SYNCHRONICITY PROPERTIES FOR MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function \( w(\lambda), \lambda > 0 \) and \( \mu \) a positive measure on \((0, \infty)\) we consider the following mapping that we call the monotonic integral transform
\[
\mathcal{M}(w, \mu)(T) := \int_0^\infty w(\lambda) T (\lambda + T)^{-1} \, d\mu(\lambda),
\]
where the integral is assumed to exist for \( T \) a positive operator on a complex Hilbert space \( H \). In this paper, we show among others that we have the representation
\[
[\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A)](B - A) = \int_0^\infty \lambda w(\lambda) \left( \int_0^1 \left[ (\lambda + (1 - t) A + tB)^{-1} (B - A) \right]^2 \, dt \right) \, d\mu(\lambda)
\]
for all \( A, B > 0 \). We also provide some sufficient conditions for the operators \( A, B > 0 \) such that the inequality
\[
A \mathcal{M}(w, \mu)(A) + B \mathcal{M}(w, \mu)(B) \geq \mathcal{M}(w, \mu)(A) B + \mathcal{M}(w, \mu)(B) A
\]
holds. Some examples for operator monotonic and operator convex functions are also given.

1. Introduction

Consider a complex Hilbert space \((H, \langle \cdot, \cdot \rangle)\). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( \langle Tx, x \rangle \geq 0 \) for all \( x \in H \) and also an operator \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible.

We have the following integral representation for the power function when \( t > 0, r \in (0, 1], \) see for instance [1, p. 145]
\[
t^{r-1} = \frac{\sin (r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} \, d\lambda.
\]
(1.1)

Observe that for \( t > 0, t \neq 1 \), we have
\[
\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln \left( \frac{u + t}{u + 1} \right)
\]
for all \( u > 0 \).

By taking the limit over \( u \to \infty \) in this equality, we derive
\[
\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},
\]

\textsuperscript{1}1991 Mathematics Subject Classification. 47A63, 47A60.

\textsuperscript{2}Key words and phrases. Operator monotone functions, Operator convex functions, Operator inequalities, Löwner-Heinz inequality, Logarithmic operator inequalities.
which gives the representation for the logarithm

\[
\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)}
\]

for all \(t > 0\).

Motivated by these representations, we introduce \([2]\), for a continuous and positive function \(w(\lambda), \lambda > 0\), the following integral transform

\[
\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \quad t > 0,
\]

where \(\mu\) is a positive measure on \((0, \infty)\) and the integral \((1.3)\) exists for all \(t > 0\).

For \(\mu\) the Lebesgue usual measure, we put

\[
\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \quad t > 0.
\]

If we take \(\mu\) to be the usual Lebesgue measure and the kernel \(w_r(\lambda) = \lambda^{r-1}\), \(r \in (0, 1]\), then

\[
t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \quad t > 0.
\]

For the same measure, if we take the kernel \(w_{\ln}(\lambda) = (\lambda + 1)^{-1}\), \(t > 0\), then we have the representation

\[
\ln t = (t - 1) \mathcal{D}(w_{\ln})(t), \quad t > 0.
\]

Assume that \(T > 0\), then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

\[
\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),
\]

where \(w\) and \(\mu\) are as above. Also, when \(\mu\) is the usual Lebesgue measure, then

\[
\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda,
\]

for \(T > 0\).

A real valued continuous function \(f\) on \((0, \infty)\) is said to be operator monotone if \(f(A) \geq f(B)\) holds for any \(A \geq B > 0\). A real valued continuous function \(f\) on \((0, \infty)\) is said to be operator monotone if \(f(A) \geq f(B)\) holds for any \(A \geq B > 0\).

We have the following representation of operator monotone functions \([8]\), see for instance \([1, p. 144-145]\):

**Theorem 1.** A function \(f : (0, \infty) \to \mathbb{R}\) is operator monotone in \((0, \infty)\) if and only if it has the representation

\[
f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda),
\]

where \(a \in \mathbb{R}, b \geq 0\) and a positive measure \(\mu\) on \((0, \infty)\) such that

\[
\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty.
\]

If \(f\) is operator monotone in \([0, \infty)\), then \(a = f(0)\) in \((1.9)\).
For examples of operator monotone functions, see [4], [5], [9], [10] and the references therein.

A real valued continuous function $f$ on an interval $I$ is said to be operator convex (operator concave) on $I$ if

\[(OC) \quad f \left( (1 - \lambda) A + \lambda B \right) \leq (\geq) (1 - \lambda) f(A) + \lambda f(B) \]

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator $A$ and $B$ on a Hilbert space $H$ whose spectra are contained in $I$. Notice that a function $f$ is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

**Theorem 2.** A function $f : (0, \infty) \to \mathbb{R}$ is operator convex in $(0, \infty)$ if and only if it has the representation

\[(1.11) \quad f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t + \lambda} d\mu(\lambda), \]

where $a, b \in \mathbb{R}$, $c \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ such that (1.2) holds. If $f$ is operator convex in $[0, \infty)$, then $a = f(0)$ and $b = f'(0)$, the right derivative, in (1.11).

For a continuous and positive function $w(\lambda), \lambda > 0$ and a positive measure $\mu$ on $(0, \infty)$, we can define the following mapping, which we call monotonic integral transform, by

\[(1.12) \quad \mathcal{M}(w, \mu) (t) := t \mathcal{D}(w, \mu) (t), \ t > 0. \]

For $t > 0$ we have

\[(1.13) \quad \mathcal{M}(w, \mu) (t) := t \mathcal{D}(w, \mu) (t) = \int_0^\infty w(\lambda) t (t + \lambda)^{-1} d\mu(\lambda) = \int_0^\infty w(\lambda) (t + \lambda - \lambda) (t + \lambda)^{-1} d\mu(\lambda) = \int_0^\infty w(\lambda) \left[1 - \lambda (t + \lambda)^{-1}\right] d\mu(\lambda). \]

If $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$, then

\[(1.14) \quad \mathcal{M}(w, \mu) (t) = \int_0^\infty w(\lambda) d\mu(\lambda) - \mathcal{D}(\ell w, \mu) (t), \]

where $\ell(t) = t, \ t > 0$.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda), \ \lambda \geq 0$ and $a > 0$. Then after some calculations, we get

\[\mathcal{D}(e_{-a}) (t) = \int_0^\infty \frac{\exp(-a\lambda)}{t + \lambda} d\lambda = E_1(at) \exp(at), \ t \geq 0 \]

and

\[\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a}. \]

This gives that

\[\mathcal{M}(e_{-a}) (t) = t \mathcal{D}(w, \mu) (t) = t E_1(at) \exp(at), \ t \geq 0, \]

where

\[E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.\]
By integration we also have
\[ D(e^{-a}, \mu) (t) = \int_0^\infty \frac{\lambda e^{-a \lambda}}{t + \lambda} d\lambda = \frac{1}{a} - tE_1(at) \exp(at) \]
for \( t > 0 \).

One observes that
\[ M(e^{-a}) (t) = \int_0^\infty w(\lambda) d\lambda - D(e^{-a}, \mu) (t), \ t > 0 \]
and the equality (1.14) is verified in this case.

If we take \( w_r(\lambda) = \lambda^{r-1}, \ r \in (0, 1] \), then \( \int_0^\infty w_r(\lambda) d\lambda = \infty \) and the equality (1.14) does not hold in this case.

For all \( T > 0 \) we have, by the continuous functional calculus for selfadjoint operators, that
\[ M(w; \mu)(T) = TD(w, \mu)(T) = \int_0^\infty w(\lambda) \left[ 1 - \lambda(T + \lambda)^{-1} \right] d\mu(\lambda). \]
This gives the representation
\[ T^r = \frac{\sin(r\pi)}{\pi} M(w_r, \mu)(T), \]
where \( w_r(\lambda) = \lambda^{r-1}, \ r \in (0, 1] \) and \( \mu \) is the usual Lebesgue norm. Also, from (1.6) we have the representation
\[ \ln T = (T - 1) D(w_1)(T), \ T > 0, \]
where \( w_1(\lambda) = (\lambda + 1)^{-1}, \ t > 0. \)

2. Some Identities

We have the following representation result:

**Theorem 3.** For all \( A, B > 0 \) we have the representations
\[ [M(w, \mu)(B) - M(w, \mu)(A)](B - A) = \int_0^\infty \lambda w(\lambda) \left( \int_0^1 \left[ (\lambda + (1 - t)A + tB)^{-1} (B - A) \right]^2 dt \right) d\mu(\lambda) \]
and
\[ (B - A) [M(w, \mu)(B) - M(w, \mu)(A)] = \int_0^\infty \lambda w(\lambda) \left( \int_0^1 \left[ (B - A)/(\lambda + (1 - t)A + tB)^{-1} \right]^2 dt \right) d\mu(\lambda). \]

**Proof.** From (1.15) we have for all \( A, B \geq 0 \) that
\[ M(w, \mu)(B) - M(w, \mu)(A) = \int_0^\infty w(\lambda) \left[ 1 - \lambda(B + \lambda)^{-1} \right] d\mu(\lambda) - \int_0^\infty w(\lambda) \left[ 1 - \lambda(A + \lambda)^{-1} \right] d\mu(\lambda) = \int_0^\infty \lambda w(\lambda) \left[ (A + \lambda)^{-1} - (B + \lambda)^{-1} \right] d\mu(\lambda). \]
Let \( T, S > 0 \). The function \( f(t) = -t^{-1} \) is operator monotone on \((0, \infty)\), operator Gâteaux differentiable and the Gâteaux derivative is given by

\[
\nabla f_T(S) := \lim_{t \to 0} \left[ \frac{f(T + tS) - f(T)}{t} \right] = T^{-1}ST^{-1}
\]

for \( T, S > 0 \).

Consider the continuous function \( f \) defined on an interval \( I \) for which the corresponding operator function is Gâteaux differentiable on the segment \([C, D] : \{(1 - t)C + tD, \, t \in [0, 1]\}\) for \( C, D \) selfadjoint operators with spectra in \( I \). We consider the auxiliary function defined on \([0, 1]\) by

\[
f_{C,D}(t) := f((1 - t)C + tD), \quad t \in [0, 1].
\]

Then we have, by the properties of the Bochner integral, that

\[
f(D) - f(C) = \int_0^1 \frac{d}{dt}(f_{C,D}(t)) \, dt = \int_0^1 \nabla f_{(1-\ell)C+tD} (D-C) \, dt.
\]

If we write this equality for the function \( f(t) = -t^{-1} \) and \( C, D > 0 \), then we get the representation

\[
C^{-1} - D^{-1} = \int_0^1 ((1 - t)C + tD)^{-1} (D - C) ((1 - t)C + tD)^{-1} \, dt.
\]

Now, if we take in (2.6) \( C = \lambda + A, \, D = \lambda + B \), then

\[
(\lambda + A)^{-1} - (\lambda + B)^{-1}
\]

\[
= \int_0^1 ((1 - t) (\lambda + A) + t (\lambda + B))^{-1} (B - A)
\]

\[
\times ((1 - t) (\lambda + A) + t (\lambda + B))^{-1} \, dt
\]

\[
= \int_0^1 (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} \, dt.
\]

By employing (2.3) and (2.7), we derive

\[
\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A)
\]

\[
= \int_0^\infty \lambda w(\lambda)
\]

\[
\times \left( \int_0^1 (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} \, dt \right) d\mu(\lambda).
\]

for all \( A, B > 0 \).

If we multiply at the right with \( B - A \), then we get

\[
[\mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A)](B - A)
\]

\[
= \int_0^\infty \lambda w(\lambda) \left( \int_0^1 (\lambda + (1 - t) A + tB)^{-1} (B - A)
\]

\[
\times (\lambda + (1 - t) A + tB)^{-1} (B - A) \, dt \right) d\mu(\lambda)
\]

\[
= \int_0^\infty \lambda w(\lambda) \left( \int_0^1 \left[ (\lambda + (1 - t) A + tB)^{-1} (B - A) \right]^2 \, dt \right) d\mu(\lambda),
\]

which proves (2.1).
The identity and inequality in (2.2) follow in a similar way and we omit the
details.

The case of operator monotone function is as follows:

**Corollary 1.** Assume that the function $f : (0, \infty) \rightarrow \mathbb{R}$ is operator monotone
in $(0, \infty)$ and it has the representation (1.9), then for all $A, B > 0$ we have the
representation

\begin{equation}
[ f(B) - f(A)] (B - A) - b (B - A)^2
= \int_0^\infty \lambda^2 \left( \int_0^1 \left[ (\lambda + (1 - t) A + tB)^{-1} (B - A) \right]^2 dt \right) \, d\mu(\lambda)
\end{equation}

and

\begin{equation}
(B - A) [ f(B) - f(A)] - b (B - A)^2
= \int_0^\infty \lambda^2 \left( \int_0^1 \left[ (B - A) (\lambda + (1 - t) A + tB)^{-1} \right]^2 dt \right) \, d\mu(\lambda).
\end{equation}

**Proof.** From (1.9) we have for $T > 0$ that

\[ f(T) - a - bT = \mathcal{M}(\ell, \mu)(T), \]

for some positive measure $\mu$, where $\ell(\lambda) = \lambda, \lambda \geq 0$. Therefore

\[ \mathcal{M}(w, \mu)(B) - \mathcal{M}(w, \mu)(A) = f(B) - f(A) - b(B - A) \]

and by (2.1) we get (2.10).

The case of operator convex functions is as follows:

**Corollary 2.** Assume that the function $f : [0, \infty) \rightarrow \mathbb{R}$ is operator convex in
$[0, \infty)$ and it has the representation (1.11), then for all $A, B > 0$ we have the
representation

\begin{equation}
f(B) + f(A) - f(B) B^{-1} A - f(A) A^{-1} B
= \int_0^\infty \lambda^2 \left( \int_0^1 \left[ (\lambda + (1 - t) A + tB)^{-1} (B - A) \right]^2 dt \right) \, d\mu(\lambda)
\end{equation}

and

\begin{equation}
f(B) + f(A) - A f(B) B^{-1} - B f(A) A^{-1}
= \int_0^\infty \lambda^2 \left( \int_0^1 \left[ (\lambda + (1 - t) A + tB) (B - A) \right]^{-1} \left( B - A \right) \right]^2 dt \, d\mu(\lambda).
\end{equation}

If $f(0) = 0$, then

\begin{equation}
f(B) + f(A) - f(B) B^{-1} A - f(A) A^{-1} B - c (B - A)^2
= \int_0^\infty \lambda^2 \left( \int_0^1 \left[ (\lambda + (1 - t) A + tB) (B - A) \right]^{-1} \left( B^{-1} A + tB \right) \right)^2 dt \, d\mu(\lambda)
\end{equation}
and

\begin{equation}
(2.15) \quad f(B) + f(A) - Af(B) B^{-1} - Bf(A) A^{-1} - c(B - A)^2
\end{equation}

\begin{equation}
= \int_{0}^{\infty} \lambda^2 \left( \int_{0}^{1} \left[ (\lambda + (1 - t) A + tB)^{-1} (B - A) \right]^2 dt \right) d\mu(\lambda).
\end{equation}

**Proof.** From (1.11) we have for \( T > 0 \) that

\((f(T) - f(0)) T^{-1} - b - cT = \mathcal{M}(\ell, \mu) (T)\),

for some positive measure \( \mu \). Therefore

\[ \mathcal{M}(\ell, \mu) (B) - \mathcal{M}(\ell, \mu) (A) = (f(B) - f(0)) B^{-1} - (f(A) - f(0)) A^{-1} - c(B - A) \]

and by (2.1) we get (2.12). \( \square \)

The case of logarithmic function is of interest and can be derived as follows:

**Proposition 1.** For all \( A, B > 0 \) we have

\begin{equation}
(2.16) \quad (\ln B - \ln A) (B - A)
\end{equation}

\begin{equation}
= \int_{0}^{\infty} \left( \int_{0}^{1} \left[ (\lambda + (1 - t) A + tB)^{-1} (B - A) \right]^2 dt \right) d\lambda.
\end{equation}

and

\begin{equation}
(2.17) \quad (B - A) (\ln B - \ln A)
\end{equation}

\begin{equation}
= \int_{0}^{\infty} \left( \int_{0}^{1} \left[ (B - A) (\lambda + (1 - t) A + tB)^{-1} \right]^2 dt \right) d\lambda.
\end{equation}

**Proof.** We have from (1.17) for \( A, B > 0 \) that

\begin{equation}
(2.18) \quad \ln B - \ln A = \int_{0}^{\infty} \frac{1}{\lambda + 1} \left[ (B - 1) (\lambda + B)^{-1} - (A - 1) (\lambda + A)^{-1} \right] d\lambda.
\end{equation}

Since

\[ (B - 1) (\lambda + B)^{-1} - (A - 1) (\lambda + A)^{-1} \]

\[ = B (\lambda + B)^{-1} - A (\lambda + A)^{-1} - (\lambda + B)^{-1} - (\lambda + A)^{-1} \]

and

\[ B (\lambda + B)^{-1} - A (\lambda + A)^{-1} \]

\[ = (B + \lambda - \lambda) (\lambda + B)^{-1} - (A + \lambda - \lambda) (\lambda + A)^{-1} \]

\[ = 1 - \lambda (\lambda + B)^{-1} - 1 + \lambda (\lambda + A)^{-1} = \lambda (\lambda + A)^{-1} - \lambda (\lambda + B)^{-1}, \]

hence

\[ (B - 1) (\lambda + B)^{-1} - (A - 1) (\lambda + A)^{-1} \]

\[ = \lambda (\lambda + A)^{-1} - \lambda (\lambda + B)^{-1} - (\lambda + B)^{-1} - (\lambda + A)^{-1} \]

\[ = (\lambda + 1) \left[ (\lambda + A)^{-1} - (\lambda + B)^{-1} \right] \]

and by (2.18) we get

\begin{equation}
(2.19) \quad \ln B - \ln A = \int_{0}^{\infty} \left[ (\lambda + A)^{-1} - (\lambda + B)^{-1} \right] d\lambda.
\end{equation}
Since, by (2.7) we have
\[(\lambda + A)^{-1} - (\lambda + B)^{-1} = \int_0^1 (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} \, dt,\]
for all \(\lambda \geq 0\), hence by (2.19) and (2.20) we get
\[\ln B - \ln A = \int_0^\infty \left( \int_0^1 (\lambda + (1 - t) A + tB)^{-1} (B - A) (\lambda + (1 - t) A + tB)^{-1} \, dt \right) d\lambda.\]
Finally, by multiplying (2.21) at right by \((B - A)\) we obtain (2.16). By multiplying (2.21) at left produces the identity (2.17). \(\square\)

3. SOME INEQUALITIES

In the following, in order to simplify terminology, when we write \(T \geq 0\) we automatically assume that the operator \(T\) is selfadjoint.

We need the following lemmas:

**Lemma 1.** Let \(A, B > 0\). The following statements are equivalent:

(i) For all \(s \geq 0\),
\[(A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) \geq 2.\]

(ii) For all \(s \geq 0\),
\[\int_0^1 \left[ ((1 - t) A + tB + s)^{-1} (B - A) \right]^2 dt \geq 0.\]

(iii) For all \(s \geq 0\),
\[(\ell_s(B) - \ell_s(A)) (B - A) \geq 0,\]
where \(\ell_s(t) = -(t + s)^{-1}, \quad t > 0.\)

**Proof.** From (2.7) we have, by multiplying at right with \((B - A)\) that
\[
\left[ (A + s)^{-1} - (B + s)^{-1} \right] (B - A) = \int_0^1 ((1 - t) A + tB + s)^{-1} (B - A) ((1 - t) A + tB + s)^{-1} (B - A) \, dt
\]
\[= \int_0^1 \left[ ((1 - t) A + tB + s)^{-1} (B - A) \right]^2 dt\]
for all \(s \geq 0\).

Also
\[
\left[ (A + s)^{-1} - (B + s)^{-1} \right] (B - A) = \left[ (A + s)^{-1} - (B + s)^{-1} \right] [B + s - (A + s)]
\]
\[= (A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) - 2\]
for all \(s \geq 0.\)
Therefore
\begin{equation}
(\ell_s(B) - \ell_s(A))(B - A)
= (A + s)^{-1}(B + s) + (B + s)^{-1}(A + s) - 2
= \int_0^1 \left[ ((1 - t)A + tB + s)^{-1}(B - A) \right]^2 dt
\end{equation}
for all \( s \geq 0 \).

The identity (3.2) reveals that the statements (i), (ii) and (iii) are equivalent. \( \square \)

In the recent note [3] Fujii and Nakamoto proved the following inequality:

**Lemma 2.** If \( C, D > 0 \) and \( CD^{-1} + DC^{-1} \) is selfadjoint, then
\begin{equation}
CD^{-1} + DC^{-1} \geq 2.
\end{equation}

**Proof.** Indeed, as shown in [3], if we put \( T = CD^{-1} \), then \( V = T + T^{-1} \) is selfadjoint by the assumption. Note that the spectrum \( \text{Sp}(T) \) of \( T \) is included in \((0, \infty)\), because \( C, D > 0 \) and \( \text{Sp}(T) = \text{Sp}(C^{1/2}D^{-1}C^{1/2}) \). Since \( \text{Sp}(V) = \{ t + \frac{1}{t}, t \in \text{Sp}(T) \} \) by the spectral mapping theorem for rational functions, hence we have \( T + T^{-1} \geq 2 \). \( \square \)

As a consequence, they proved that, if \( (i') \ Operator A(B + s)^{-1} + B(A + s)^{-1} \) is selfadjoint for all \( s \geq 0 \), then
\((B - A)(f(B) - f(A)) \geq 0\)
for all \( f \) operator monotone functions on \((0, \infty)\).

**Lemma 3.** Let \( A, B > 0 \), then the statements (i) and (i') are equivalent.

**Proof.** Notice that for all \( s \geq 0 \),
\begin{equation}
(A + s)^{-1}(B + s) + (B + s)^{-1}(A + s)
= (A + s)^{-1}B + (B + s)^{-1}A + s(A + s)^{-1} + s(B + s)^{-1}.
\end{equation}
Also, the operator \( s(A + s)^{-1} + s(B + s)^{-1} \) is selfadjoint for \( s \geq 0 \).

If the statement (i) holds, then \( (A + s)^{-1}(B + s) + (B + s)^{-1}(A + s) \) is selfadjoint and by (3.4) we must have that \( (A + s)^{-1}B + (B + s)^{-1}A \) is selfadjoint, which shows that
\((A + s)^{-1}B + (B + s)^{-1}A)^* = B(A + s)^{-1} + A(B + s)^{-1}\)
is selfadjoint, namely (i') is true.

If the statement (i') holds, then by (3.4) we get
\((A + s)^{-1}(B + s) + (B + s)^{-1}(A + s)\)
is selfadjoint and by (3.3) for \( C = (A + s)^{-1}, D = (B + s)^{-1} \) we obtain the inequality (3.1), namely (i) is true. \( \square \)

We define the class of operators
\[ \mathfrak{C}_{(0, \infty)}(H) := \{ (A, B) \mid A, B > 0 \text{ and satisfy condition } (i') \} . \]
We observe that if \( (A, B) \in \mathfrak{C}_{(0, \infty)}(H) \) then \( (B, A) \in \mathfrak{C}_{(0, \infty)}(H) \).
Also if $AB = BA$, $A$, $B > 0$, then
$$U_s := (A + s)^{-1} (B + s)$$
and
$$U_s^{-1} = (B + s)^{-1} (A + s)$$
are selfadjoint and since $U_s + U_s^{-1} \geq 2$, $s \geq 0$ we derive that
\((A, B) \in C_{(0, \infty)}(H)\). Therefore, if $C_{(0, \infty)}(H)$ is the class of all pairs of commutative operators $A$, $B > 0$, then we have
\begin{equation}
0 \neq C_{(0, \infty)}(H) \subset C_{(0, \infty)}(H).
\end{equation}

We have:

**Theorem 4.** Let $(A, B) \in C_{(0, \infty)}(H)$. Then
\begin{equation}
[M(w, \mu)(B) - M(w, \mu)(A)](B - A) = (B - A)[M(w, \mu)(B) - M(w, \mu)(A)] \geq 0
\end{equation}
and
\begin{equation}
AM(w, \mu)(A) + BM(w, \mu)(B) \geq M(w, \mu)(A)B + M(w, \mu)(B)A.
\end{equation}

**Proof.** By (ii) from Lemma 1 we have
\begin{equation}
f(A) \leq (B + tA)^{-1} \leq 2(1 - t)B + tA
\end{equation}
and by (2.7) we obtain
\begin{equation}
\int_0^\infty \lambda w(\lambda) \left( \int_0^1 \left[ (\lambda + (1 - t)B + tA)^{-1} (B - A) \right]^2 dt \right) d\mu(\lambda) \geq 0
\end{equation}
and by (2.7) we obtain
\begin{equation}
[M(w, \mu)(B) - M(w, \mu)(A)](B - A) \geq 0.
\end{equation}
Since $L := [M(w, \mu)(B) - M(w, \mu)(A)](B - A)$ is selfadjoint then
\begin{equation}
L^* = \left\{ [M(w, \mu)(B) - M(w, \mu)(A)](B - A) \right\}^*
\end{equation}
\begin{equation}
= (B - A)^* [M(w, \mu)(B) - M(w, \mu)(A)]^*
\end{equation}
\begin{equation}
= (B - A)[M(w, \mu)(B) - M(w, \mu)(A)] = L
\end{equation}
and the theorem is proved. \(\Box\)

**Corollary 3.** Assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and it has the representation $(1.9)$, then for all $(A, B) \in C_{(0, \infty)}(H)$
\begin{equation}
f(B)B + Af(A) - f(B)A - f(A)B \geq b (B - A)^2 \geq 0.
\end{equation}

If $(A, B) \in C_{(0, \infty)}(H)$ and $r \in (0, 1)$, then
\begin{equation}
B^{r+1} + A^{r+1} \geq B^r A + A^r B
\end{equation}
and
\begin{equation}
B \ln B + A \ln A \geq A \ln B + B \ln A.
\end{equation}

**Corollary 4.** Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ and it has the representation $(1.11)$, then for all $(A, B) \in C_{(0, \infty)}(H)$
\begin{equation}
f(B) + f(A) - f(B) - f(A) \geq c (B - A)^2 \geq 0.
\end{equation}

If $(A, B) \in C_{(0, \infty)}(H)$, then by (3.8) for $f(t) = -\ln(t + 1)$,
\begin{equation}
\ln(B + 1) B^{\ln A + \ln(A + 1)} A^{-1} B \geq \ln(B + 1) + \ln(A + 1).
\end{equation}

**Acknowledgement.** The author would like to thank Professors M. Fujii and R. Nakamoto for the private note [3].
References


[6] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function $f$ on $[0, \infty)$. J. Math. Inequal. 9 (2015), no. 1, 47–52.


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