LIPSCHITZ TYPE INEQUALITIES FOR MONOTONIC
INTEGRAL TRANSFORM OF POSITIVE OPERATORS WITH
APPLICATIONS

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Abstract. For a continuous and positive function \( w(\lambda), \lambda > 0 \) and \( \mu \) a positive measure on \((0, \infty)\) we consider the following monotonic integral transform

\[
\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T(\lambda + T)^{-1} d\mu(\lambda),
\]

where the integral is assumed to exist for \( T \) a positive operator on a complex Hilbert space \( \mathcal{H} \).

Assume that \( A \geq m_1 > 0, B \geq m_2 > 0 \), then we show that

\[
\| \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \| \leq \| B - A \| \begin{cases}
\frac{\mathcal{M}(w, \mu)(m_2) - \mathcal{M}(w, \mu)(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\mathcal{M}'(w, \mu)(m) & \text{if } m_1 = m_2 = m,
\end{cases}
\]

where \( \mathcal{M}'(w, \mu)(t) \) is the derivative of \( \mathcal{M}(w, \mu) \) as a function of \( t \). If the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\), then

\[
\| f(B) - f(A) \| \leq \| B - A \| \begin{cases}
\frac{f(m_2) - f(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
f'(m) & \text{if } m_1 = m_2 = m.
\end{cases}
\]

In particular we have the power inequalities

\[
\| B^r - A^r \| \leq \| B - A \| \begin{cases}
\frac{m_2^r - m_1^r}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
r(m_2^r - m_1^r) & \text{if } m_1 = m_2 = m,
\end{cases}
\]

and the logarithmic inequalities

\[
\| \ln B - \ln A \| \leq \| B - A \| \begin{cases}
\frac{\ln m_2 - \ln m_1}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\frac{1}{m} & \text{if } m_1 = m_2 = m.
\end{cases}
\]

Some applications for operator convex functions and midpoint and trapezoid norm inequalities are also provided.

1. Introduction

Let \( \mathcal{B}(\mathcal{H}) \) be the Banach algebra of bounded linear operators on a complex Hilbert space \( \mathcal{H} \). The absolute value of an operator \( A \) is the positive operator \( |A| \) defined as \( |A| := (A^*A)^{1/2} \).

It is known that [3] in the infinite-dimensional case the map \( f(A) := |A| \) is not Lipschitz continuous on \( \mathcal{B}(\mathcal{H}) \) with the usual operator norm, i.e. there is no

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constant $L > 0$ such that
\[ \|A| - |B|\| \leq L \|A - B\| \]
for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [7], [8] and Kato in [14], the following inequality holds
\begin{equation}
\|A| - |B|\| \leq \frac{2}{\pi} \|A - B\| \left( 2 + \log \left( \frac{\|A\| + \|B\|}{\|A - B\|} \right) \right)
\end{equation}
for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with Hilbert-Schmidt norm $\|C\|_{HS} := (\text{tr} C^*C)^{1/2}$ of an operator $C$, then the following inequality is true [1]
\begin{equation}
\|A| - |B|\|_{HS} \leq \sqrt{2} \|A - B\|_{HS}
\end{equation}
for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general $A$ and $B$. If $A$ and $B$ are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if $A$ is an invertible operator, then for all operators $B$ in a neighborhood of $A$ we have
\begin{equation}
\|A| - |B|\| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O \left( \|A - B\|^3 \right),
\end{equation}
where
\[ a_1 = \|A^{-1}\| \|A\| \quad \text{and} \quad a_2 = \|A^{-1}\| + \|A^{-1}\|^3 \|A\|^2. \]

In [2] the author also obtained the following Lipschitz type inequality
\begin{equation}
\|f(A) - f(B)\| \leq f'(a) \|A - B\|
\end{equation}
where $f$ is an operator monotone function on $(0, \infty)$ and $A, B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A - B\|$ for different classes of measurable functions $f$ for which the function of operator can be defined. For some results on this topic, see [4], [9] and the references therein.

We have the following representation of operator monotone functions [15], see for instance [5, p. 144-145]:

**Theorem 1.** A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation
\begin{equation}
f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda),
\end{equation}
where $a \in \mathbb{R}$, $b \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ such that
\begin{equation}
\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty.
\end{equation}
If $f$ is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.5).

A real valued continuous function $f$ on an interval $I$ is said to be operator convex (operator concave) on $I$ if
\begin{equation}
f((1-\lambda)A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B)
\end{equation}
in the operator order, for all \( \lambda \in [0, 1] \) and for every selfadjoint operator \( A \) and \( B \) on a Hilbert space \( H \) whose spectra are contained in \( I \). Notice that a function \( f \) is operator concave if \( -f \) is operator convex.

We have the following representation of operator convex functions [5, p. 147]:

**Theorem 2.** A function \( f : (0, \infty) \to \mathbb{R} \) is operator convex in \( (0, \infty) \) if and only if it has the representation

\[
 f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t + \lambda} d\mu(\lambda),
\]

where \( a, b \in \mathbb{R}, c \geq 0 \) and a positive measure \( \mu \) on \( (0, \infty) \) such that (1.2) holds. If \( f \) is operator convex in \( [0, \infty) \), then \( a = f(0) \) and \( b = f'_+(0) \), the right derivative, in (1.5).

We have the following integral representation for the power function when \( t > 0 \), \( r \in (0, 1] \), see for instance [5, p. 145]

\[
 t^{r-1} = \frac{\sin(r \pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda.
\]

Motivated by these representations, we introduce, for a continuous and positive function \( w(\lambda), \lambda > 0 \), the following integral transform

\[
 D(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \quad t > 0,
\]

where \( \mu \) is a positive measure on \( (0, \infty) \) and the integral (1.8) exists for all \( t > 0 \).

For \( \mu \) the Lebesgue usual measure, we put

\[
 D(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \quad t > 0.
\]

Now, assume that \( T > 0 \), then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

\[
 D(w, \mu)(T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\mu(\lambda),
\]

where \( w \) and \( \mu \) are as above. Also, when \( \mu \) is the usual Lebesgue measure, then

\[
 D(w)(T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\lambda,
\]

for \( T > 0 \).

If we take \( \mu \) to be the usual Lebesgue measure and the kernel \( w_r(\lambda) = \lambda^{r-1}, r \in (0, 1] \), then

\[
 t^{r-1} = \frac{\sin(r \pi)}{\pi} D(w_r)(t), \quad t > 0.
\]

For a continuous and positive function \( w(\lambda), \lambda > 0 \) and a positive measure \( \mu \) on \( (0, \infty) \), we can define the following mapping, which we call monotonic integral transform, by

\[
 M(w, \mu)(t) := tD(w, \mu)(t), \quad t > 0.
\]
For \( t > 0 \) we have
\[
M(w, \mu)(t) := tD(w, \mu)(t) = \int_0^\infty w(\lambda) t (t + \lambda)^{-1} d\mu(\lambda) \\
= \int_0^\infty w(\lambda) (t + \lambda - \lambda) (t + \lambda)^{-1} d\mu(\lambda) \\
= \int_0^\infty w(\lambda) \left[ 1 - \lambda (t + \lambda)^{-1} \right] d\mu(\lambda).
\]

If \( \int_0^\infty w(\lambda) d\mu(\lambda) < \infty \), then
\[
M(w, \mu)(t) = \int_0^\infty w(\lambda) d\mu(\lambda) - D(\ell w, \mu)(t),
\]
where \( \ell(t) = t, t > 0 \).

Consider the kernel \( e_{-a}(\lambda) := \exp(-a\lambda), \lambda \geq 0 \) and \( a > 0 \). Then, after some calculations, we get
\[
D(e_{-a})(t) = \int_0^\infty \exp(-a\lambda) t + \lambda d\lambda = E_1(at) \exp(at), t \geq 0
\]
and
\[
\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a},
\]
where the exponential integral is defined by
\[
E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.
\]
This gives that
\[
M(e_{-a})(t) = t D(w, \mu)(t) = t E_1(at) \exp(at), t \geq 0.
\]

By integration we also have
\[
D(\ell e_{-a}, \mu)(t) = \int_0^\infty \frac{\lambda \exp(-a\lambda)}{t + \lambda} d\lambda = \frac{1}{a} - t E_1(at) \exp(at)
\]
for \( t > 0 \).

One observes that
\[
M(e_{-a})(t) = \int_0^\infty w(\lambda) d\lambda - D(\ell e_{-a}, \mu)(t), t > 0
\]
and the equality (1.15) is verified in this case.

If we take \( w_r(\lambda) = \lambda^{r-1}, r \in (0, 1] \), then \( \int_0^\infty w_r(\lambda) d\lambda = \infty \) and the equality (1.15) does not hold in this case.

For all \( T > 0 \) we have, by the continuous functional calculus for selfadjoint operators, that
\[
M(w, \mu)(T) = T D(w, \mu)(T) = \int_0^\infty w(\lambda) \left[ 1 - \lambda (T + \lambda)^{-1} \right] d\mu(\lambda).
\]
This gives the representation
\[
T^r = \frac{\sin(r\pi)}{\pi} M(w_r, \mu)(T),
\]
where \( w_r(\lambda) = \lambda^{r-1}, r \in (0, 1] \) and \( \mu \) is the usual Lebesgue norm.
Assume that \( A = m_1 > 0, B = m_2 > 0 \), then we show that
\[
\| \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) \| \leq \| B - A \| \begin{cases} \frac{\mathcal{M}(w, \mu)(m_2) - \mathcal{M}(w, \mu)(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \mathcal{M}'(w, \mu)(m) & \text{if } m_1 = m_2 = m, \end{cases}
\]
where \( \mathcal{M}'(w, \mu)(t) \) is the derivative of \( \mathcal{M}(w, \mu) \) as a function of \( t \). If the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \( (0, \infty) \), then
\[
\| f(B) - f(A) \| \leq \| B - A \| \begin{cases} \frac{f(m_2) - f(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ f'(m) & \text{if } m_1 = m_2 = m. \end{cases}
\]
In particular we have the power inequalities
\[
\| B^r - A^r \| \leq \| B - A \| \begin{cases} \frac{m_2^r - m_1^r}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ rm^{r-1} & \text{if } m_1 = m_2 = m, \end{cases}
\]
and the logarithmic inequalities
\[
\| \ln B - \ln A \| \leq \| B - A \| \begin{cases} \frac{\ln m_2 - \ln m_1}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \frac{1}{m} & \text{if } m_1 = m_2 = m. \end{cases}
\]
Some applications for operator convex functions and midpoint and trapezoid norm inequalities are also provided.

2. Main Results

We have the following equality that is of interest in itself:

**Lemma 1.** For all \( A, B > 0 \) we have the representation
\[
\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) = \int_0^1 \left( \int_0^1 (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} dt \right) \times \lambda w(\lambda) \ d\mu(\lambda).
\]

**Proof.** From (1.16) we have for all \( A, B \geq 0 \) that
\[
\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) = \int_0^\infty w(\lambda) \left[ 1 - \lambda (B + \lambda)^{-1} \right] d\mu(\lambda) - \int_0^\infty w(\lambda) \left[ 1 - \lambda (A + \lambda)^{-1} \right] d\mu(\lambda)
\]
\[
= \int_0^\infty \lambda w(\lambda) \left[ (A + \lambda)^{-1} - (B + \lambda)^{-1} \right] d\mu(\lambda).
\]
Let \( T, S > 0 \). The function \( f(t) = -t^{-1} \) is operator monotone on \((0, \infty)\), operator Gâteaux differentiable and the Gâteaux derivative is given by
\[
\nabla f_T(S) := \lim_{\delta T \to 0} \left[ \frac{f(T + \delta T) - f(T)}{\delta T} \right] = T^{-1} ST^{-1}
\]
for \( T, S > 0 \).
Consider the continuous function $f$ defined on an interval $I$ for which the corresponding operator function is Gâteaux differentiable on the segment $[C, D] = \{(1 - t)C + tD, \ t \in [0, 1]\}$ for $C$, $D$ selfadjoint operators with spectra in $I$. We consider the auxiliary function defined on $[0, 1]$ by
\[
f_{C,D}(t) := f((1 - t)C + tD), \ t \in [0, 1].
\]
Then we have, by the properties of the Bochner integral, that
\[
(2.4) \quad f(D) - f(C) = \int_0^1 \frac{d}{dt} f_{C,D}(t) \, dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C) \, dt.
\]
If we write this equality for the function $f(t) = -t^{-1}$ and $C, D > 0$, then we get the representation
\[
(2.5) \quad C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1} (D-C)((1-t)C + tD)^{-1} \, dt.
\]
Now, if we take in (2.5) $C = \lambda + A$, $D = \lambda + B$, then
\[
(2.6) \quad (\lambda + A)^{-1} - (\lambda + B)^{-1}
\]
\[
= \int_0^1 ((1-t)(\lambda + A) + t(\lambda + B))^{-1} (B-A)
\]
\[
\times ((1-t)(\lambda + A) + t(\lambda + B))^{-1} \, dt
\]
\[
= \int_0^1 (\lambda + (1-t)A + tB)^{-1} (B-A)(\lambda + (1-t)A + tB)^{-1} \, dt.
\]
By employing (2.2) and (2.6), we derive (2.1).

**Corollary 1.** Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and it has the representation (1.5), then for all $A, B > 0$ we have the equality
\[
(2.7) \quad f(B) - f(A) = \int_0^\infty \left( \int_0^1 (\lambda + (1-t)A + tB)^{-1} (B-A)(\lambda + (1-t)A + tB)^{-1} \, dt \right) \lambda^2 \, d\mu(\lambda).
\]

**Proof.** From (1.5) we have for $T > 0$ that
\[
f(T) - a - bT = M(\ell, \mu)(T),
\]
for some positive measure $\mu$, where $\ell(\lambda) = \lambda, \ \lambda \geq 0$. Therefore
\[
M(\ell, \mu)(B) - M(\ell, \mu)(A) = f(B) - f(A) - b(B - A)
\]
and by (2.1) we get (2.7).

**Corollary 2.** Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator convex in $(0, \infty)$ and it has the identity (1.3), then for all $A, B > 0$ we have the identity
\[
(2.8) \quad f(B)B^{-1} - f(A)A^{-1} - a(B^{-1} - A^{-1}) - c(B - A)
\]
\[
= \int_0^\infty \left( \int_0^1 (\lambda + (1-t)A + tB)^{-1} (B-A)(\lambda + (1-t)A + tB)^{-1} \, dt \right) \lambda^2 \, d\mu(\lambda).
\]
Proof. From (1.7) we have for $T > 0$ that
\[
(f(T) - a)T^{-1} - b - cT = M(\ell, \mu)(T),
\]
for some positive measure $\mu$. Therefore
\[
M(\ell, \mu)(B) - M(\ell, \mu)(A) = (f(B) - a)B^{-1} - (f(A) - a)A^{-1} - c(B - A)
\]
and by (2.1) we get (2.8).

\[\square\]

Remark 1. If $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$, then we can take $a = f(0)$ and by (2.8) we get
\[
\begin{align*}
(2.9) \quad f(B)B^{-1} - f(A)A^{-1} - f(0)(B^{-1} - A^{-1}) - c(B - A) \\
= & \int_0^\infty \left( \int_0^1 (\lambda + (1 - t)A + tB)^{-1}(B - A)(\lambda + (1 - t)A + tB)^{-1}dt \right) \\
& \times \lambda^2 d\mu(\lambda). 
\end{align*}
\]

Remark 2. From the representation (2.1) we observe that if $B \geq A > 0$, then
\[
M(w, \mu)(B) \geq M(w, \mu)(A)
\]
which means that $M(w, \mu)$ is operator monotone on $(0, \infty)$, see also [6].

We have the following Lipschitz type inequality:

**Theorem 3.** Assume that $A \geq m_1 > 0$, $B \geq m_2 > 0$, then
\[
\begin{align*}
(2.10) \quad & \|M(w, \mu)(B) - M(w, \mu)(A)\| \\
\leq & \|B - A\| \left\{ \begin{array}{ll}
\frac{M(w, \mu)(m_2) - M(w, \mu)(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
M'(w, \mu)(m) & \text{if } m_1 = m_2 = m,
\end{array} \right.
\end{align*}
\]

where $M'(w, \mu)(t)$ is the derivative of $M(w, \mu)$ as a function of $t$.

**Proof.** From the identity (2.6) we get by taking the norm that
\[
\begin{align*}
(2.11) \quad & \|M(w, \mu)(B) - M(w, \mu)(A)\| \\
\leq & \int_0^\infty \left( \int_0^1 (\lambda + (1 - t)A + tB)^{-1}(B - A)(\lambda + (1 - t)A + tB)^{-1}dt \right) \\
& \times \lambda w(\lambda) d\mu(\lambda) \\
\leq & \int_0^\infty \left( \int_0^1 \|\lambda + (1 - t)A + tB\|^{-1}(B - A)(\lambda + (1 - t)A + tB)^{-1}dt \right) \\
& \times \lambda w(\lambda) d\mu(\lambda) \\
\leq & \|B - A\| \int_0^\infty \lambda w(\lambda) \left( \int_0^1 \|\lambda + (1 - t)A + tB\|^{-1}dt \right) d\mu(\lambda)
\end{align*}
\]

for all $A, B > 0$.

Assume that $m_2 > m_1$. Then
\[
(1 - t)A + tB + \lambda \geq (1 - t)m_1 + tm_2 + \lambda,
\]
which implies that
\[
((1 - t)A + tB + \lambda)^{-1} \leq ((1 - t)m_1 + tm_2 + \lambda)^{-1},
\]
and

\[(2.12) \quad \left\| (1-t) A + tB + \lambda \right\|^{-2} \leq ((1-t) m_1 + tm_2 + \lambda)^{-2}\]

for all \( t \in [0, 1] \) and \( \lambda \geq 0 \).

Therefore, by integrating (2.12) we derive

\[
\int_0^\infty \lambda w(\lambda) \left( \int_0^1 \left\| (1-t) A + tB + \lambda \right\|^{-2} dt \right) dw(\lambda)
\leq \int_0^\infty \lambda w(\lambda) \left( \int_0^1 ((1-t) m_1 + tm_2 + \lambda)^{-1} dt \right) dw(\lambda)
\]

\[
= \frac{1}{m_2 - m_1} \int_0^\infty \lambda w(\lambda) \left( \int_0^1 ((1-t) m_1 + tm_2 + \lambda)^{-1} dt \right) dw(\lambda)
= \frac{1}{m_2 - m_1} \left[ M(w, \mu)(m_2) - M(w, \mu)(m_1) \right] \quad \text{(by (2.1))}
\]

and by (2.11) we deduce

\[(2.13) \quad \| M(w, \mu)(B) - M(w, \mu)(A) \| \leq \frac{1}{m_2 - m_1} \left[ M(w, \mu)(m_2) - M(w, \mu)(m_1) \right].\]

The case \( m_2 < m_1 \) goes in a similar way and we also obtain (2.13). Let \( \epsilon > 0 \). Then \( B + \epsilon \geq m + \epsilon > m \). From (2.13) we get

\[
\| M(w, \mu)(B + \epsilon) - M(w, \mu)(A) \|
\leq \frac{1}{m + \epsilon - m} \left[ M(w, \mu)(m + \epsilon) - M(w, \mu)(m) \right]
\]

and by taking the limit over \( \epsilon \to 0^+ \), using the continuity and differentiability of \( M(w, \mu) \) we deduce the second part of (2.10). \( \square \)

**Corollary 3.** Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) and it has the representation (1.5). If \( A \geq m_1 > 0, B \geq m_2 > 0 \), then,

\[(2.14) \quad \| f(B) - f(A) - b(B - A) \|
\leq \| B - A \| \left\{ \begin{array}{ll}
(f(m_2) - f(m_1)) - b) & \text{if } m_1 \neq m_2, \\
(f'(m) - bm) & \text{if } m_1 = m_2 = m.
\end{array} \right.\]

**Proof.** From (1.5) we have for \( T > 0 \) that

\[
f(T) - a - bT = M(\ell, \mu)(T),
\]

for some positive measure \( \mu \), where \( \ell(\lambda) = \lambda, \lambda \geq 0 \). Therefore

\[
M(\ell, \mu)(B) - M(w, \mu)(A) = f(B) - f(A) - b(B - A),
\]

\[
M(\ell, \mu)(m_2) - M(w, \mu)(m_1) = f(m_2) - f(m_1) - b(m_2 - m_1)
\]

and

\[
M'(\ell, \mu)(m) = f'(m) - bm.
\]
By (2.10) we obtain
\[
\left\| f(B) - f(A) - b(B - A) \right\| \leq \|B - A\| \left\{ \begin{array}{ll}
\frac{f(m_2) - f(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
(f'(m) - bm) & \text{if } m_1 = m_2 = m,
\end{array} \right.
\]
which is equivalent to (2.14).

By the properties of the norm, we have
\[
\left\| f(B) - f(A) \right\| - b \left\| B - A \right\| \leq \left\| f(B) - f(A) - b(B - A) \right\| \leq \|B - A\| \left\{ \begin{array}{ll}
\frac{f(m_2) - f(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
(f'(m) - bm) & \text{if } m_1 = m_2 = m,
\end{array} \right.
\]
which implies the following inequalities in which the nonnegative parameter $b$ is not involved
\[
\left\| f(B) - f(A) \right\| \leq \|B - A\| \left\{ \begin{array}{ll}
\frac{f(m_2) - f(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
(f'(m) - bm) & \text{if } m_1 = m_2 = m,
\end{array} \right.
\]
where the function $f : (0, \infty) \rightarrow \mathbb{R}$ is operator monotone in $(0, \infty)$.

By employing this inequality for power and logarithmic functions we can state the following results of interest:

**Proposition 1.** If $A \geq m_1 > 0$, $B \geq m_2 > 0$, then for $r \in (0, 1]$ we have the power inequalities
\[
\left\| B^r - A^r \right\| \leq \|B - A\| \left\{ \begin{array}{ll}
m_2^r - m_1^r & \text{if } m_1 \neq m_2, \\
r m_2^{r-1} & \text{if } m_1 = m_2 = m,
\end{array} \right.
\]
and the logarithmic inequalities
\[
\left\| \ln B - \ln A \right\| \leq \|B - A\| \left\{ \begin{array}{ll}
\frac{\ln m_2 - \ln m_1}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\frac{1}{m} & \text{if } m_1 = m_2 = m.
\end{array} \right.
\]

**Corollary 4.** Assume that $f : [0, \infty) \rightarrow \mathbb{R}$ is operator convex in $[0, \infty)$ that has the representation (1.7). If $A \geq m_1 > 0$, $B \geq m_2 > 0$, then
\[
\left\| f(B)B^{-1} - f(A)A^{-1} - f(0)(B^{-1} - A^{-1}) - c(B - A) \right\| \leq \|B - A\| \left\{ \begin{array}{ll}
\frac{f(m_2)m_2^{-1} - f(m_1)m_1^{-1} - f(0)(m_2^{-1} - m_1^{-1})}{m_2 - m_1} - c & \text{if } m_1 \neq m_2, \\
\frac{f'(m)m - f(m) + f(0)}{m^2} - c & \text{if } m_1 = m_2 = m.
\end{array} \right.
\]
If \( f(0) = 0 \), then we have the simpler inequalities

\[
\| f(B)B^{-1} - f(A)A^{-1} - c(B - A) \|
\leq \| B - A \| \left\{ \begin{array}{ll}
\frac{f(m_2)m_2^{-1} - f(m_1)m_1^{-1}}{m_2 - m_1} - c & \text{if } m_1 \neq m_2, \\
\frac{f'(m)m - f(m)}{m^2} - c & \text{if } m_1 = m_2 = m.
\end{array} \right.
\]

(2.19)

\[
\text{Proof. From (1.7) we have for } T > 0 \text{ that}
\]

\[
(f(T) - f(0))T^{-1} - f'_+(0) - cT = \mathcal{M}(\ell, \mu)(T),
\]

for some positive measure \( \mu \). Therefore

\[
\mathcal{M}(\ell, \mu)(B) - \mathcal{M}(\ell, \mu)(A)
= f(B)B^{-1} - f(A)A^{-1} - f(0)(B^{-1} - A^{-1}) - c(B - A),
\]

\[
\mathcal{M}(\ell, \mu)(m_2) - \mathcal{M}(\ell, \mu)(m_1)
= f(m_2)m_2^{-1} - f(m_1)m_1^{-1} - f(0)(m_2^{-1} - m_1^{-1}) - c(m_2 - m_1)
\]

and

\[
\mathcal{M}(\ell, \mu)(m) = \frac{f'(m)m - f(m) + f(0)}{m^2} - c.
\]

Then by (2.10) we get

\[
\| f(B)B^{-1} - f(A)A^{-1} - f(0)(B^{-1} - A^{-1}) - c(B - A) \|
\leq \| B - A \| \left\{ \begin{array}{ll}
\frac{f(m_2)m_2^{-1} - f(m_1)m_1^{-1} - f(0)(m_2^{-1} - m_1^{-1})}{m_2 - m_1} - c & \text{if } m_1 \neq m_2, \\
\frac{f'(m)m - f(m) + f(0)}{m^2} - c & \text{if } m_1 = m_2 = m,
\end{array} \right.
\]

and the inequality (2.18) is obtained. \qed

By the properties of the norm, we have

\[
\| f(B)B^{-1} - f(A)A^{-1} - f(0)(B^{-1} - A^{-1}) - c(B - A) \|
\leq \| B - A \| \left\{ \begin{array}{ll}
\frac{f(m_2)m_2^{-1} - f(m_1)m_1^{-1} - f(0)(m_2^{-1} - m_1^{-1})}{m_2 - m_1} - c & \text{if } m_1 \neq m_2, \\
\frac{f'(m)m - f(m) + f(0)}{m^2} - c & \text{if } m_1 = m_2 = m,
\end{array} \right.
\]

which implies the following inequalities in which the nonnegative parameter \( c \) is not involved

(2.20)

\[
\| f(B)B^{-1} - f(A)A^{-1} - f(0)(B^{-1} - A^{-1}) \|
\leq \| B - A \| \left\{ \begin{array}{ll}
\frac{f(m_2)m_2^{-1} - f(m_1)m_1^{-1} - f(0)(m_2^{-1} - m_1^{-1})}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\frac{f'(m)m - f(m) + f(0)}{m^2} & \text{if } m_1 = m_2 = m.
\end{array} \right.
\]

By applying this inequality to the operator convex function \( f(t) = -\ln(t + 1) \),
then we can state the following result:
Proposition 2. If $A \geq m_1 > 0$, $B \geq m_2 > 0$, then we have the logarithmic inequalities

\begin{equation}
\| B^{-1} \ln (B + 1) - A^{-1} \ln (A + 1) \| \leq \| B - A \| \begin{cases} 
\frac{m_2^{-1} \ln (m_2 + 1) - m_2^{-1} \ln (m_2 + 1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\
\frac{\ln (m + 1) - m (m + 1)^{-1}}{m^2} & \text{if } m_1 = m_2 = m.
\end{cases}
\end{equation}

3. Midpoint and Trapezoid Inequalities

We have the following midpoint type inequalities:

Proposition 3. For all $A, B \geq m > 0$ we have the midpoint inequality

\begin{equation}
\int_0^1 \mathcal{M} (w, \mu) ((1 - t) A + t B) dt - \mathcal{M} (w, \mu) \left( \frac{A + B}{2} \right) \leq \frac{1}{4} \mathcal{M}' (w, \mu) (m) \| B - A \|.
\end{equation}

Proof. Since $A, B \geq m$, hence $\frac{A + B}{2} \geq m > 0$ and $(1 - t) A + t B \geq m > 0$ for all $t \in [0, 1]$ and by (2.10)

\begin{equation}
\mathcal{M} (w, \mu) ((1 - t) A + t B) - \mathcal{M} (w, \mu) \left( \frac{A + B}{2} \right) \\
\leq \mathcal{M}' (w, \mu) (m) \left( (1 - t) A + t B - \frac{A + B}{2} \right) \\
= \mathcal{M}' (w, \mu) (m) \left| t - \frac{1}{2} \right| \| B - A \|
\end{equation}

for all $t \in [0, 1]$.

Taking the integral in (3.2), we get

\begin{equation}
\int_0^1 \mathcal{M} (w, \mu) ((1 - t) A + t B) dt - \mathcal{M} (w, \mu) \left( \frac{A + B}{2} \right) \leq \int_0^1 \mathcal{M} (w, \mu) ((1 - t) A + t B) - \mathcal{M} (w, \mu) \left( \frac{A + B}{2} \right) dt \\
\leq \mathcal{M}' (w, \mu) (m) \| B - A \| \int_0^1 \left| t - \frac{1}{2} \right| dt = \frac{1}{4} \mathcal{M}' (w, \mu) (m) \| B - A \|
\end{equation}

and the inequality (3.1) is proved. \hfill \square

We have the following midpoint type inequalities:

Proposition 4. For all $A, B \geq m > 0$ we have the trapezoid inequality

\begin{equation}
\left\| \mathcal{M} (w, \mu) (A) + \mathcal{M} (w, \mu) (B) \right\| - \int_0^1 \mathcal{M} (w, \mu) ((1 - t) A + t B) dt \leq \frac{1}{4} \mathcal{M}' (w, \mu) (m) \| B - A \|.
\end{equation}
Proof. Since $A, B \geq m$, hence $(1 - s) A + s \frac{A + B}{2}, s \frac{A + B}{2} + (1 - s) B \geq m > 0$ for all $s \in [0, 1]$ and by (3) we get

\begin{align}
(3.4) \quad & \left\| \mathcal{M}(w, \mu) (A) - \mathcal{M}(w, \mu) \left( (1 - s) A + s \frac{A + B}{2} \right) \right\| \\
& \leq \frac{1}{2} \mathcal{M}'(w, \mu)(m) \| B - A \| s
\end{align}

and

\begin{align}
(3.5) \quad & \left\| \mathcal{M}(w, \mu) (B) - \mathcal{M}(w, \mu) \left( s \frac{A + B}{2} + (1 - s) B \right) \right\| \\
& \leq \frac{1}{2} \mathcal{M}'(w, \mu)(m) \| B - A \| s.
\end{align}

From (3.4) and (3.5) we derive by addition, division by 2 and triangle inequality that

\begin{align}
\left\| \frac{\mathcal{M}(w, \mu)(A) + \mathcal{M}(w, \mu)(B)}{2} \\
- \frac{1}{2} \left[ \mathcal{M}(w, \mu) \left( (1 - s) A + s \frac{A + B}{2} \right) + \mathcal{M}(w, \mu) \left( s \frac{A + B}{2} + (1 - s) B \right) \right] ds \right\| \\
\leq \frac{1}{2} \mathcal{M}'(w, \mu)(m) \| B - A \| s
\end{align}

for all $s \in [0, 1]$.

By taking the integral and using its properties, we derive

\begin{align}
(3.6) \quad & \left\| \frac{\mathcal{M}(w, \mu)(A) + \mathcal{M}(w, \mu)(B)}{2} \\
& - \frac{1}{2} \left[ \mathcal{M}(w, \mu) \left( (1 - t) A + t \frac{A + B}{2} \right) \\
& + \mathcal{M}(w, \mu) \left( t \frac{A + B}{2} + (1 - t) B \right) ds \right] \right\| \\
& \leq \frac{1}{2} \mathcal{M}'(w, \mu)(m) \| B - A \| \int_0^1 s ds = \frac{1}{4} \mathcal{M}'(w, \mu)(m) \| B - A \|.
\end{align}

Now, using the change of variable $t = 2s$ we have

\begin{align}
\frac{1}{2} \int_0^1 \mathcal{M}(w, \mu) \left( (1 - t) A + t \frac{A + B}{2} \right) dt = \int_0^{1/2} \mathcal{M}(w, \mu) ((1 - s) A + s B) ds
\end{align}

and by the change of variable $t = 1 - v$ we have

\begin{align}
\frac{1}{2} \int_0^1 \mathcal{M}(w, \mu) \left( t \frac{A + B}{2} + (1 - t) A \right) dt \\
= \frac{1}{2} \int_0^1 \mathcal{M}(w, \mu) \left( (1 - v) \frac{A + B}{2} + vB \right) dv.
\end{align}

Moreover, if we make the change of variable $v = 2s - 1$ we also have

\begin{align}
\frac{1}{2} \int_0^1 \mathcal{M}(w, \mu) \left( (1 - v) \frac{A + B}{2} + vB \right) dv = \int_{1/2}^1 \mathcal{M}(w, \mu) ((1 - s) A + s B) ds.
\end{align}
Therefore
\[
\frac{1}{2} \int_0^1 \left[ \mathcal{M}(w, \mu) \left( (1 - s) A + s \frac{A + B}{2} \right) + \mathcal{M}(w, \mu) \left( s \frac{A + B}{2} + (1 - s) B \right) \right] ds
\]
\[
= \int_0^{1/2} \mathcal{M}(w, \mu) ((1 - s) A + sB) ds + \int_{1/2}^1 \mathcal{M}(w, \mu) ((1 - s) A + sB) ds
\]
\[
= \int_0^1 \mathcal{M}(w, \mu) ((1 - s) A + sB) ds
\]
and by (3.6) we deduce the desired result (3.3).

The case of operator monotone functions is as follows:

**Corollary 5.** Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) and it has the representation (1.5). If \( A, B \geq m > 0 \), then we have the midpoint inequality

\[
(3.7) \quad \left\| \int_0^1 f ((1 - t) A + tB) dt - f \left( \frac{A + B}{2} \right) \right\|
\leq \frac{1}{4} [f'(m) - b] \|B - A\| \leq \frac{1}{4} f'(m) \|B - A\|
\]

and the trapezoid inequality

\[
(3.8) \quad \left\| \frac{f(A) + f(B)}{2} - \int_0^1 f ((1 - t) A + tB) dt \right\|
\leq \frac{1}{4} [f'(m) - b] \|B - A\| \leq \frac{1}{4} f'(m) \|B - A\|.
\]

**Proof.** From (1.5) we have for \( T > 0 \) that

\[ f(T) - a - bT = \mathcal{M}(\ell, \mu)(T), \]

for some positive measure \( \mu \), where \( \ell(\lambda) = \lambda, \lambda \geq 0 \).

Therefore

\[
\int_0^1 \mathcal{M}(\ell, \mu) ((1 - t) A + tB) dt = \int_0^1 f ((1 - t) A + tB) dt - a - b \left( \frac{A + B}{2} \right),
\]

\[
\mathcal{M}(\ell, \mu) \left( \frac{A + B}{2} \right) = f \left( \frac{A + B}{2} \right) - a - b \left( \frac{A + B}{2} \right)
\]

and

\[
\mathcal{M}'(\ell, \mu)(m) = f'(m) - b.
\]

From (3.1) we derive (3.7).

Since

\[
\mathcal{M}(\ell, \mu)(A) = f(A) - a - bA, \text{ and } \mathcal{M}(\ell, \mu)(B) = f(B) - a - bB,
\]

then by (3.3) we derive (3.8).

**Remark 3.** If \( A, B \geq m > 0 \), then we have the midpoint inequality and the trapezoid inequality for power function with exponent \( r \in (0, 1) \)

\[
(3.9) \quad \left\| \int_0^1 ((1 - t) A + tB)^r dt - \left( \frac{A + B}{2} \right)^r \right\| \leq \frac{1}{4} rm^{r-1} \|B - A\|
\]
(3.10) \[ \left\| \frac{A^r + B^r}{2} - \int_0^1 ((1-t)A + tB)^r \, dt \right\| \leq \frac{1}{4} rm^{r-1} \|B - A\|. \]

The following inequalities for logarithm also hold

(3.11) \[ \left\| \int_0^1 \ln ((1-t)A + tB) \, dt - \ln \left( \frac{A + B}{2} \right) \right\| \leq \frac{1}{4m} \|B - A\| \]

and

(3.12) \[ \left\| \frac{\ln A + \ln B}{2} - \int_0^1 \ln ((1-t)A + tB) \, dt \right\| \leq \frac{1}{4m} \|B - A\|. \]

**Corollary 6.** Assume that \( f : [0, \infty) \to \mathbb{R} \) is operator convex in \([0, \infty)\) that has the representation (1.7). If \( A \geq m > 0, B \geq m > 0 \), then

(3.13) \[ \left\| \int_0^1 f((1-t)A + tB)((1-t)A + tB)^{-1} \, dt - f\left( \frac{A + B}{2} \right) \left( \frac{A + B}{2} \right)^{-1} \right\| \]

\[ = f(0) \left( \int_0^1 ((1-t)A + tB)^{-1} \, dt - \left( \frac{A + B}{2} \right)^{-1} \right) \]

\[ \leq \frac{1}{4} \left( \frac{f'(m) m - f(m) + f(0)}{m^2} - c \right) \|B - A\| \]

\[ \leq \frac{f'(m) m - f(m) + f(0)}{4m^2} \|B - A\|. \]

and

(3.14) \[ \left\| \frac{f(A)A^{-1} + f(B)B^{-1}}{2} - \int_0^1 f((1-t)A + tB)((1-t)A + tB)^{-1} \, dt \right\| \]

\[ = f(0) \left( \frac{A^{-1} + B^{-1}}{2} - \int_0^1 ((1-t)A + tB)^{-1} \, dt \right) \]

\[ \leq \frac{1}{4} \left( \frac{f'(m) m - f(m) + f(0)}{m^2} - c \right) \|B - A\| \]

\[ \leq \frac{f'(m) m - f(m) + f(0)}{4m^2} \|B - A\|. \]

**Proof.** From (1.7) we have for \( T > 0 \) that

\[ \mathcal{M}(\ell, \mu)(T) = (f(T) - f(0)) T^{-1} - f^*_+(0) - cT, \]

for some positive measure \( \mu \). Therefore

\[ \int_0^1 \mathcal{M}(\ell, \mu)((1-t)A + tB) \, dt \]

\[ = \int_0^1 f((1-t)A + tB)((1-t)A + tB)^{-1} \, dt - f(0) \int_0^1 ((1-t)A + tB)^{-1} \, dt \]

\[ - f^*_+(0) - c \left( \frac{A + B}{2} \right), \]
\[
\mathcal{M}(\ell, \mu) \left( \frac{A + B}{2} \right) = f \left( \frac{A + B}{2} \right) \left( \frac{A + B}{2} \right)^{-1} - f(0) \left( \frac{A + B}{2} \right)^{-1} - f'(0) - c \left( \frac{A + B}{2} \right),
\]

and

\[
\mathcal{M}(\ell, \mu) (m) = \frac{f'(m) m - f(m) + f(0)}{m^2} - c.
\]

By utilising (3.1) we get (3.13).

Since

\[
\mathcal{M}(\ell, \mu) (A) = (f (A) - f (0)) A^{-1} - f'_+ (0) - c A
\]

and

\[
\mathcal{M}(\ell, \mu) (B) = (f (B) - f (0)) B^{-1} - f'_+ (0) - c B,
\]

hence by (3.3) we get (3.14).

\[
\text{Remark 4. In the case when } f(0) = 0 \text{ in Corollary 6, we have the simpler inequalities}
\]

\[
\left\| \int_0^1 f ((1 - t) A + tB) ((1 - t) A + tB)^{-1} dt - f \left( \frac{A + B}{2} \right) \left( \frac{A + B}{2} \right)^{-1} \right\| \leq \frac{1}{4} \left( \frac{f'(m) m - f(m)}{m^2} - c \right) \| B - A \| \leq \frac{f'(m) m - f(m)}{4m^2} \| B - A \|
\]

and

\[
\left\| \frac{f (A) A^{-1} + f (B) B^{-1}}{2} - \int_0^1 f ((1 - t) A + tB) ((1 - t) A + tB)^{-1} dt \right\| \leq \frac{1}{4} \left( \frac{f'(m) m - f(m)}{m^2} - c \right) \| B - A \| \leq \frac{f'(m) m - f(m)}{4m^2} \| B - A \|.
\]

If in these inequalities we take the operator convex function \( f(t) = -\ln (t + 1) \), then we get

\[
\left\| \int_0^1 \ln ((1 - t) A + tB + 1) ((1 - t) A + tB)^{-1} dt - \ln \left( \frac{A + B}{2} + 1 \right) \left( \frac{A + B}{2} \right)^{-1} \right\| \leq \frac{\ln (m + 1) - m (m + 1)^{-1}}{m^2} \| B - A \|
\]

and

\[
\left\| \frac{A^{-1} \ln (A + 1) + B^{-1} \ln (B + 1)}{2} - \int_0^1 \ln ((1 - t) A + tB + 1) ((1 - t) A + tB)^{-1} dt \right\| \leq \frac{\ln (m + 1) - m (m + 1)^{-1}}{m^2} \| B - A \|.
\]
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