SYNCHRONICITY PROPERTIES FOR MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following mapping that we call the *monotonic integral transform*

$$\mathcal{M}(w,\mu)(T) := \int_0^\infty w(\lambda)\, T(\lambda + T)^{-1}\, d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. In this paper, we show among others that we have the representation

$$\begin{aligned}&[\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)](B-A)\\ &= \int_0^\infty \lambda w(\lambda)\left(\int_0^1 \left[(\lambda + (1-t)A + tB)^{-1}(B-A)\right]^2 dt\right) d\mu(\lambda)\end{aligned}$$

for all A, $B > 0$. We also provide some sufficient conditions for the operators A, $B > 0$ such that the inequality

$$A\mathcal{M}(w,\mu)(A) + B\mathcal{M}(w,\mu)(B) \geq \mathcal{M}(w,\mu)(A)\,B + \mathcal{M}(w,\mu)(B)\,A$$

holds. Some examples for operator monotonic and operator convex functions are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda + t}\, d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t-1} + \frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator monotone functions, Operator convex functions, Operator inequalities, Löwner-Heinz inequality. Logarithmic operator inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 83, 11 pp. Received 09/07/20
Revised 20/08/20

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce [2], for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, then we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.9}$$

where $a\in R$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.9).

For examples of operator monotone functions, see [4], [5], [9], [10] and the references therein.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left(\left(1-\lambda\right)A+\lambda B\right)\leq\left(\geq\right)\left(1-\lambda\right)f\left(A\right)+\lambda f\left(B\right) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f\left(t\right)=a+bt+ct^{2}+\int_{0}^{\infty}\frac{t^{2}\lambda}{t+\lambda}d\mu\left(\lambda\right), \tag{1.11}$$

where a, $b\in\mathbb{R}$, $c\geq0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f\left(0\right)$ and $b=f_{+}^{\prime}\left(0\right)$, the right derivative, in (1.11).

For a continuous and positive function $w\left(\lambda\right)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w,\mu)\left(t\right):=t\mathcal{D}(w,\mu)\left(t\right),\ t>0. \tag{1.12}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{M}(w,\mu)\left(t\right)&:=t\mathcal{D}(w,\mu)\left(t\right)=\int_{0}^{\infty}w\left(\lambda\right)t\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right)\\&=\int_{0}^{\infty}w\left(\lambda\right)\left(t+\lambda-\lambda\right)\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right)\\&=\int_{0}^{\infty}w\left(\lambda\right)\left[1-\lambda\left(t+\lambda\right)^{-1}\right]d\mu\left(\lambda\right).\end{aligned} \tag{1.13}$$

If $\int_{0}^{\infty}w\left(\lambda\right)d\mu\left(\lambda\right)<\infty$, then

$$\mathcal{M}(w,\mu)\left(t\right)=\int_{0}^{\infty}w\left(\lambda\right)d\mu\left(\lambda\right)-\mathcal{D}(\ell w,\mu)\left(t\right), \tag{1.14}$$

where $\ell\left(t\right)=t$, $t>0$.

Consider the kernel $e_{-a}\left(\lambda\right):=\exp\left(-a\lambda\right)$, $\lambda\geq0$ and $a>0$. Then after some calculations, we get

$$\mathcal{D}(e_{-a})\left(t\right)=\int_{0}^{\infty}\frac{\exp\left(-a\lambda\right)}{t+\lambda}d\lambda=E_{1}\left(at\right)\exp\left(at\right),\ t\geq0$$

and

$$\int_{0}^{\infty}w\left(\lambda\right)d\lambda=\int_{0}^{\infty}\exp\left(-a\lambda\right)d\lambda=\frac{1}{a}.$$

This gives that

$$\mathcal{M}(e_{-a})\left(t\right)=t\mathcal{D}(w,\mu)\left(t\right)=tE_{1}\left(at\right)\exp\left(at\right),\ t\geq0,$$

where

$$E_{1}\left(t\right):=\int_{t}^{\infty}\frac{e^{-u}}{u}du.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t)=\int_0^\infty \frac{\lambda\exp(-a\lambda)}{t+\lambda}d\lambda=\frac{1}{a}-tE_1(at)\exp(at)$$

for $t>0$.

One observes that

$$\mathcal{M}(e_{-a})(t)=\int_0^\infty w(\lambda)\,d\lambda-\mathcal{D}(\ell e_{-a},\mu)(t),\ t>0$$

and the equality (1.14) is verified in this case.

If we take $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda=\infty$ and the equality (1.14) does not hold in this case.

For all $T>0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T)=T\mathcal{D}(w,\mu)(T)=\int_0^\infty w(\lambda)\left[1-\lambda(T+\lambda)^{-1}\right]d\mu(\lambda). \tag{1.15}$$

This gives the representation

$$T^r=\frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T), \tag{1.16}$$

where $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$ and μ is the usual Lebesgue norm. Also, from (1.6) we have the representation

$$\ln T=(T-1)\,\mathcal{D}(w_{\ln})(T),\ T>0, \tag{1.17}$$

where $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$.

2. Some Identities

We have the following representation result:

Theorem 3. *For all A, $B>0$ we have the representations*

$$\begin{aligned}&[\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)](B-A)\\&=\int_0^\infty \lambda w(\lambda)\left(\int_0^1\left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2dt\right)d\mu(\lambda)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}&(B-A)[\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)]\\&=\int_0^\infty \lambda w(\lambda)\left(\int_0^1\left[(B-A)(\lambda+(1-t)A+tB)^{-1}\right]^2dt\right)d\mu(\lambda).\end{aligned} \tag{2.2}$$

Proof. From (1.15) we have for all A, $B\geq 0$ that

$$\begin{aligned}&\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\&=\int_0^\infty w(\lambda)\left[1-\lambda(B+\lambda)^{-1}\right]d\mu(\lambda)-\int_0^\infty w(\lambda)\left[1-\lambda(A+\lambda)^{-1}\right]d\mu(\lambda)\\&=\int_0^\infty \lambda w(\lambda)\left[(A+\lambda)^{-1}-(B+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned} \tag{2.3}$$

Let $T,\ S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{2.4}$$

for $T,\ S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Now, if we take in (2.6) $C=\lambda+A$, $D=\lambda+B$, then

$$\begin{aligned}&(\lambda+A)^{-1}-(\lambda+B)^{-1}\\&=\int_0^1((1-t)(\lambda+A)+t(\lambda+B))^{-1}(B-A)\\&\times((1-t)(\lambda+A)+t(\lambda+B))^{-1}\,dt\\&=\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,dt.\end{aligned} \tag{2.7}$$

By employing (2.3) and (2.7), we derive

$$\begin{aligned}&\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\\&=\int_0^\infty\lambda w(\lambda)\\&\times\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,dt\right)d\mu(\lambda).\end{aligned} \tag{2.8}$$

for all $A,B>0$.

If we multiply at the right with $B-A$, then we get

$$\begin{aligned}&[\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)](B-A)\\&=\int_0^\infty\lambda w(\lambda)\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\\&\left.\times(\lambda+(1-t)A+tB)^{-1}(B-A)\,dt\right)d\mu(\lambda)\\&=\int_0^\infty\lambda w(\lambda)\left(\int_0^1\left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2dt\right)d\mu(\lambda),\end{aligned} \tag{2.9}$$

which proves (2.1).

The identity and inequality in (2.2) follow in a similar way and we omit the details. □

The case of operator monotone function is as follows:

Corollary 1. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and it has the representation (1.9), then for all* A, $B>0$ *we have the representation*

$$[f(B)-f(A)](B-A)-b(B-A)^2 = \int_0^\infty \lambda^2\left(\int_0^1 \left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2 dt\right)d\mu(\lambda) \tag{2.10}$$

and

$$(B-A)[f(B)-f(A)]-b(B-A)^2 = \int_0^\infty \lambda^2\left(\int_0^1 \left[(B-A)(\lambda+(1-t)A+tB)^{-1}\right]^2 dt\right)d\mu(\lambda). \tag{2.11}$$

Proof. From (1.9) we have for $T>0$ that

$$f(T)-a-bT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$. Therefore

$$\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)=f(B)-f(A)-b(B-A)$$

and by (2.1) we get (2.10). □

The case of operator convex functions is as follows:

Corollary 2. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator convex in* $[0,\infty)$ *and it has the representation (1.11), then for all* A, $B>0$ *we have the representation*

$$\begin{aligned} & f(B)+f(A)-f(B)B^{-1}A-f(A)A^{-1}B \\ & -f(0)\left(2-A^{-1}B-B^{-1}A\right)-c(B-A)^2 \\ & =\int_0^\infty \lambda^2\left(\int_0^1 \left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2 dt\right)d\mu(\lambda) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} & f(B)+f(A)-Af(B)B^{-1}-Bf(A)A^{-1} \\ & -f(0)\left(2-BA^{-1}-AB^{-1}\right)-c(B-A)^2 \\ & =\int_0^\infty \lambda^2\left(\int_0^1 \left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2 dt\right)d\mu(\lambda). \end{aligned} \tag{2.13}$$

If $f(0)=0$, *then*

$$\begin{aligned} & f(B)+f(A)-f(B)B^{-1}A-f(A)A^{-1}B-c(B-A)^2 \\ & =\int_0^\infty \lambda^2\left(\int_0^1 \left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2 dt\right)d\mu(\lambda) \end{aligned} \tag{2.14}$$

and

$$f(B)+f(A)-Af(B)B^{-1}-Bf(A)A^{-1}-c(B-A)^2 = \int_0^\infty \lambda^2\left(\int_0^1 \left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2 dt\right)d\mu(\lambda). \tag{2.15}$$

Proof. From (1.11) we have for $T>0$ that

$$(f(T)-f(0))T^{-1}-b-cT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ. Therefore

$$\mathcal{M}(\ell,\mu)(B)-\mathcal{M}(\ell,\mu)(A)=(f(B)-f(0))B^{-1}-(f(A)-f(0))A^{-1}-c(B-A)$$

and by (2.1) we get (2.12). □

The case of logarithmic function is of interest and can be derived as follows:

Proposition 1. *For all A, $B>0$ we have*

$$(\ln B-\ln A)(B-A) = \int_0^\infty \left(\int_0^1 \left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2 dt\right)d\lambda \tag{2.16}$$

and

$$(B-A)(\ln B-\ln A) = \int_0^\infty \left(\int_0^1 \left[(B-A)(\lambda+(1-t)A+tB)^{-1}\right]^2 dt\right)d\lambda. \tag{2.17}$$

Proof. We have from (1.17) for A, $B>0$ that

$$\ln B-\ln A=\int_0^\infty \frac{1}{\lambda+1}\left[(B-1)(\lambda+B)^{-1}-(A-1)(\lambda+A)^{-1}\right]d\lambda. \tag{2.18}$$

Since

$$\begin{aligned}&(B-1)(\lambda+B)^{-1}-(A-1)(\lambda+A)^{-1}\\&=B(\lambda+B)^{-1}-A(\lambda+A)^{-1}-\left((\lambda+B)^{-1}-(\lambda+A)^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}&B(\lambda+B)^{-1}-A(\lambda+A)^{-1}\\&=(B+\lambda-\lambda)(\lambda+B)^{-1}-(A+\lambda-\lambda)(\lambda+A)^{-1}\\&=1-\lambda(\lambda+B)^{-1}-1+\lambda(\lambda+A)^{-1}=\lambda(\lambda+A)^{-1}-\lambda(\lambda+B)^{-1},\end{aligned}$$

hence

$$\begin{aligned}&(B-1)(\lambda+B)^{-1}-(A-1)(\lambda+A)^{-1}\\&=\lambda(\lambda+A)^{-1}-\lambda(\lambda+B)^{-1}-\left((\lambda+B)^{-1}-(\lambda+A)^{-1}\right)\\&=(\lambda+1)\left[(\lambda+A)^{-1}-(\lambda+B)^{-1}\right]\end{aligned}$$

and by (2.18) we get

$$\ln B-\ln A=\int_0^\infty \left[(\lambda+A)^{-1}-(\lambda+B)^{-1}\right]d\lambda. \tag{2.19}$$

Since, by (2.7) we have

$$(\lambda + A)^{-1} - (\lambda + B)^{-1} = \int_0^1 (\lambda + (1-t)A + tB)^{-1} (B - A) (\lambda + (1-t)A + tB)^{-1} dt, \tag{2.20}$$

for all $\lambda \geq 0$, hence by (2.19) and (2.20) we get

$$\ln B - \ln A = \int_0^\infty \left(\int_0^1 (\lambda + (1-t)A + tB)^{-1} (B - A) (\lambda + (1-t)A + tB)^{-1} dt \right) d\lambda. \tag{2.21}$$

Finally, by multiplying (2.21) at right by $(B - A)$ we obtain (2.16). By multiplying (2.21) at left produces the identity (2.17). □

3. Some Inequalities

In the following, in order to simplify terminology, when we write $T \geq 0$ we automatically assume that the operator T is selfadjoint.

We need the following lemmas:

Lemma 1. *Let A, $B > 0$. The following statements are equivalent:*

(i) *For all $s \geq 0$,*

$$(A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) \geq 2. \tag{3.1}$$

(ii) *For all $s \geq 0$,*

$$\int_0^1 \left[((1-t)A + tB + s)^{-1} (B - A) \right]^2 dt \geq 0.$$

(iii) *For all $s \geq 0$,*

$$(\ell_s(B) - \ell_s(A))(B - A) \geq 0,$$

where $\ell_s(t) = -(t + s)^{-1}$, $t > 0$.

Proof. From (2.7) we have, by multiplying at right with $B - A$ that

$$\begin{aligned} &\left[(A + s)^{-1} - (B + s)^{-1} \right] (B - A) \\ &= \int_0^1 ((1-t)A + tB + s)^{-1} (B - A) ((1-t)A + tB + s)^{-1} (B - A) dt \\ &= \int_0^1 \left[((1-t)A + tB + s)^{-1} (B - A) \right]^2 dt \end{aligned}$$

for all $s \geq 0$.

Also

$$\begin{aligned} &\left[(A + s)^{-1} - (B + s)^{-1} \right] (B - A) \\ &= \left[(A + s)^{-1} - (B + s)^{-1} \right] [B + s - (A + s)] \\ &= (A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) - 2 \end{aligned}$$

for all $s \geq 0$.

Therefore

$$
\begin{aligned}
(3.2)\qquad & (\ell_s(B) - \ell_s(A))(B - A) \\
&= (A+s)^{-1}(B+s) + (B+s)^{-1}(A+s) - 2 \\
&= \int_0^1 \left[((1-t)A + tB + s)^{-1}(B-A)\right]^2 dt
\end{aligned}
$$

for all $s \geq 0$.

The identity (3.2) reveals that the statements (i), (ii) and (iii) are equivalent. □

In the recent note [3] Fujii and Nakamoto proved the following inequality:

Lemma 2. *If $C,\ D > 0$ and $CD^{-1} + DC^{-1}$ is selfadjoint, then*

$$
CD^{-1} + DC^{-1} \geq 2. \tag{3.3}
$$

Proof. Indeed, as shown in [3], if we put $T = CD^{-1}$, then $V = T + T^{-1}$ is selfadjoint by the assumption. Note that the spectrum $\operatorname{Sp}(T)$ of T is included in $(0, \infty)$, because $C,\ D > 0$ and $\operatorname{Sp}(T) = \operatorname{Sp}\left(C^{1/2}D^{-1}C^{1/2}\right)$. Since $\operatorname{Sp}(V) = \left\{t + \frac{1}{t},\ t \in \operatorname{Sp}(T)\right\}$ by the spectral mapping theorem for rational functions, hence we have $T + T^{-1} \geq 2$. □

As a consequence, they proved that, if

(i') Operator $A(B+s)^{-1} + B(A+s)^{-1}$ is selfadjoint for all $s \geq 0$,

then

$$
(B - A)(f(B) - f(A)) \geq 0
$$

for all f operator monotone functions on $(0, \infty)$.

Lemma 3. *Let $A,\ B > 0$, then the statements (i) and (i') are equivalent.*

Proof. Notice that for all $s \geq 0$,

$$
\begin{aligned}
(3.4)\qquad & (A+s)^{-1}(B+s) + (B+s)^{-1}(A+s) \\
&= (A+s)^{-1}B + (B+s)^{-1}A + s(A+s)^{-1} + s(B+s)^{-1}.
\end{aligned}
$$

Also, the operator $s(A+s)^{-1} + s(B+s)^{-1}$ is selfadjoint for $s \geq 0$.

If the statement *(i)* holds, then $(A+s)^{-1}(B+s) + (B+s)^{-1}(A+s)$ is selfadjoint and by (3.4) we must have that $(A+s)^{-1}B + (B+s)^{-1}A$ is selfadjoint, which shows that

$$
\left((A+s)^{-1}B + (B+s)^{-1}A\right)^* = B(A+s)^{-1} + A(B+s)^{-1}
$$

is selfadjoint, namely *(i')* is true.

If the statement *(i')* holds, then by (3.4) we get

$$
(A+s)^{-1}(B+s) + (B+s)^{-1}(A+s)
$$

is selfadjoint and by (3.3) for $C = (A+s)^{-1}$, $D = (B+s)^{-1}$ we obtain the inequality (3.1), namely *(i)* is true. □

We define the class of operators

$$
\mathfrak{Cl}_{(0,\infty)}(H) := \{(A, B) \mid A,\ B > 0 \text{ and satisfy condition } (i')\}.
$$

We observe that if $(A, B) \in \mathfrak{Cl}_{(0,\infty)}(H)$ then $(B, A) \in \mathfrak{Cl}_{(0,\infty)}(H)$.

Also if $AB = BA$, $A, B > 0$, then $U_s := (A+s)^{-1}(B+s)$ and $U_s^{-1} = (B+s)^{-1}(A+s)$ are selfadjoint and since $U_s + U_s^{-1} \geq 2$, $s \geq 0$ we derive that $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$. Therefore, if $\mathfrak{Co}_{(0,\infty)}(H)$ is the class of all pairs of commutative operators $A, B > 0$, then we have

$$\emptyset \neq \mathfrak{Co}_{(0,\infty)}(H) \subset \mathfrak{Cl}_{(0,\infty)}(H). \tag{3.5}$$

We have:

Theorem 4. *Let* $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$. *Then*

$$[\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)](B-A) = (B-A)[\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)] \geq 0$$

and

$$A\mathcal{M}(w,\mu)(A) + B\mathcal{M}(w,\mu)(B) \geq \mathcal{M}(w,\mu)(A)B + \mathcal{M}(w,\mu)(B)A. \tag{3.6}$$

Proof. By (ii) from Lemma 1 we have

$$\int_0^\infty \lambda w(\lambda)\left(\int_0^1 \left[(\lambda + (1-t)B + tA)^{-1}(B-A)\right]^2 dt\right) d\mu(\lambda) \geq 0$$

and by (2.7) we obtain

$$[\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)](B-A) \geq 0.$$

Since $L := [\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)](B-A)$ is selfadjoint then

$$\begin{aligned} L^* &= \{[\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)](B-A)\}^* \\ &= (B-A)^*[\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)]^* \\ &= (B-A)[\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)] = L \end{aligned}$$

and the theorem is proved. □

Corollary 3. *Assume that the function* $f : (0,\infty) \to \mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and it has the representation (1.9), then for all* $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$

$$f(B)B + Af(A) - f(B)A - f(A)B \geq b(B-A)^2 \geq 0. \tag{3.7}$$

If $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$ and $r \in (0,1]$, then

$$B^{r+1} + A^{r+1} \geq B^r A + A^r B$$

and

$$B\ln B + A\ln A \geq A\ln B + B\ln A.$$

Corollary 4. *Assume that the function* $f : [0,\infty) \to \mathbb{R}$ *is operator convex in* $[0,\infty)$ *and it has the representation (1.11), then for all* $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$

$$\begin{aligned} &f(B) + f(A) - f(B)B^{-1}A - f(A)A^{-1}B - f(0)\left(2 - A^{-1}B - B^{-1}A\right) \\ &\geq c(B-A)^2 \geq 0. \end{aligned} \tag{3.8}$$

If $(A,B) \in \mathfrak{Cl}_{(0,\infty)}(H)$, then by (3.8) for $f(t) = -\ln(t+1)$,

$$\ln(B+1)B^{-1}A + \ln(A+1)A^{-1}B \geq \ln(B+1) + \ln(A+1).$$

Acknowledgement. The author would like to thank Professors M. Fujii and R. Nakamoto for the private note [3].

References

[1] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] S. S. Dragomir, Simple operator asynchronicity of an integral transform with applications, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 71, 12 pp. [Online https://rgmia.org/papers/v23/v23a71.pdf].

[3] M. Fujii and R. Nakamoto, Note on Dragomir's theorems, *Private note* 5 August 2020. Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 113, 4 pp.

[4] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[5] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.

[6] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[7] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[8] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[9] M. S. Moslehian, H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.

[10] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

[11] Incomplete Gamma and Related Functions, Definitions, *Digital Library of Mathematical Functions*, NIST. [Online https://dlmf.nist.gov/8.2].

[12] Incomplete Gamma and Related Functions, Integral Representations, *Digital Library of Mathematical Functions,* NIST. [Online https://dlmf.nist.gov/8.6].

[13] Generalized Exponential Integral, *Digital Library of Mathematical Functions,* NIST. [Online https://dlmf.nist.gov/8.19#E1].

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.