aligned}$$

Remark 4. *If $\beta\geq A\geq\alpha>0$, $B>0$ with $M\geq B-A\geq m>0$ for some constants α, β, m, M, then for $r\in(0,1]$*

$$\begin{aligned}0&\leq\frac{m^2}{M^2}P(M,\beta,r)\leq\frac{m^2}{M}P(M,\beta,r)(B-A)^{-1}\\&\leq\int_0^1((1-t)A+tB)^r\,tdt-\frac{1}{2}\int_0^1 f((1-t)A+tB)^r\,dt\\&\leq\frac{M^2}{m}P(m,\beta,\alpha)(B-A)^{-1}\leq\frac{M^2}{m^2}P(m,\beta,\alpha).\end{aligned}\tag{3.5}$$

For a, $b>0$ we also define

$$L(a,b):=\int_0^1\left(t-\frac{1}{2}\right)\ln(at+b)\,dt>0.$$

Calculation provides the following expression for $L(a,b)$,

$$L(a,b)=\frac{1}{4}+\frac{b}{2a}-\frac{b(b+a)}{2a^2}\ln\left(\frac{b+a}{b}\right).$$

Remark 5. *If $\beta\geq A\geq\alpha>0$, $B>0$ with $M\geq B-A\geq m>0$ for some constants α, β, m, M, then*

$$\begin{aligned}0&\leq\frac{m^2}{M^2}L(M,\beta)\leq\frac{m^2}{M}L(M,\beta)(B-A)^{-1}\\&\leq\int_0^1\ln((1-t)A+tB)\,tdt-\frac{1}{2}\int_0^1\ln((1-t)A+tB)\,dt\\&\leq\frac{M^2}{m}L(m,\alpha)(B-A)^{-1}\leq\frac{M^2}{m^2}L(m,\alpha).\end{aligned}\tag{3.6}$$

The interested reader may state similar results for operator convex functions. However the details are not presented here.

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