SUBADDITIVITY OF MONOTONIC INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Assume that $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$, then for all A, $B > 0$ we show that

$$\int_0^\infty w(\lambda) d\mu(\lambda) + \mathcal{M}(w, \mu)(A + B)$$
$$\geq \mathcal{M}(w, \mu)(A) + \mathcal{M}(w, \mu)(B) (\geq 0).$$

For all A, $B > 0$ with $BA + AB \geq k$ for some real constant k, we have the reverse inequality

$$\mathcal{M}(w, \mu)(A) + \mathcal{M}(w, \mu)(B)$$
$$\geq \mathcal{M}(w, \mu)(A + B) - k\mathcal{D}'(w, \mu)(A + B),$$

where $\mathcal{D}'(w, \mu)$ is the derivative of

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty w(\lambda) (\lambda + t)^{-1} d\mu(\lambda)$$

as a function of $t > 0$. Some examples for power function and integral transforms $\mathcal{M}(\cdot, \cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [9], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.1}$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator convex functions, Operator inequalities, Löwner-Heinz inequality. Logarithmic operator inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 82, 15 pp. Received 09/07/20

where $a \in R$, $b \geq 0$ *and a positive measure* μ *on* $(0, \infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.2}$$

If f *is operator monotone in* $[0, \infty)$, *then* $a = f(0)$ *in (1.1).*

The interested reader may find several examples of monotone operator functions in [4], [5], [6], [11] and [12].

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda) A + \lambda B) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function* $f : (0, \infty) \to \mathbb{R}$ *is operator convex in* $(0, \infty)$ *if and only if it has the representation*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t+\lambda} d\mu(\lambda), \tag{1.3}$$

where $a, b \in \mathbb{R}$, $c \geq 0$ *and a positive measure* μ *on* $(0, \infty)$ *such that (1.2) holds. If* f *is operator convex in* $[0, \infty)$, *then* $a = f(0)$ *and* $b = f'_+(0)$, *the right derivative, in (1.1).*

Assume that $A, B \geq 0$. In the recent paper [10], Moslehian and Najafi showed that $AB + BA$ is positive if and only if the following *operator subadditivity property* holds

$$f(A+B) \leq f(A) + f(B) \tag{1.4}$$

for all nonnegative operator monotone functions f on $[0, \infty)$. For some interesting consequences of this result see [10].

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda.$$

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.5}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.5) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.6}$$

Now, assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.8}$$

for $T > 0$.

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \quad t > 0. \tag{1.9}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, then we have the representation [2]

$$\ln t = (t - 1) \mathcal{D}(w_{\ln})(t), \quad t > 0. \tag{1.10}$$

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t), \quad t > 0. \tag{1.11}$$

For $t > 0$ we have

$$\begin{aligned} \mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t) &= \int_0^\infty w(\lambda) t (t + \lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)(t + \lambda - \lambda)(t + \lambda)^{-1} d\mu(\lambda) \\ &= \int_0^\infty w(\lambda) \left[1 - \lambda (t + \lambda)^{-1}\right] d\mu(\lambda). \end{aligned} \tag{1.12}$$

If $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$, then

$$\mathcal{M}(w, \mu)(t) = \int_0^\infty w(\lambda) d\mu(\lambda) - \mathcal{D}(\ell w, \mu)(t), \tag{1.13}$$

where $\ell(t) = t$, $t > 0$.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then, after some calculations, we get

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t + \lambda} d\lambda = E_1(at) \exp(at), \quad t \geq 0$$

and

$$\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a}.$$

This gives that

$$\mathcal{M}(e_{-a})(t) = t\mathcal{D}(w, \mu)(t) = tE_1(at) \exp(at), \quad t \geq 0,$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a}, \mu)(t) = \int_0^\infty \frac{\lambda \exp(-a\lambda)}{t + \lambda} d\lambda = \frac{1}{a} - tE_1(at) \exp(at)$$

for $t > 0$.

One observes that

$$\mathcal{M}(e_{-a})(t)=\int_0^\infty w(\lambda)\,d\lambda-\mathcal{D}(\ell e_{-a},\mu)(t),\ t>0$$

and the equality (1.13) is verified in this case.

If we take $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda=\infty$ and the equality (1.13) does not hold in this case.

For all $T>0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T)=T\mathcal{D}(w,\mu)(T)=\int_0^\infty w(\lambda)\left[1-\lambda(T+\lambda)^{-1}\right]d\mu(\lambda). \tag{1.14}$$

This gives the representation

$$T^r=\frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r)(T),\ T>0 \tag{1.15}$$

where $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$. Also, from (1.10) we have the representation

$$T\ln T=(T-1)\,\mathcal{M}(w_{\ln})(T),\ T>0, \tag{1.16}$$

where $w_{\ln}(\lambda)=(\lambda+1)^{-1}$.

In the recent paper [3] we obtained the following result:

Theorem 3. *For all A, $B>0$ we have the representation*

$$\begin{aligned}&\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\&=\int_0^\infty\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right)\\&\times\lambda w(\lambda)\,d\mu(\lambda).\end{aligned} \tag{1.17}$$

If $B\ge A>0$, then

$$\mathcal{M}(w,\mu)(B)\ge\mathcal{M}(w,\mu)(A), \tag{1.18}$$

namely $\mathcal{M}(w,\mu)$ is operator monotone on $(0,\infty)$.

Assume that $\int_0^\infty w(\lambda)\,d\mu(\lambda)<\infty$, then for all A, $B>0$ we show that

$$\int_0^\infty w(\lambda)\,d\mu(\lambda)+\mathcal{M}(w,\mu)(A+B)\ge\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)\,(\ge 0).$$

For all A, $B>0$ with $BA+AB\ge k$ for some real constant k, we have the reverse inequality

$$\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)\ge\mathcal{M}(w,\mu)(A+B)-k\mathcal{D}'(w,\mu)(A+B),$$

where $\mathcal{D}'(w,\mu)$ is the derivative of

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty w(\lambda)(\lambda+t)^{-1}\,d\mu(\lambda)$$

as a function of $t>0$. Some examples for power function and integral transforms $\mathcal{M}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

We have the following perturbed superadditivity property:

Theorem 4. *Assume that* $\int_0^\infty w(\lambda)\, d\mu(\lambda) < \infty$, *then for all* $A, B > 0$ *we have*

$$\int_0^\infty w(\lambda)\, d\mu(\lambda) + \mathcal{M}(w,\mu)(A+B) \geq \mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B)\,(\geq 0)\,. \tag{2.1}$$

Proof. For $A, B > 0$ and $\lambda \geq 0$, define

$$K_\lambda := (A+\lambda)^{-1} + (B+\lambda)^{-1} - (A+B+\lambda)^{-1}\,. \tag{2.2}$$

Therefore

$$\begin{aligned} &(A+B+\lambda) K_\lambda (A+B+\lambda) \\ &= (A+B+\lambda)(A+\lambda)^{-1}(A+B+\lambda) \\ &+ (A+B+\lambda)(B+\lambda)^{-1}(A+B+\lambda) - A - B - \lambda \\ &= \left(1 + B(A+\lambda)^{-1}\right)(A+\lambda+B) \\ &+ \left(A(B+\lambda)^{-1} + 1\right)(A+B+\lambda) - A - B - \lambda \\ &= A + \lambda + B + B + B(A+\lambda)^{-1} B \\ &+ A(B+\lambda)^{-1} A + A + A + B + \lambda - A - B - \lambda \\ &= B(A+\lambda)^{-1} B + A(B+\lambda)^{-1} A + 2(A+B) + \lambda \\ &=: L_\lambda. \end{aligned} \tag{2.3}$$

Since $A, B > 0$ and $\lambda \geq 0$, then $(A+\lambda)^{-1} > 0$, $(B+\lambda)^{-1} > 0$ which imply that $B(A+\lambda)^{-1} B > 0$ and $A(B+\lambda)^{-1} A > 0$. Therefore $L_\lambda > 0$ for all $\lambda \geq 0$.

By multiplying both sides of (2.3) with $(A+B+\lambda)^{-1}$ we get

$$K_\lambda = (A+B+\lambda)^{-1} L_\lambda (A+B+\lambda)^{-1} > 0$$

for all $\lambda \geq 0$.

We have

$$\begin{aligned} &\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A+B) \\ &= \int_0^\infty w(\lambda)\left(1 - \lambda(A+\lambda)^{-1}\right) d\mu(\lambda) \\ &+ \int_0^\infty w(\lambda)\left(1 - \lambda(B+\lambda)^{-1}\right) d\mu(\lambda) \\ &- \int_0^\infty w(\lambda)\left(1 - \lambda(A+B+\lambda)^{-1}\right) d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)(1 - \lambda K_\lambda) d\mu(\lambda)\,. \end{aligned} \tag{2.4}$$

Since $\int_0^\infty w(\lambda)\, d\mu(\lambda) < \infty$, then

$$\int_0^\infty w(\lambda)(1 - \lambda K_\lambda) d\mu(\lambda) = \int_0^\infty w(\lambda)\, d\mu(\lambda) - \int_0^\infty w(\lambda)\, \lambda K_\lambda d\mu(\lambda)$$

and we get

$$\begin{aligned}
&\int_0^\infty w(\lambda)\,d\mu(\lambda) + \mathcal{M}(w,\mu)(A+B) - \mathcal{M}(w,\mu)(A) - \mathcal{M}(w,\mu)(B) \\
&= \int_0^\infty w(\lambda)\,\lambda K_\lambda d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\,\lambda (A+B+\lambda)^{-1} L_\lambda (A+B+\lambda)^{-1} d\mu(\lambda) \geq 0
\end{aligned}$$

proving the required inequality (2.1). □

Corollary 1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.1). If the expectation $\int_0^\infty \lambda d\mu(\lambda)$ is finite, then*

$$\int_0^\infty \lambda d\mu(\lambda) + f(0) \geq f(A) + f(B) - f(A+B) \tag{2.5}$$

for all A, $B>0$.

Proof. From (1.1) we have

$$\mathcal{M}(\ell,\mu)(T) = f(t) - f(0) - bT,$$

for $T>0$.

Therefore

$$\mathcal{M}(w,\mu)(A+B) = f(A+B) - f(0) - b(A+B),$$
$$\mathcal{M}(w,\mu)(A) = f(A) - f(0) - bA$$

and

$$\mathcal{M}(w,\mu)(B) = f(B) - f(0) - bB$$

and by (2.1) we get

$$\begin{aligned}
&\int_0^\infty \lambda d\mu(\lambda) + f(A+B) - f(0) - b(A+B) \\
&\geq f(A) - f(0) - bA + f(B) - f(0) - bB,
\end{aligned}$$

which is equivalent to (2.5). □

Corollary 2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.3). If the expectation $\int_0^\infty \lambda d\mu(\lambda)$ is finite, then*

$$\begin{aligned}
&\int_0^\infty \lambda d\mu(\lambda) + f'_+(0) + f(0)\left[A^{-1} + B^{-1} - (A+B)^{-1}\right] \\
&\geq f(A)A^{-1} + f(B)B^{-1} - f(A+B)(A+B)^{-1}
\end{aligned} \tag{2.6}$$

for all A, $B>0$.

If $f(0)=0$, then we have the simpler inequality

$$\int_0^\infty \lambda d\mu(\lambda) + f'_+(0) \geq f(A)A^{-1} + f(B)B^{-1} - f(A+B)(A+B)^{-1}. \tag{2.7}$$

Proof. From (1.3) we have

$$\mathcal{M}(\ell,\mu)(T) = f(T)T^{-1} - f(0)T^{-1} - f'_+(0) - cT,$$

for $T>0$.

Therefore

$$\mathcal{M}(\ell,\mu)(A+B) = f(A+B)(A+B)^{-1} - f(0)(A+B)^{-1} - f'_{+}(0) - c(A+B),$$

$$\mathcal{M}(\ell,\mu)(A) = f(A)A^{-1} - f(0)A^{-1} - f'_{+}(0) - cA,$$

and

$$\mathcal{M}(\ell,\mu)(B) = f(B)B^{-1} - f(0)B^{-1} - f'_{+}(0) - cB,$$

and by (2.1) we get

$$\begin{aligned}&\int_0^\infty \lambda d\mu(\lambda) + f(A+B)(A+B)^{-1} - f(0)(A+B)^{-1} - f'_{+}(0) - c(A+B)\\ &\geq f(A)A^{-1} - f(0)A^{-1} - f'_{+}(0) - cA + f(B)B^{-1} - f(0)B^{-1} - f'_{+}(0) - cB\end{aligned}$$

namely

$$\begin{aligned}&\int_0^\infty \lambda d\mu(\lambda) + f(A+B)(A+B)^{-1} - f(0)(A+B)^{-1}\\ &\geq f(A)A^{-1} - f(0)A^{-1} - f'_{+}(0) + f(B)B^{-1} - f(0)B^{-1},\end{aligned}$$

which is equivalent to (2.6). □

Theorem 5. *For all* A, $B>0$ *we have*

$$\begin{aligned}&\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A+B)\\ &\geq \int_0^\infty w(\lambda)(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda).\end{aligned} \tag{2.8}$$

If $BA+AB\geq 0$, *then*

$$\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B) \geq \mathcal{M}(w,\mu)(A+B), \tag{2.9}$$

namely $\mathcal{M}(w,\mu)$ *is operator subadditive.*

Proof. For A, $B>0$ and $\lambda\geq 0$, define

$$W_\lambda := 1 - \lambda K_\lambda,$$

where K_λ is given in (2.2).

We have successively

$$\begin{aligned}
&(A+B+\lambda)W_\lambda(A+B+\lambda)\\
&=(A+B+\lambda)(1-\lambda K_\lambda)(A+B+\lambda)\\
&=(A+B+\lambda)^2-\lambda(A+B+\lambda)K_\lambda(A+B+\lambda)\\
&=(A+B+\lambda)(A+B+\lambda)\\
&-\lambda\left[B(A+\lambda)^{-1}B+A(B+\lambda)^{-1}A+2(A+B)+\lambda\right]\\
&=A^2+BA+\lambda A+AB+B^2+\lambda B+\lambda A+\lambda B+\lambda^2\\
&-\lambda B(A+\lambda)^{-1}B-\lambda A(B+\lambda)^{-1}A-2\lambda(A+B)-\lambda^2\\
&=A^2+B^2+BA+AB-\lambda B(A+\lambda)^{-1}B-\lambda A(B+\lambda)^{-1}A\\
&=A(B+\lambda)^{-1}(B+\lambda)A-\lambda A(B+\lambda)^{-1}A\\
&+B(A+\lambda)^{-1}(A+\lambda)B-\lambda B(A+\lambda)^{-1}B\\
&+BA+AB\\
&=A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB,
\end{aligned}$$

therefore

$$\begin{aligned}
W_\lambda&=(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB\right]\\
&\times(A+B+\lambda)^{-1}.
\end{aligned}$$

We also have the representation

$$\begin{aligned}
&\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A+B)\\
&=\int_0^\infty w(\lambda)W_\lambda d\mu(\lambda)\\
&=\int_0^\infty w(\lambda)(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\right](A+B+\lambda)^{-1}\\
&+\int_0^\infty w(\lambda)(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda).
\end{aligned}$$

Since $(B+\lambda)^{-1}B>0$, $(A+\lambda)^{-1}A>0$, hence

$$A(B+\lambda)^{-1}BA>0, B(A+\lambda)^{-1}AB>0,$$

which implies that

$$(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\right](A+B+\lambda)^{-1}>0.$$

If we multiply this by $w(\lambda)\geq 0$ and integrate, then we get the desired inequality (2.8). □

Remark 1. *By Theorem 3 we observe that* $\mathcal{M}(w,\mu)$ *is operator monotone and nonnegative. If* $BA+AB\geq 0$*, then by utilising Moslehian and Najafi's result (1.4) we also deduce (2.9).*

Corollary 3. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.1). For all A, $B>0$ we have*

$$\begin{aligned}&f(A)+f(B)-f(A+B)-f(0)\\&\geq\int_0^\infty\lambda(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda).\end{aligned}\tag{2.10}$$

If $BA+AB\geq0$, then

$$f(A)+f(B)\geq f(A+B)+f(0).\tag{2.11}$$

Proof. Since

$$\mathcal{M}(\ell,\mu)(A+B)=f(A+B)-f(0)-b(A+B),$$

$$\mathcal{M}(\ell,\mu)(A)=f(A)-f(0)-bA$$

and

$$\mathcal{M}(\ell,\mu)(B)=f(B)-f(0)-bB$$

hence by Theorem 5 we obtain the desired inequalities. □

Corollary 4. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.3). For all A, $B>0$ we have*

$$\begin{aligned}&f(A)A^{-1}+f(B)B^{-1}-f(A+B)(A+B)^{-1}\\&-f(0)\left[B^{-1}+A^{-1}-(A+B)^{-1}\right]-f'_+(0)\\&\geq\int_0^\infty\lambda(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda).\end{aligned}\tag{2.12}$$

If $f(0)=0$, then we have the simpler inequality

$$\begin{aligned}&f(A)A^{-1}+f(B)B^{-1}-f(A+B)(A+B)^{-1}-f'_+(0)\\&\geq\int_0^\infty\lambda(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda).\end{aligned}\tag{2.13}$$

If $BA+AB\geq0$, then

$$\begin{aligned}&f(A)A^{-1}+f(B)B^{-1}-f(A+B)(A+B)^{-1}\\&\geq f(0)\left[B^{-1}+A^{-1}-(A+B)^{-1}\right]+f'_+(0).\end{aligned}\tag{2.14}$$

If $f(0)=0$, then

$$f(A)A^{-1}+f(B)B^{-1}\geq f(A+B)(A+B)^{-1}+f'_+(0).\tag{2.15}$$

Proof. Since

$$\mathcal{M}(\ell,\mu)(A+B)=f(A+B)(A+B)^{-1}-f(0)(A+B)^{-1}-f'_+(0)-c(A+B),$$

$$\mathcal{M}(\ell,\mu)(A)=f(A)A^{-1}-f(0)A^{-1}-f'_+(0)-cA,$$

and

$$\mathcal{M}(\ell,\mu)(B)=f(B)B^{-1}-f(0)B^{-1}-f'_+(0)-cB,$$

hence by Theorem 5 we obtain the desired inequalities. □

Theorem 6. *For all $A,\ B>0$ with $BA+AB\geq k$ for some real constant k, we have*

$$\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A+B)\geq -k\mathcal{D}'(w,\mu)(A+B), \tag{2.16}$$

where $\mathcal{D}'(w,\mu)$ is the derivative of $\mathcal{D}(w,\mu)$ as a function of $t>0$.

If $k\geq 0$, then

$$\begin{aligned}\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A+B)\\ \geq -kD'(w,\mu)(A+B)\geq 0.\end{aligned} \tag{2.17}$$

Proof. Let $t>0$. For $h\neq 0$ small,

$$\begin{aligned}\frac{\mathcal{D}(w,\mu)(t+h)-\mathcal{D}(w,\mu)(t)}{h} &= \frac{1}{h}\int_0^\infty \left(\frac{w(\lambda)}{t+h+\lambda}-\frac{w(\lambda)}{t+\lambda}\right)d\mu(\lambda)\\ &= -\int_0^\infty \frac{w(\lambda)}{(t+h+\lambda)(t+\lambda)}d\mu(\lambda).\end{aligned}$$

By taking the limit over $h\to 0$ and using the properties of limits and integrals, we get the derivative of $\mathcal{D}(w,\mu)$ as

$$\mathcal{D}'(w,\mu)(t)=-\int_0^\infty \frac{w(\lambda)}{(t+\lambda)^2}d\mu(\lambda)\leq 0,\ t>0. \tag{2.18}$$

If $T>0$, then by using the continuous functional calculus for selfadjoint operators, we have

$$\mathcal{D}'(w,\mu)(T)=-\int_0^\infty w(\lambda)(T+\lambda)^{-2}d\mu(\lambda)\leq 0.$$

Observe that, since $BA+AB\geq k$, hence by multiplying both sides with $(A+B+\lambda)^{-1}$ for $\lambda\geq 0$, we get

$$(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}\geq k(A+B+\lambda)^{-2}.$$

By multiplying with $w(\lambda)\geq 0$ and integrate, then we get

$$\begin{aligned}&\int_0^\infty w(\lambda)(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}\\ &\geq k\int_0^\infty w(\lambda)(A+B+\lambda)^{-2}d\mu(\lambda)=-k\mathcal{D}'(w,\mu)(A+B)\end{aligned}$$

for $A,\ B>0$.

By making use of (2.8) we derive the inequality (2.16). □

Remark 2. *The symmetrized product of two operators $A,\ B\in B(H)$ is defined by $S(A,B)=AB+BA$. In general, the symmetrized product of two operators $A,\ B$ is not positive (see for instance [11]). Also Gustafson [7] showed that if $0\leq m\leq A\leq M$ and $0\leq n\leq B\leq N$, then we have the lower bound*

$$S(A,B)\geq 2mn-\frac{1}{4}(M-m)(N-n)=:k, \tag{2.19}$$

which can take positive or negative values depending on the parameters m, M, n, N.

Corollary 5. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *and has the representation (1.1). For all* A, $B>0$ *with* $S(A,B)\geq k$ *for some real constant* k, *we have*

$$\begin{aligned}(2.20)\qquad & f(A)+f(B)-f(A+B)-f(0)\\ &\geq k\left[f(A+B)-f'(A+B)(A+B)-f(0)\right](A+B)^{-2}\end{aligned}$$

If $k\geq 0$, *then*

$$\begin{aligned}(2.21)\qquad & f(A)+f(B)-f(A+B)-f(0)\\ &\geq k\left[f(A+B)-f'(A+B)(A+B)-f(0)\right](A+B)^{-2}\geq 0.\end{aligned}$$

Proof. We have that

$$\mathcal{D}(\ell,\mu)(t)=\frac{f(t)-f(0)}{t}-b,\ t>0$$

for some positive measure $\mu(\lambda)$ and nonnegative b.

From this,

$$\mathcal{D}'(\ell,\mu)(t)=\frac{f'(t)\,t-f(t)+f(0)}{t^2},\ \ t>0,$$

which gives

$$\mathcal{D}'(\ell,\mu)(A+B)=\left[f'(A+B)(A+B)-f(A+B)+f(0)\right](A+B)^{-2}.$$

The desired inequalities then follow by Theorem 6. □

Remark 3. *If* $f(0)=0$ *in (2.21), then we get the simpler inequality*

$$\begin{aligned}(2.22)\qquad & f(A)+f(B)-f(A+B)\\ &\geq k\left[f(A+B)-f'(A+B)(A+B)\right](A+B)^{-2}\geq 0\end{aligned}$$

if A, $B>0$ *with* $S(A,B)\geq k\geq 0$.

Corollary 6. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator convex in* $[0,\infty)$ *and has the representation (1.3). For all* A, $B>0$ *with* $S(A,B)\geq k$,

$$\begin{aligned}(2.23)\quad & f(A)A^{-1}+f(B)B^{-1}-f(A+B)(A+B)^{-1}\\ &-f(0)\left[B^{-1}+A^{-1}-(A+B)^{-1}\right]-f'_+(0)\\ &\geq 2k\left(\left[f(A+B)-f(0)\right](A+B)^{-1}-\frac{f'(A+B)+f'_+(0)}{2}\right)(A+B)^{-2}.\end{aligned}$$

If $f(0)=0$, *then we have the simpler inequality*

$$\begin{aligned}(2.24)\qquad & f(A)A^{-1}+f(B)B^{-1}-f(A+B)(A+B)^{-1}-f'_+(0)\\ &\geq 2k\left(f(A+B)(A+B)^{-1}-\frac{f'(A+B)+f'_+(0)}{2}\right)(A+B)^{-2}.\end{aligned}$$

Proof. From (1.3) we have

$$\mathcal{D}(\ell,\mu)(t)=\frac{f(t)-f(0)-f'_+(0)\,t}{t^2}-c,\ t>0.$$

This gives

$$\mathcal{D}'(w,\mu)(t) = \frac{\left(f'(t) - f'_+(0)\right)t^2 - 2t\left(f(t) - f(0) - f'_+(0)\,t\right)}{t^4} = \frac{2}{t^2}\left(\frac{f'(t) + f'_+(0)}{2} - \frac{f(t) - f(0)}{t}\right).$$

Therefore

$$\mathcal{D}'(w,\mu)(A+B) = 2\left(\frac{f'(A+B) + f'_+(0)}{2} - [f(A+B) - f(0)](A+B)^{-1}\right)(A+B)^{-2}$$

and by (2.16) we derive (2.23). □

Remark 4. *If A, $B > 0$ with $S(A,B) \geq k \geq 0$ and $f(0) = 0$, then*

$$f(A)A^{-1} + f(B)B^{-1} - f(A+B)(A+B)^{-1} - f'_+(0) \geq 2k\left(f(A+B)(A+B)^{-1} - \frac{f'(A+B) + f'_+(0)}{2}\right)(A+B)^{-2} \geq 0. \tag{2.25}$$

3. Some Examples

We define the *upper incomplete Gamma function* as [13]

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z = 0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [14]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{3.1}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (3.1) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t)$$

and

$$\mathcal{M}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \Gamma(1-a)t^{1-a}e^t\Gamma(a,t) \tag{3.2}$$

for $a < 1$ and $t > 0$.

We also have

$$\int_0^\infty w_{\cdot^{-a}e^{-\cdot}}(\lambda)\,d\lambda = \Gamma(1-a), \text{ for } a < 1. \tag{3.3}$$

For $a = 0$ in (3.2) we get

$$\mathcal{M}\left(w_{e^{-\cdot}}\right)(t) = t\int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)te^t\Gamma(0,t) = te^tE_1(t) \tag{3.4}$$

for $t > 0$, where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du. \tag{3.5}$$

For all $A,\ B > 0$, we have by Theorem 4 applied for $w_{\cdot^{-a}e^{-\cdot}}(\lambda)$ that

$$\begin{aligned} &1 + (A+B)^{1-a} \exp(A+B)\, \Gamma(a, (A+B)) \\ &\geq A^{1-a} \exp(A)\, \Gamma(a, A) + B^{1-a} \exp(B)\, \Gamma(a, B) \end{aligned} \tag{3.6}$$

for $a < 1$.

If $a = 0$ we get from (3.6) that

$$1 + (A+B) \exp(A+B)\, E_1(A+B) \geq A \exp(A)\, E_1(A) + B \exp(B)\, E_1(B) \tag{3.7}$$

for all $A,\ B > 0$.

If we assume that $A,\ B > 0$, then we have by Theorem 5 for $w_{e^{-\cdot}}(\lambda)$ that

$$\begin{aligned} &A \exp(A)\, E_1(A) + B \exp(B)\, E_1(B) - (A+B) \exp(A+B)\, E_1(A+B) \\ &\geq \int_0^\infty \lambda^{-a} e^{-\lambda} (A+B+\lambda)^{-1} (BA+AB) (A+B+\lambda)^{-1} d\lambda. \end{aligned} \tag{3.8}$$

If $S(A, B) \geq 0$, then

$$A \exp(A)\, E_1(A) + B \exp(B)\, E_1(B) \geq (A+B) \exp(A+B)\, E_1(A+B). \tag{3.9}$$

Since $E_1'(t) = -\frac{e^{-t}}{t}$, $t > 0$, then

$$\mathcal{D}'(w_{e^{-\cdot}})(t) = \left(e^t E_1(t)\right)' = e^t E_1(t) - e^t \frac{e^{-t}}{t} = e^t E_1(t) - \frac{1}{t}.$$

Therefore

$$\mathcal{D}'(w_{e^{-\cdot}})(A+B) = \exp(A+B)\, E_1(A+B) - (A+B)^{-1}$$

for $A,\ B > 0$.

For all $A,\ B > 0$ with $S(A, B) \geq k$ for some real constant k, we have by Theorem 6 for $w_{e^{-\cdot}}$ that

$$\begin{aligned} &A \exp(A)\, E_1(A) + B \exp(B)\, E_1(B) - (A+B) \exp(A+B)\, E_1(A+B) \\ &\geq k \left[(A+B)^{-1} - \exp(A+B)\, E_1(A+B)\right]. \end{aligned} \tag{3.10}$$

If $k \geq 0$, then we obtain an improvement of (3.9) as

$$\begin{aligned} &A \exp(A)\, E_1(A) + B \exp(B)\, E_1(B) - (A+B) \exp(A+B)\, E_1(A+B) \\ &\geq k \left[(A+B)^{-1} - \exp(A+B)\, E_1(A+B)\right] \geq 0. \end{aligned} \tag{3.11}$$

Consider the operator monotone function $f(t) = t^r$, $r \in (0, 1]$. For all $A,\ B > 0$ with $S(A, B) \geq k$ for some real constant k, we have by (2.20) that

$$A^r + B^r - (A+B)^r \geq (1-r)\, k\, (A+B)^{r-2}. \tag{3.12}$$

If $0 \leq m \leq A \leq M$ and $0 \leq n \leq B \leq N$, then by (2.19) we get

$$A^r + B^r - (A+B)^r \geq \frac{1}{4}(1-r)\left[8mn - (M-m)(N-n)\right](A+B)^{r-2}, \tag{3.13}$$

where $r \in (0, 1]$.

Consider the operator monotone function $f(t) = \ln(t+1)$. Then by (2.20) we derive

$$\ln(A+1) + \ln(B+1) - \ln(A+B+1) \geq k\left[\ln(A+B+1) - (A+B)(A+B+1)^{-1}\right](A+B)^{-2} \tag{3.14}$$

for all A, $B > 0$ with $S(A,B) \geq k$.

By choosing the function $f(t) = -\ln(t+1)$, which is operator convex, we obtain from (2.24) that

$$1 + (A+B)^{-1}\ln(A+B+1) - A^{-1}\ln(A+1) - B^{-1}\ln(B+1) \geq 2k\left(\frac{(A+B+1)^{-1} - 1}{2} - (A+B)^{-1}\ln(A+B+1)\right)(A+B)^{-2}, \tag{3.15}$$

for all A, $B > 0$ with $S(A,B) \geq k$.

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we derive that

$$\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+a^2\right)} d\lambda = \frac{\pi t}{2a\left(t^2+a^2\right)} - \frac{\ln t - \ln a}{t^2+a^2}$$

for $t > 0$ and $a > 0$.

For $a = 1$ we have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+1\right)} d\lambda = \frac{\pi t}{2\left(t^2+1\right)} - \frac{\ln t}{t^2+1}$$

for $t > 0$.

If $T > 0$ and $a > 0$, then

$$\left(T^2+a^2\right)^{-1}\left(\frac{\pi}{2a}T - \ln T + \ln a\right) = \int_0^\infty \frac{1}{\left(\lambda^2+a^2\right)}(\lambda+T)^{-1} d\lambda \tag{3.16}$$

and, in particular,

$$\left(T^2+1\right)^{-1}\left(\frac{\pi}{2}T - \ln T\right) = \int_0^\infty \frac{1}{\left(\lambda^2+1\right)}(\lambda+T)^{-1} d\lambda. \tag{3.17}$$

By using the above general results the interested reader may state similar operator inequalities for the functions in (3.16) and (3.17). The details are omitted.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.