LIPSCHITZ TYPE INEQUALITIES FOR MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS WITH APPLICATIONS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w,\mu)(T) := \int_0^\infty w(\lambda) T (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Assume that $A \geq m_1 > 0$, $B \geq m_2 > 0$, then we show that

$$\begin{aligned} &\|\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)\| \\ &\leq \|B - A\| \begin{cases} \frac{\mathcal{M}(w,\mu)(m_2) - \mathcal{M}(w,\mu)(m_1)}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ \mathcal{M}'(w,\mu)(m) \text{ if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

where $\mathcal{M}'(w,\mu)(t)$ is the derivative of $\mathcal{M}(w,\mu)$ as a function of t. If the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$, then

$$\|f(B) - f(A)\| \leq \|B - A\| \begin{cases} \frac{f(m_2) - f(m_1)}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ f'(m) \text{ if } m_1 = m_2 = m. \end{cases}$$

In particular we have the power inequalities

$$\|B^r - A^r\| \leq \|B - A\| \begin{cases} \frac{m_2^r - m_1^r}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ r m^{r-1} \text{ if } m_1 = m_2 = m, \end{cases}$$

and the logarithmic inequalities

$$\|\ln B - \ln A\| \leq \|B - A\| \begin{cases} \frac{\ln m_2 - \ln m_1}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ \frac{1}{m} \text{ if } m_1 = m_2 = m. \end{cases}$$

Some applications for operator convex functions and midpoint and trapezoid norm inequalities are also provided.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [3] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no

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constant $L>0$ such that

$$\| |A| - |B| \| \le L \|A-B\|$$

for any $A,\ B\in\mathcal{B}(H)$.

However, as shown by Farforovskaya in [7], [8] and Kato in [14], the following inequality holds

$$\| |A| - |B| \| \le \frac{2}{\pi}\|A-B\|\left(2+\log\left(\frac{\|A\|+\|B\|}{\|A-B\|}\right)\right) \tag{1.1}$$

for any $A,\ B\in\mathcal{B}(H)$ with $A\neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS}:=(\operatorname{tr}C^*C)^{1/2}$ of an operator C, then the following inequality is true [1]

$$\| |A| - |B| \|_{HS} \le \sqrt{2}\,\|A-B\|_{HS} \tag{1.2}$$

for any $A,\ B\in\mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [3] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\| |A| - |B| \| \le a_1\|A-B\| + a_2\|A-B\|^2 + O\left(\|A-B\|^3\right), \tag{1.3}$$

where

$$a_1 = \left\|A^{-1}\right\|\|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3\|A\|^2.$$

In [2] the author also obtained the following *Lipschitz type inequality*

$$\|f(A)-f(B)\| \le f'(a)\|A-B\| \tag{1.4}$$

where f is an *operator monotone function* on $(0,\infty)$ and $A,\ B\ge a>0$.

One of the problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [4], [9] and the references therein.

We have the following representation of operator monotone functions [15], see for instance [5, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.5}$$

where $a\in R$, $b\ge 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda) < \infty. \tag{1.6}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.5).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B) \le (\ge)\,(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [5, p. 147]:

Theorem 2. *A function $f : (0,\infty) \to \mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda), \tag{1.7}$$

where a, $b \in \mathbb{R}$, $c \geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a = f(0)$ and $b = f'_+(0)$, the right derivative, in (1.5).

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [5, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda.$$

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.8}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.8) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.9}$$

Now, assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.10}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.11}$$

for $T > 0$.

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \ t > 0. \tag{1.12}$$

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w,\mu)(t) := t\mathcal{D}(w,\mu)(t), \ t > 0. \tag{1.13}$$

For $t>0$ we have

$$\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t)=\int_0^\infty w(\lambda)\,t\,(t+\lambda)^{-1}\,d\mu(\lambda) \tag{1.14}$$
$$=\int_0^\infty w(\lambda)(t+\lambda-\lambda)(t+\lambda)^{-1}\,d\mu(\lambda)$$
$$=\int_0^\infty w(\lambda)\left[1-\lambda(t+\lambda)^{-1}\right]d\mu(\lambda).$$

If $\int_0^\infty w(\lambda)\,d\mu(\lambda)<\infty$, then

$$\mathcal{M}(w,\mu)(t)=\int_0^\infty w(\lambda)\,d\mu(\lambda)-\mathcal{D}(\ell w,\mu)(t), \tag{1.15}$$

where $\ell(t)=t$, $t>0$.

Consider the kernel $e_{-a}(\lambda):=\exp(-a\lambda)$, $\lambda\geq 0$ and $a>0$. Then, after some calculations, we get

$$\mathcal{D}(e_{-a})(t)=\int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda}d\lambda=E_1(at)\exp(at),\ \ t\geq 0$$

and

$$\int_0^\infty w(\lambda)\,d\lambda=\int_0^\infty \exp(-a\lambda)\,d\lambda=\frac{1}{a},$$

where the *exponential integral* is defined by

$$E_1(t):=\int_t^\infty \frac{e^{-u}}{u}du.$$

This gives that

$$\mathcal{M}(e_{-a})(t)=t\mathcal{D}(w,\mu)(t)=tE_1(at)\exp(at),\ \ t\geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t)=\int_0^\infty \frac{\lambda\exp(-a\lambda)}{t+\lambda}d\lambda=\frac{1}{a}-tE_1(at)\exp(at)$$

for $t>0$.

One observes that

$$\mathcal{M}(e_{-a})(t)=\int_0^\infty w(\lambda)\,d\lambda-\mathcal{D}(\ell e_{-a},\mu)(t),\ \ t>0$$

and the equality (1.15) is verified in this case.

If we take $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda=\infty$ and the equality (1.15) does not hold in this case.

For all $T>0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T)=T\mathcal{D}(w,\mu)(T)=\int_0^\infty w(\lambda)\left[1-\lambda(T+\lambda)^{-1}\right]d\mu(\lambda). \tag{1.16}$$

This gives the representation

$$T^r=\frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T),$$

where $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$ and μ is the usual Lebesgue norm.

Assume that $A \geq m_1 > 0$, $B \geq m_2 > 0$, then we show that

$$\begin{aligned}
&\left\| \mathcal{M}\left(w,\mu\right)\left(B\right) - \mathcal{M}\left(w,\mu\right)\left(A\right) \right\| \\
&\leq \left\| B - A \right\| \begin{cases} \frac{\mathcal{M}(w,\mu)(m_2) - \mathcal{M}(w,\mu)(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ \mathcal{M}'\left(w,\mu\right)\left(m\right) & \text{if } m_1 = m_2 = m, \end{cases}
\end{aligned}$$

where $\mathcal{M}'\left(w,\mu\right)\left(t\right)$ is the derivative of $\mathcal{M}\left(w,\mu\right)$ as a function of t. If the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$, then

$$\left\| f\left(B\right) - f\left(A\right) \right\| \leq \left\| B - A \right\| \begin{cases} \frac{f(m_2) - f(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ f'\left(m\right) & \text{if } m_1 = m_2 = m. \end{cases}$$

In particular we have the power inequalities

$$\left\| B^r - A^r \right\| \leq \left\| B - A \right\| \begin{cases} \frac{m_2^r - m_1^r}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ r m^{r-1} & \text{if } m_1 = m_2 = m, \end{cases}$$

and the logarithmic inequalities

$$\left\| \ln B - \ln A \right\| \leq \left\| B - A \right\| \begin{cases} \frac{\ln m_2 - \ln m_1}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ \frac{1}{m} & \text{if } m_1 = m_2 = m. \end{cases}$$

Some applications for operator convex functions and midpoint and trapezoid norm inequalities are also provided.

2. Main Results

We have the following equality that is of interest in itself:

Lemma 1. *For all A, $B > 0$ we have the representation*

$$\begin{aligned}
&\mathcal{M}(w,\mu)\left(B\right) - \mathcal{M}(w,\mu)\left(A\right) \\
&= \int_0^\infty \left(\int_0^1 \left(\lambda + (1-t)\,A + tB\right)^{-1} \left(B - A\right) \left(\lambda + (1-t)\,A + tB\right)^{-1} dt \right) \\
&\times \lambda w\left(\lambda\right) d\mu\left(\lambda\right).
\end{aligned} \tag{2.1}$$

Proof. From (1.16) we have for all A, $B \geq 0$ that

$$\begin{aligned}
&\mathcal{M}(w,\mu)\left(B\right) - \mathcal{M}(w,\mu)\left(A\right) \\
&= \int_0^\infty w\left(\lambda\right) \left[1 - \lambda\left(B + \lambda\right)^{-1}\right] d\mu\left(\lambda\right) - \int_0^\infty w\left(\lambda\right) \left[1 - \lambda\left(A + \lambda\right)^{-1}\right] d\mu\left(\lambda\right) \\
&= \int_0^\infty \lambda w\left(\lambda\right) \left[\left(A + \lambda\right)^{-1} - \left(B + \lambda\right)^{-1}\right] d\mu\left(\lambda\right).
\end{aligned} \tag{2.2}$$

Let T, $S > 0$. The function $f\left(t\right) = -t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T\left(S\right) := \lim_{t \to 0} \left[\frac{f\left(T + tS\right) - f\left(T\right)}{t} \right] = T^{-1} S T^{-1} \tag{2.3}$$

for T, $S > 0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C, D]$: $\{(1-t)C + tD,\ t \in [0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C + tD),\ t \in [0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.4}$$

If we write this equality for the function $f(t) = -t^{-1}$ and C, $D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-t)C + tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.5}$$

Now, if we take in (2.5) $C = \lambda + A$, $D = \lambda + B$, then

$$\begin{aligned}(2.6)\qquad & (\lambda + A)^{-1} - (\lambda + B)^{-1} \\ &= \int_0^1 ((1-t)(\lambda + A) + t(\lambda + B))^{-1}(B-A) \\ &\times ((1-t)(\lambda + A) + t(\lambda + B))^{-1}\,dt \\ &= \int_0^1 (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1}\,dt.\end{aligned}$$

By employing (2.2) and (2.6), we derive (2.1). □

Corollary 1. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and it has the representation (1.5), then for all A, $B > 0$ we have the equality*

$$\begin{aligned}(2.7)\qquad & f(B) - f(A) - b(B-A) \\ &= \int_0^\infty \left(\int_0^1 (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1}\,dt\right) \\ &\times \lambda^2 d\mu(\lambda).\end{aligned}$$

Proof. From (1.5) we have for $T > 0$ that

$$f(T) - a - bT = \mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda \geq 0$. Therefore

$$\mathcal{M}(\ell,\mu)(B) - \mathcal{M}(\ell,\mu)(A) = f(B) - f(A) - b(B-A)$$

and by (2.1) we get (2.7). □

Corollary 2. *Assume that the function $f : (0,\infty) \to \mathbb{R}$ is operator convex in $(0,\infty)$ and it has the representation (1.3), then for all A, $B > 0$ we have the identity*

$$\begin{aligned}(2.8)\qquad & f(B)B^{-1} - f(A)A^{-1} - a\left(B^{-1} - A^{-1}\right) - c(B-A) \\ &= \int_0^\infty \left(\int_0^1 (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1}\,dt\right) \\ &\times \lambda^2 d\mu(\lambda).\end{aligned}$$

Proof. From (1.7) we have for $T>0$ that

$$(f(T)-a)T^{-1}-b-cT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ. Therefore

$$\mathcal{M}(\ell,\mu)(B)-\mathcal{M}(\ell,\mu)(A)=(f(B)-a)B^{-1}-(f(A)-a)A^{-1}-c(B-A)$$

and by (2.1) we get (2.8). □

Remark 1. *If $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$, then we can take $a=f(0)$ and by (2.8) we get*

$$\begin{aligned}(2.9)\quad & f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\\ &=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right)\\ &\times\lambda^2 d\mu(\lambda).\end{aligned}$$

Remark 2. *From the representation (2.1) we observe that if $B\geq A>0$, then $\mathcal{M}(w,\mu)(B)\geq\mathcal{M}(w,\mu)(A)$ which means that $\mathcal{M}(w,\mu)$ is operator monotone on $(0,\infty)$, see also* [6].

We have the following Lipschitz type inequality:

Theorem 3. *Assume that $A\geq m_1>0$, $B\geq m_2>0$, then*

$$\begin{aligned}(2.10)\quad & \|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\|\\ &\leq\|B-A\|\begin{cases}\frac{\mathcal{M}(w,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ \mathcal{M}'(w,\mu)(m) & \text{if } m_1=m_2=m,\end{cases}\end{aligned}$$

where $\mathcal{M}'(w,\mu)(t)$ is the derivative of $\mathcal{M}(w,\mu)$ as a function of t.

Proof. From the identity (2.6) we get by taking the norm that

$$\begin{aligned}(2.11)\quad & \|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\|\\ &\leq\int_0^\infty\left\|\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right\|\\ &\times\lambda w(\lambda)d\mu(\lambda)\\ &\leq\int_0^\infty\left(\int_0^1\left\|(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\right\|dt\right)\\ &\times\lambda w(\lambda)d\mu(\lambda)\\ &\leq\|B-A\|\int_0^\infty\lambda w(\lambda)\left(\int_0^1\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^2dt\right)d\mu(\lambda)\end{aligned}$$

for all A, $B>0$.

Assume that $m_2>m_1$. Then

$$(1-t)A+tB+\lambda\geq(1-t)m_1+tm_2+\lambda,$$

which implies that

$$((1-t)A+tB+\lambda)^{-1}\leq((1-t)m_1+tm_2+\lambda)^{-1},$$

and

$$\left\|\left((1-t)A+tB+\lambda\right)^{-1}\right\|^{2}\leq\left((1-t)m_{1}+tm_{2}+\lambda\right)^{-2} \tag{2.12}$$

for all $t\in[0,1]$ and $\lambda\geq 0$.

Therefore, by integrating (2.12) we derive

$$\begin{aligned}&\int_{0}^{\infty}\lambda w(\lambda)\left(\int_{0}^{1}\left\|\left((1-t)A+tB+\lambda\right)^{-1}\right\|^{2}dt\right)dw(\lambda)\\&\leq\int_{0}^{\infty}\lambda w(\lambda)\left(\int_{0}^{1}\left((1-t)m_{1}+tm_{2}+\lambda\right)^{-2}dt\right)dw(\lambda)\\&=\frac{1}{m_{2}-m_{1}}\int_{0}^{\infty}\lambda w(\lambda)\left(\int_{0}^{1}\left((1-t)m_{1}+tm_{2}+\lambda\right)^{-1}\right.\\&\left.\times(m_{2}-m_{1})\left((1-t)m_{1}+tm_{2}+\lambda\right)^{-1}dt\right)dw(\lambda)\\&=\frac{1}{m_{2}-m_{1}}\left[\mathcal{M}(w,\mu)(m_{2})-\mathcal{M}(w,\mu)(m_{1})\right]\ \text{(by (2.1))}\end{aligned}$$

and by (2.11) we deduce

$$\left\|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\right\|\leq\frac{1}{m_{2}-m_{1}}\left[\mathcal{M}(w,\mu)(m_{2})-\mathcal{M}(w,\mu)(m_{2})\right]. \tag{2.13}$$

The case $m_2<m_1$ goes in a similar way and we also obtain (2.13).

Let $\epsilon>0$. Then $B+\epsilon\geq m+\epsilon>m$. From (2.13) we get

$$\begin{aligned}&\left\|\mathcal{M}(w,\mu)(B+\epsilon)-\mathcal{M}(w,\mu)(A)\right\|\\&\leq\frac{1}{m+\epsilon-m}\left[\mathcal{M}(w,\mu)(m+\epsilon)-\mathcal{M}(w,\mu)(m)\right]\end{aligned}$$

and by taking the limit over $\epsilon\to 0+$, using the continuity and differentiability of $\mathcal{M}(w,\mu)$ we deduce the second part of (2.10). □

Corollary 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and it has the representation (1.5). If $A\geq m_1>0$, $B\geq m_2>0$, then,*

$$\begin{aligned}&\left\|f(B)-f(A)-b(B-A)\right\|\\&\leq\|B-A\|\begin{cases}\left(\frac{f(m_2)-f(m_1)}{m_2-m_1}-b\right) & \text{if } m_1\neq m_2,\\[2ex] \left(f'(m)-bm\right) & \text{if } m_1=m_2=m.\end{cases}\end{aligned} \tag{2.14}$$

Proof. From (1.5) we have for $T>0$ that

$$f(T)-a-bT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$. Therefore

$$\mathcal{M}(\ell,\mu)(B)-\mathcal{M}(w,\mu)(A)=f(B)-f(A)-b(B-A),$$

$$\mathcal{M}(\ell,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)=f(m_2)-f(m_1)-b(m_2-m_1)$$

and

$$\mathcal{M}'(\ell,\mu)(m)=f(m)-bm.$$

By (2.10) we obtain

$$\begin{aligned}&\|f(B)-f(A)-b(B-A)\|\\&\leq\|B-A\|\begin{cases}\left(\frac{f(m_2)-f(m_1)}{m_2-m_1}-b\right) \text{ if } m_1\neq m_2,\\ \\ (f'(m)-bm) \text{ if } m_1=m_2=m,\end{cases}\end{aligned}$$

which is equivalent to (2.14). □

By the properties of the norm, we have

$$\begin{aligned}&\|f(B)-f(A)\|-b\|B-A\|\\&\leq\|f(B)-f(A)-b(B-A)\|\\&\leq\|B-A\|\begin{cases}\left(\frac{f(m_2)-f(m_1)}{m_2-m_1}-b\right) \text{ if } m_1\neq m_2,\\ \\ (f'(m)-bm) \text{ if } m_1=m_2=m,\end{cases}\end{aligned}$$

which implies the following inequalities in which the nonnegative parameter b is not involved

$$\|f(B)-f(A)\|\leq\|B-A\|\begin{cases}\frac{f(m_2)-f(m_1)}{m_2-m_1} \text{ if } m_1\neq m_2,\\ \\ f'(m) \text{ if } m_1=m_2=m,\end{cases} \tag{2.15}$$

where the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$.

By employing this inequality for power and logarithmic functions we can state the following results of interest:

Proposition 1. *If $A\geq m_1>0$, $B\geq m_2>0$, then for $r\in(0,1]$ we have the power inequalities*

$$\|B^r-A^r\|\leq\|B-A\|\begin{cases}\frac{m_2^r-m_1^r}{m_2-m_1} \text{ if } m_1\neq m_2,\\ \\ rm^{r-1} \text{ if } m_1=m_2=m,\end{cases} \tag{2.16}$$

and the logarithmic inequalities

$$\|\ln B-\ln A\|\leq\|B-A\|\begin{cases}\frac{\ln m_2-\ln m_1}{m_2-m_1} \text{ if } m_1\neq m_2,\\ \\ \frac{1}{m} \text{ if } m_1=m_2=m.\end{cases} \tag{2.17}$$

Corollary 4. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ that has the representation (1.7). If $A\geq m_1>0$, $B\geq m_2>0$, then*

$$\begin{aligned}&\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\right\|\\&\leq\|B-A\|\begin{cases}\left(\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1}-c\right) \text{ if } m_1\neq m_2,\\ \\ \left(\frac{f'(m)m-f(m)+f(0)}{m^2}-c\right) \text{ if } m_1=m_2=m.\end{cases}\end{aligned} \tag{2.18}$$

If $f(0)=0$, *then we have the simpler inequalities*

$$\left\|f(B)B^{-1}-f(A)A^{-1}-c(B-A)\right\| \tag{2.19}$$
$$\leq\|B-A\|\begin{cases}\left(\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}}{m_2-m_1}-c\right) & \text{if } m_1\neq m_2,\\ \\ \left(\frac{f'(m)m-f(m)}{m^2}-c\right) & \text{if } m_1=m_2=m.\end{cases}$$

Proof. From (1.7) we have for $T>0$ that

$$(f(T)-f(0))T^{-1}-f'_+(0)-cT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ. Therefore

$$\mathcal{M}(\ell,\mu)(B)-\mathcal{M}(w,\mu)(A)$$
$$=f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A),$$

$$\mathcal{M}(\ell,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)$$
$$=f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)-c(m_2-m_1)$$

and

$$\mathcal{M}(\ell,\mu)(m)=\frac{f'(m)m-f(m)+f(0)}{m^2}-c.$$

Then by (2.10) we get

$$\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\right\|$$
$$\leq\|B-A\|\begin{cases}\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1}-c \text{ if } m_1\neq m_2,\\ \\ \left(\frac{f'(m)m-f(m)+f(0)}{m^2}-c\right) \text{ if } m_1=m_2=m,\end{cases}$$

and the inequality (2.18) is obtained. □

By the properties of the norm, we have

$$\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\right\|-c\|B-A\|$$
$$\leq\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\right\|$$
$$\leq\|B-A\|\begin{cases}\left(\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1}-c\right) \text{ if } m_1\neq m_2,\\ \\ \left(\frac{f'(m)m-f(m)+f(0)}{m^2}-c\right) \text{ if } m_1=m_2=m,\end{cases}$$

which implies the following inequalities in which the nonnegative parameter c is not involved

$$\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\right\| \tag{2.20}$$
$$\leq\|B-A\|\begin{cases}\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1} \text{ if } m_1\neq m_2,\\ \\ \frac{f'(m)m-f(m)+f(0)}{m^2} \text{ if } m_1=m_2=m.\end{cases}$$

By applying this inequality to the operator convex function $f(t)=-\ln(t+1)$, then we can state the following result:

Proposition 2. *If $A \geq m_1 > 0$, $B \geq m_2 > 0$, then we have the logarithmic inequalities*

$$\left\|B^{-1}\ln(B+1) - A^{-1}\ln(A+1)\right\| \leq \|B-A\| \begin{cases} \frac{m_1^{-1}\ln(m_1+1) - m_2^{-1}\ln(m_2+1)}{m_2-m_1} & \text{if } m_1 \neq m_2, \\ \\ \frac{\ln(m+1) - m(m+1)^{-1}}{m^2} & \text{if } m_1 = m_2 = m. \end{cases} \tag{2.21}$$

3. Midpoint and Trapezoid Inequalities

We have the following midpoint type inequalities:

Proposition 3. *For all A, $B \geq m > 0$ we have the midpoint inequality*

$$\left\|\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \leq \frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\|. \tag{3.1}$$

Proof. Since A, $B \geq m$, hence $\frac{A+B}{2} \geq m > 0$ and $(1-t)A+tB \geq m > 0$ for all $t \in [0,1]$ and by (2.10)

$$\begin{aligned} &\left\|\mathcal{M}(w,\mu)((1-t)A+tB) - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \\ &\leq \mathcal{M}'(w,\mu)(m)\left\|(1-t)A+tB-\frac{A+B}{2}\right\| \\ &= \mathcal{M}'(w,\mu)(m)\left|t-\frac{1}{2}\right|\|B-A\| \end{aligned} \tag{3.2}$$

for all $t \in [0,1]$.

Taking the integral in (3.2), we get

$$\begin{aligned} &\left\|\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\| \\ &\leq \int_0^1 \left\|\mathcal{M}(w,\mu)((1-t)A+tB) - \mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\| dt \\ &\leq \mathcal{M}'(w,\mu)(m)\|B-A\| \int_0^1 \left|t-\frac{1}{2}\right| dt = \frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\| \end{aligned}$$

and the inequality (3.1) is proved. □

We have the following midpoint type inequalities:

Proposition 4. *For all A, $B \geq m > 0$ we have the trapezoid inequality*

$$\left\|\frac{\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B)}{2} - \int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt\right\| \leq \frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\|. \tag{3.3}$$

Proof. Since A, $B\geq m$, hence $(1-s)A+s\frac{A+B}{2}$, $s\frac{A+B}{2}+(1-s)B\geq m>0$ for all $s\in[0,1]$ and by (3) we get

$$\left\|\mathcal{M}(w,\mu)(A)-\mathcal{M}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)\right\| \leq \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|s \tag{3.4}$$

and

$$\left\|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right\| \leq \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|s. \tag{3.5}$$

From (3.4) and (3.5) we derive by addition, division by 2 and triangle inequality that

$$\left\|\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2} -\frac{1}{2}\left[\mathcal{M}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)+\mathcal{M}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right]\right\| \leq \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|s$$

for all $s\in[0,1]$.

By taking the integral and using its properties, we derive

$$\left\|\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2} -\frac{1}{2}\left[\int_0^1\mathcal{M}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right) +\mathcal{M}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)ds\right]\right\| \leq \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|\int_0^1 sds=\frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\|. \tag{3.6}$$

Now, using the change of variable $t=2s$ we have

$$\frac{1}{2}\int_0^1\mathcal{M}(w,\mu)\left((1-t)A+t\frac{A+B}{2}\right)dt=\int_0^{1/2}\mathcal{M}(w,\mu)((1-s)A+sB)\,ds$$

and by the change of variable $t=1-v$ we have

$$\frac{1}{2}\int_0^1\mathcal{M}(w,\mu)\left(t\frac{A+B}{2}+(1-t)A\right)dt =\frac{1}{2}\int_0^1\mathcal{M}(w,\mu)\left((1-v)\frac{A+B}{2}+vB\right)dv.$$

Moreover, if we make the change of variable $v=2s-1$ we also have

$$\frac{1}{2}\int_0^1\mathcal{M}(w,\mu)\left((1-v)\frac{A+B}{2}+vB\right)dv=\int_{1/2}^1\mathcal{M}(w,\mu)((1-s)A+sB)\,ds.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[\mathcal{M}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)+\mathcal{M}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right]ds\\
&=\int_0^{1/2}\mathcal{M}(w,\mu)((1-s)A+sB)\,dt+\int_{1/2}^1\mathcal{M}(w,\mu)((1-s)A+sB)\,ds\\
&=\int_0^1\mathcal{M}(w,\mu)((1-s)A+sB)\,ds
\end{aligned}$$

and by (3.6) we deduce the desired result (3.3). □

The case of operator monotone functions is as follows:

Corollary 5. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and it has the representation (1.5). If A, $B\geq m>0$, then we have the midpoint inequality*

$$\begin{aligned}
&\left\|\int_0^1 f((1-t)A+tB)\,dt-f\left(\frac{A+B}{2}\right)\right\| \qquad (3.7)\\
&\leq\frac{1}{4}[f'(m)-b]\|B-A\|\leq\frac{1}{4}f'(m)\|B-A\|
\end{aligned}$$

and the trapezoid inequality

$$\begin{aligned}
&\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\| \qquad (3.8)\\
&\leq\frac{1}{4}[f'(m)-b]\|B-A\|\leq\frac{1}{4}f'(m)\|B-A\|.
\end{aligned}$$

Proof. From (1.5) we have for $T>0$ that

$$f(T)-a-bT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda\geq0$.

Therefore

$$\int_0^1\mathcal{M}(\ell,\mu)((1-t)A+tB)\,dt=\int_0^1 f((1-t)A+tB)\,dt-a-b\left(\frac{A+B}{2}\right),$$

$$\mathcal{M}(\ell,\mu)\left(\frac{A+B}{2}\right)=f\left(\frac{A+B}{2}\right)-a-b\left(\frac{A+B}{2}\right)$$

and

$$\mathcal{M}'(\ell,\mu)(m)=f'(m)-b.$$

From (3.1) we derive (3.7).

Since

$$\mathcal{M}(\ell,\mu)(A)=f(A)-a-bA,\ \text{and}\ \mathcal{M}(\ell,\mu)(B)=f(B)-a-bB,$$

then by (3.3) we derive (3.8). □

Remark 3. *If A, $B\geq m>0$, then we have the midpoint inequality and the trapezoid inequality for power function with exponent $r\in(0,1]$*

$$\left\|\int_0^1((1-t)A+tB)^r\,dt-\left(\frac{A+B}{2}\right)^r\right\|\leq\frac{1}{4}rm^{r-1}\|B-A\| \qquad (3.9)$$

and

$$\left\| \frac{A^r + B^r}{2} - \int_0^1 ((1-t)A + tB)^r \, dt \right\| \le \frac{1}{4} r m^{r-1} \|B - A\| . \tag{3.10}$$

The following inequalities for logarithm also hold

$$\left\| \int_0^1 \ln((1-t)A + tB) \, dt - \ln\left(\frac{A+B}{2}\right) \right\| \le \frac{1}{4m} \|B - A\| \tag{3.11}$$

and

$$\left\| \frac{\ln A + \ln B}{2} - \int_0^1 \ln((1-t)A + tB) \, dt \right\| \le \frac{1}{4m} \|B - A\| . \tag{3.12}$$

Corollary 6. *Assume that $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ that has the representation (1.7). If $A \ge m > 0$, $B \ge m > 0$, then*

$$\begin{aligned} &\left\| \int_0^1 f((1-t)A + tB)((1-t)A + tB)^{-1} dt - f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1} \right. \\ &\left. -f(0)\left(\int_0^1 ((1-t)A + tB)^{-1} dt - \left(\frac{A+B}{2}\right)^{-1}\right) \right\| \\ &\le \frac{1}{4}\left(\frac{f'(m)m - f(m) + f(0)}{m^2} - c\right) \|B - A\| \\ &\le \frac{f'(m)m - f(m) + f(0)}{4m^2} \|B - A\| \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\left\| \frac{f(A)A^{-1} + f(B)B^{-1}}{2} - \int_0^1 f((1-t)A + tB)((1-t)A + tB)^{-1} dt \right. \\ &\left. -f(0)\left(\frac{A^{-1} + B^{-1}}{2} - \int_0^1 ((1-t)A + tB)^{-1} dt\right) \right\| \\ &\le \frac{1}{4}\left(\frac{f'(m)m - f(m) + f(0)}{m^2} - c\right) \|B - A\| \\ &\le \frac{f'(m)m - f(m) + f(0)}{4m^2} \|B - A\| . \end{aligned} \tag{3.14}$$

Proof. From (1.7) we have for $T > 0$ that

$$\mathcal{M}(\ell, \mu)(T) = (f(T) - f(0))T^{-1} - f'_+(0) - cT,$$

for some positive measure μ. Therefore

$$\begin{aligned} &\int_0^1 \mathcal{M}(\ell, \mu)((1-t)A + tB) \, dt \\ &= \int_0^1 f((1-t)A + tB)((1-t)A + tB)^{-1} dt - f(0)\int_0^1 ((1-t)A + tB)^{-1} dt \\ &- f'_+(0) - c\left(\frac{A+B}{2}\right), \end{aligned}$$

$$\mathcal{M}(\ell,\mu)\left(\frac{A+B}{2}\right)=f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}-f(0)\left(\frac{A+B}{2}\right)^{-1}-f'_{+}(0)-c\left(\frac{A+B}{2}\right),$$

and

$$\mathcal{M}(\ell,\mu)(m)=\frac{f'(m)\,m-f(m)+f(0)}{m^2}-c.$$

By utilising (3.1) we get (3.13).

Since

$$\mathcal{M}(\ell,\mu)(A)=(f(A)-f(0))\,A^{-1}-f'_{+}(0)-cA$$

and

$$\mathcal{M}(\ell,\mu)(B)=(f(B)-f(0))\,B^{-1}-f'_{+}(0)-cB,$$

hence by (3.3) we get (3.14). □

Remark 4. *In the case when $f(0)=0$ in Corollary 6, we have the simpler inequalities*

$$\left\|\int_0^1 f((1-t)A+tB)((1-t)A+tB)^{-1}dt-f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}\right\| \le \frac{1}{4}\left(\frac{f'(m)\,m-f(m)}{m^2}-c\right)\|B-A\|\le\frac{f'(m)\,m-f(m)}{4m^2}\|B-A\| \tag{3.15}$$

and

$$\left\|\frac{f(A)A^{-1}+f(B)B^{-1}}{2}-\int_0^1 f((1-t)A+tB)((1-t)A+tB)^{-1}dt\right\| \le \frac{1}{4}\left(\frac{f'(m)\,m-f(m)}{m^2}-c\right)\|B-A\|\le\frac{f'(m)\,m-f(m)}{4m^2}\|B-A\|. \tag{3.16}$$

If in these inequalities we take the operator convex function $f(t)=-\ln(t+1)$, then we get

$$\left\|\int_0^1 \ln((1-t)A+tB+1)((1-t)A+tB)^{-1}dt-\ln\left(\frac{A+B}{2}+1\right)\left(\frac{A+B}{2}\right)^{-1}\right\| \le\frac{\ln(m+1)-m(m+1)^{-1}}{m^2}\|B-A\| \tag{3.17}$$

and

$$\left\|\frac{A^{-1}\ln(A+1)+B^{-1}\ln(B+1)}{2}-\int_0^1 \ln((1-t)A+tB+1)((1-t)A+tB)^{-1}dt\right\| \le\frac{\ln(m+1)-m(m+1)^{-1}}{m^2}\|B-A\|. \tag{3.18}$$

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