REVERSES OF DAVIS-CHOI-JENSEN’S INEQUALITY FOR THE MONOTONIC INTEGRAL TRANSFORM

SILVESTRU SEVER DRAGOMIR¹,²

Abstract. Some applications for operator monotone and operator convex functions are also provided.

1. Introduction

Let $H$ be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on $H$. We denote by $\mathcal{B}_b(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on $H$ and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on $H$.

Let $H, K$ be complex Hilbert spaces. Following [2] (see also [6, p. 18]) we can introduce the following definition:

Definition 1. A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any $\lambda, \mu \in \mathbb{C}$ and $A, B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e., if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e., $\Phi(1) = 1$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map $\Phi$ preserves the order relation, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = (\Phi(A))^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1 \leq A \leq \beta 1$, then $\alpha \Phi(1) \leq \Phi(A) \leq \beta \Phi(1)$.

If the map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Phi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1) \Psi \Psi^{-1/2}(1)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function $f$ on an interval $I$ is said to be operator convex (concave) on $I$ if

$$f((1 - \lambda) A + \lambda B) \leq (\geq) \ (1 - \lambda) f(A) + \lambda f(B)$$

for all $\lambda \in [0, 1]$ and for every selfadjoint operators $A, B \in \mathcal{B}(H)$ whose spectra are contained in $I$.

The following Jensen’s type result is well known [2]:

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Theorem 1 (Davis-Choi-Jensen’s Inequality). Let \( f : I \to \mathbb{R} \) be an operator convex function on the interval \( I \) and \( \Phi \in \mathcal{F}_N [\mathcal{B}(H), \mathcal{B}(K)] \), then for any selfadjoint operator \( A \) whose spectrum is contained in \( I \) we have
\[
(1.1) \quad f(\Phi(A)) \leq \Phi(f(A)).
\]

We observe that if \( \Psi \in \mathcal{F}[\mathcal{B}(H), \mathcal{B}(K)] \) with \( \Psi(1) \in \mathcal{B}^{++}(K) \), then by taking \( \Phi = \Psi^{1/2}(1) \Psi^{-1/2}(1) \) in (1.1) we get
\[
f\left(\Psi^{-1/2}(1) \Psi(A) \Psi^{-1/2}(1)\right) \leq \Psi^{-1/2}(1) \Psi(f(A)) \Psi^{-1/2}(1).
\]

If we multiply both sides of this inequality by \( \Psi^{1/2}(1) \) we get the following Davis-Choi-Jensen’s inequality for general positive linear maps:
\[
(1.2) \quad \Psi^{1/2}(1) f\left(\Psi^{-1/2}(1) \Psi(A) \Psi^{-1/2}(1)\right) \Psi^{1/2}(1) \leq \Psi(f(A)).
\]

A real valued continuous function \( f \) on \((0, 1]\) is said to be operator monotone if \( f(A) \geq f(B) \) holds for any \( A \geq B > 0 \).

We have the following integral representation for the power function when \( t > 0 \), \( r \in (0, 1] \), see for instance [1, p. 145]
\[
(1.3) \quad t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^{\infty} \frac{\lambda^{r-1}}{\lambda + t} d\lambda.
\]

Observe that for \( t > 0, t \neq 1 \), we have
\[
\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln \left(\frac{u + t}{u + 1}\right)
\]
for all \( u > 0 \).

By taking the limit over \( u \to \infty \) in this equality, we derive
\[
\frac{\ln t}{t-1} = \int_0^{\infty} \frac{d\lambda}{(\lambda + t)(\lambda + 1)},
\]
which gives the representation for the logarithm
\[
(1.4) \quad \ln t = (t-1) \int_0^{\infty} \frac{d\lambda}{(\lambda + t)(\lambda + 1)}
\]
for all \( t > 0 \).

Motivated by these representations, we introduce, for a continuous and positive function \( w(\lambda), \lambda > 0 \), the following integral transform
\[
(1.5) \quad \mathcal{D}(w, \mu)(t) := \int_0^{\infty} \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0,
\]
where \( \mu \) is a positive measure on \((0, \infty)\) and the integral (1.5) exists for all \( t > 0 \).

For \( \mu \), the Lebesgue usual measure, we put
\[
(1.6) \quad \mathcal{D}(w)(t) := \int_0^{\infty} \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0.
\]

If we take \( \mu \) to be the usual Lebesgue measure and the kernel \( w_r(\lambda) = \lambda^{r-1} \), \( r \in (0, 1] \), then
\[
(1.7) \quad t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \ t > 0.
\]
For the same measure, if we take the kernel \( w_{in}(\lambda) = (\lambda + 1)^{-1} \), \( t > 0 \), we have the representation

\[
\ln t = (t - 1) D(w_{in})(t), \quad t > 0.
\]

Assume that \( T > 0 \), then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

\[
D(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),
\]

where \( w \) and \( \mu \) are as above. Also, when \( \mu \) is the usual Lebesgue measure, then

\[
D(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda,
\]

for \( T > 0 \).

A real valued continuous function \( f \) on \((0, \infty)\) is said to be operator monotone if \( f(A) \geq f(B) \) holds for any \( A \geq B > 0 \).

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

**Theorem 2.** A function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) if and only if it has the representation

\[
f(t) = a + bt + \int_0^\infty \frac{t \lambda}{t + \lambda} d\mu(\lambda),
\]

where \( a \in \mathbb{R}, b \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that

\[
\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty.
\]

If \( f \) is operator monotone in \([0, \infty)\), then \( a = f(0) \) in (1.11).

A real valued continuous function \( f \) on an interval \( I \) is said to be operator convex (operator concave) on \( I \) if

\[
f((1 - \lambda)A + \lambda B) \leq (\geq) (1 - \lambda)f(A) + \lambda f(B)
\]

in the operator order, for all \( \lambda \in [0, 1] \) and for every selfadjoint operator \( A \) and \( B \) on a Hilbert space \( H \) whose spectra are contained in \( I \). Notice that a function \( f \) is operator concave if \( -f \) is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

**Theorem 3.** A function \( f : (0, \infty) \to \mathbb{R} \) is operator convex in \((0, \infty)\) if and only if it has the representation

\[
f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t + \lambda} d\mu(\lambda),
\]

where \( a, b \in \mathbb{R}, c \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that (1.4) holds. If \( f \) is operator convex in \([0, \infty)\), then \( a = f(0) \) and \( b = f'_+(0) \), the right derivative, in (1.13).

For a continuous and positive function \( w(\lambda), \lambda > 0 \) and a positive measure \( \mu \) on \((0, \infty)\), we can define the following mapping, which we call monotonic integral transform, by

\[
\mathcal{M}(w, \mu)(t) := tD(w, \mu)(t), \quad t > 0.
\]
For $t > 0$ we have

$$
(1.15) \quad \mathcal{M}(w, \mu) (t) := tD(w, \mu) (t) = \int_0^\infty w(\lambda) t \lambda (t + \lambda)^{-1} d\mu(\lambda)
$$

$$
= \int_0^\infty w(\lambda) (t + \lambda - \lambda)(t + \lambda)^{-1} d\mu(\lambda)
$$

$$
= \int_0^\infty w(\lambda) \left[1 - \lambda (t + \lambda)^{-1}\right] d\mu(\lambda).
$$

If $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$, then

$$
(1.16) \quad \mathcal{M}(w, \mu) (t) = \int_0^\infty w(\lambda) d\mu(\lambda) - \mathcal{D}(t w, \mu) (t),
$$

where $\ell(t) = t, t > 0$.

Consider the kernel $e_{-a} (\lambda) := \exp (-a\lambda), \lambda \geq 0$ and $a > 0$. Then after some calculations, we get

$$
\mathcal{D}(e_{-a}) (t) = \int_0^\infty \exp (-a\lambda) \frac{t}{t + \lambda} d\lambda = E_1 (at) \exp (at), t \geq 0
$$

and

$$
\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp (-a\lambda) d\lambda = \frac{1}{a},
$$

where

$$
E_1 (t) := \int_t^\infty \frac{e^{-u}}{u} du.
$$

This gives that

$$
\mathcal{M}(e_{-a}) (t) = t \mathcal{D}(w, \mu) (t) = t E_1 (at) \exp (at), t \geq 0.
$$

By integration we also have

$$
\mathcal{D}(\ell e_{-a}, \mu) (t) = \int_0^\infty \lambda \exp (-a\lambda) \frac{t}{t + \lambda} d\lambda = \frac{1}{a} - t E_1 (at) \exp (at)
$$

for $t > 0$.

One observes that

$$
\mathcal{M}(e_{-a}) (t) = \int_0^\infty w(\lambda) d\lambda - \mathcal{D}(e_{-a}, \mu) (t), t > 0
$$

and the equality (1.16) is verified in this case.

If we take $w_r (\lambda) = \lambda^{-r}, r \in (0, 1]$, then $\int_0^\infty w_r (\lambda) d\lambda = \infty$ and the equality (1.16) does not hold in this case.

For all $T > 0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$
(1.17) \quad \mathcal{M}(w, \mu) (T) = T \mathcal{D}(w, \mu) (T) = \int_0^\infty w(\lambda) \left[1 - \lambda (T + \lambda)^{-1}\right] d\mu(\lambda).
$$

This gives the representation

$$
(1.18) \quad T^r = \frac{\sin (r\pi)}{\pi} \mathcal{M}(w_r, \mu) (T),
$$

where $w_r (\lambda) = \lambda^{-r}, r \in (0, 1]$ and $\mu$ is the usual Lebesgue norm. Also, from (1.8) we have the representation

$$
(1.19) \quad T \ln T = (T - 1) \mathcal{M}(w_{\ln}, \mu) (T), T > 0,
$$
where \( w_{1n}(\lambda) = (\lambda + 1)^{-1} \), \( t > 0 \).

In this paper we show among others that, if the linear map \( \Phi : \mathcal{B}(H) \to \mathcal{B}(K) \)

is positive, preserves the operator order and is normalised while \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \) for some scalars \( m < M \), then

\[
0 \leq \mathcal{M}(w, \mu)(\Phi(A)) - \Phi[\mathcal{M}(w, \mu)(A)] \\
\leq \frac{M - m}{(\sqrt{M} + \sqrt{m})^2} \left[ \mathcal{M}(w, \mu)(M) - \mathcal{M}(w, \mu)(m) \right],
\]

\[
0 \leq \mathcal{M}(w, \mu)(\Phi(A)) - \Phi[\mathcal{M}(w, \mu)(A)] \\
\leq \frac{1}{4} (M - m) [\mathcal{M}(w, \mu)(M) - \mathcal{M}(w, \mu)(m)] [\Phi(A)]^{-1}
\]

and

\[
0 \leq \mathcal{M}(w, \mu)(\Phi(A)) - \Phi[\mathcal{M}(w, \mu)(A)] \\
\leq \mathcal{M}(w, \mu) \left( \frac{m + M}{2} \right) - \frac{\mathcal{M}(w, \mu)(m) + \mathcal{M}(w, \mu)(M)}{2}.
\]

2. Main Results

We recall the following reverse inequalities [6, p. 29]:

**Lemma 1.** Let \( \Phi \in \Psi_N[\mathcal{B}(H), \mathcal{B}(K)] \) and \( A \) a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \) for some scalars \( m < M \). Then

\[
0 \leq \Phi(A^{-1}) - [\Phi(A)]^{-1} \leq \frac{(\sqrt{M} - \sqrt{m})^2}{mM}
\]

and

\[
\Phi(A^{-1}) \leq \frac{(M + m)^2}{4mM} [\Phi(A)]^{-1},
\]

or, equivalently

\[
0 \leq \Phi(A^{-1}) - [\Phi(A)]^{-1} \leq \frac{(M - m)^2}{4mM} [\Phi(A)]^{-1}.
\]

We have the following main result:

**Theorem 4.** Let \( \Phi \in \Psi_N[\mathcal{B}(H), \mathcal{B}(K)] \) and \( A \) a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \). Then

\[
0 \leq \mathcal{M}(w, \mu)(\Phi(A)) - \Phi[\mathcal{M}(w, \mu)(A)] \\
\leq \frac{M - m}{(\sqrt{M} + \sqrt{m})^2} \left[ \mathcal{M}(w, \mu)(M) - \mathcal{M}(w, \mu)(m) \right]
\]

and

\[
0 \leq \mathcal{M}(w, \mu)(\Phi(A)) - \Phi[\mathcal{M}(w, \mu)(A)] \\
\leq \frac{1}{4} (M - m) [\mathcal{M}(w, \mu)(M) - \mathcal{M}(w, \mu)(m)] [\Phi(A)]^{-1}
\]

\[
\leq \frac{1}{4} \left( \frac{M}{m} - 1 \right) [\mathcal{M}(w, \mu)(M) - \mathcal{M}(w, \mu)(m)].
\]
**Proof.** We have by the properties of $\Phi \in \mathfrak{H}_N [B(H), B(K)]$ and of Bochner integral that
\[
\Phi [\mathcal{M}(w, \mu) (A)] = \int_0^\infty w(\lambda) \left[ 1 - \lambda \Phi \left( (A + \lambda)^{-1} \right) \right] d\mu(\lambda)
\]
and
\[
\mathcal{M}(w, \mu) (\Phi (A)) = \int_0^\infty w(\lambda) \left[ 1 - \lambda \left( \Phi (A + \lambda)^{-1} \right) \right] d\mu(\lambda),
\]
which implies that
\[
\mathcal{M}(w, \mu) (\Phi (A)) - \Phi [\mathcal{M}(w, \mu) (A)]
\]
\[
= \int_0^\infty w(\lambda) \left[ 1 - \lambda \left( \Phi (A + \lambda)^{-1} \right) \right] d\mu(\lambda)
\]
\[
- \int_0^\infty w(\lambda) \left[ 1 - \lambda \Phi \left( (A + \lambda)^{-1} \right) \right] d\mu(\lambda)
\]
\[
= \int_0^\infty \lambda w(\lambda) \left( \Phi \left( (\lambda + A)^{-1} \right) - (\lambda + \Phi (A))^{-1} \right) d\mu(\lambda).
\]
Since the function $f(t) = t^{-1}$ is operator convex, then by (1.1) we have
\[
\Phi \left( (\lambda + A)^{-1} \right) - (\lambda + \Phi (A))^{-1} \geq 0
\]
for all $\lambda \geq 0$, which by multiplication with $\lambda w(\lambda) \geq 0$ and integration gives, by (2.6), the first inequality in (2.4).

Since $M + \lambda \geq A + \lambda \geq m + \lambda > 0$ for all $\lambda \geq 0$, then by (2.1) we get
\[
0 \leq \Phi \left( (\lambda + A)^{-1} \right) - (\lambda + \Phi (A))^{-1} \leq \frac{(\sqrt{M + \lambda} - \sqrt{m + \lambda})^2}{(m + \lambda)(M + \lambda)}
\]
\[
= \frac{(M - m)^2}{(\sqrt{M + \lambda} + \sqrt{m + \lambda})^2 (m + \lambda)(M + \lambda)}
\]
\[
\leq \frac{(M - m)^2}{(\sqrt{M} + \sqrt{m})^2 (m + \lambda)(M + \lambda)}
\]
for all $\lambda \geq 0$.

If we multiply by $\lambda w(\lambda) \geq 0$ and integrate, then we get by (2.6) that
\[
0 \leq \mathcal{M}(w, \mu) (\Phi (A)) - \Phi [\mathcal{M}(w, \mu) (A)]
\]
\[
\leq \frac{(M - m)^2}{(\sqrt{M} + \sqrt{m})^2} \int_0^\infty \frac{\lambda w(\lambda) d\mu(\lambda)}{(m + \lambda)(M + \lambda)}.
\]
Observe that
\[
\mathcal{M}(w, \mu) (M) - \mathcal{M}(w, \mu) (m)
\]
\[
= \int_0^\infty w(\lambda) \left[ 1 - \lambda (M + \lambda)^{-1} \right] d\mu(\lambda) - \int_0^\infty w(\lambda) \left[ 1 - \lambda (m + \lambda)^{-1} \right] d\mu(\lambda)
\]
\[
= \int_0^\infty \lambda w(\lambda) \left( \frac{1}{m + \lambda} - \frac{1}{M + \lambda} \right) d\mu(\lambda) = (M - m) \int_0^\infty \frac{\lambda w(\lambda) d\mu(\lambda)}{(m + \lambda)(M + \lambda)}
\]
and by (2.8) we derive (2.4).
From (2.3) we get
\[
0 \leq \Phi \left( (A + \lambda)^{-1} - [\Phi (A + \lambda)]^{-1} \leq \frac{(M - m)^2}{4(m + \lambda)(M + \lambda)} [\Phi (A + \lambda)]^{-1}
\]
for all \( \lambda \geq 0 \), which by multiplication with \( \lambda w(\lambda) \geq 0 \) and integration gives
\[
0 \leq \int_0^\infty w(\lambda) \left( \Phi \left[ (\lambda + T)^{-1} - (\lambda + \Phi (A))^{-1} \right] \right) d\mu(\lambda)
\]
\[
\leq \int_0^\infty \frac{\lambda w(\lambda) (M - m)^2}{4(m + \lambda)(M + \lambda)} [\Phi (A)]^{-1}
\]
\[
= \frac{(M - m)^2}{4} \int_0^\infty \frac{\lambda w(\lambda)}{(m + \lambda)(M + \lambda)} [\Phi (A)]^{-1}
\]
\[
= \frac{M - m}{4} \left[ \mathcal{M}(w, \mu) (M) - \mathcal{M}(w, \mu) (m) \right] [\Phi (A)]^{-1}
\]
\[
\leq \frac{M - m}{4m} \left[ \mathcal{M}(w, \mu) (M) - \mathcal{M}(w, \mu) (m) \right].
\]
By utilising (2.6) and (2.10) we get (2.5).

**Corollary 1.** Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) that has the representation (1.11). If \( \Phi \in \mathfrak{B}_N [\mathcal{B}(H), \mathcal{B}(K)] \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \), then
\[
0 \leq f(\Phi (A)) - \Phi [f (A)] \leq \frac{M - m}{\left( \sqrt{M} + \sqrt{m} \right)^2} \left[ f (M) - f (m) - b (M - m) \right]
\]
and
\[
0 \leq f(\Phi (A)) - \Phi [f (A)] \leq \frac{1}{4} (M - m) [f (M) - f (m) - b (M - m)] [\Phi (A)]^{-1}
\]
\[
\leq \frac{1}{4} (M - m) [f (M) - f (m)] [\Phi (A)]^{-1} \leq \frac{1}{4} \left( \frac{M}{m} - 1 \right) [f (M) - f (m)].
\]

**Proof.** From (1.11) we have
\[
\mathcal{M}(\ell, \mu) (t) = f(t) - a - bt,
\]
where \( \ell (\lambda) = \lambda, a \in R, b \geq 0 \) and \( \mu \) is a positive measure.
Since
\[
\mathcal{M}(w, \mu) (\Phi (A)) = f(\Phi (A)) - a - b\Phi (A),
\]
\[
\Phi [\mathcal{M}(w, \mu) (A)] = \Phi [f (A)] - a - b\Phi (A)
\]
and
\[
\mathcal{M}(w, \mu) (M) - \mathcal{M}(w, \mu) (m) = f(M) - f(m) - b(M - m),
\]
hence by (2.4) and (2.5) we get (2.11) and (2.12).
Remark 1. If we write the inequalities (2.11) and (2.12) for the operator monotone function \( f(t) = t^r, \ r \in (0, 1] \), then we get

\[
0 \leq \Phi^r(A) - \Phi(A^r) \leq \frac{M - m}{(\sqrt{M} + \sqrt{m})^2} (M^r - m^r)
\]

and

\[
0 \leq \Phi^r(A) - \Phi(A^r) \leq \frac{1}{4} (M - m) (M^r - m^r) [\Phi(A)]^{-1}
\]

\[
\leq \frac{1}{4} \left( \frac{M}{m} - 1 \right) (M^r - m^r),
\]

where \( \Phi \in \mathcal{P}_N [\mathcal{B}(H), \mathcal{B}(K)] \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \).

If we write the inequalities (2.11) and (2.12) for the operator monotone function \( f(t) = \ln t \), then we get

\[
0 \leq \ln \Phi(A) - \Phi(\ln A) \leq \frac{M - m}{(\sqrt{M} + \sqrt{m})^2} \ln \left( \frac{M}{m} \right)
\]

and

\[
0 \leq \ln \Phi(A) - \Phi(\ln A) \leq \frac{1}{4} (M - m) \ln \left( \frac{M}{m} \right) [\Phi(A)]^{-1}
\]

\[
\leq \frac{1}{4} \left( \frac{M}{m} - 1 \right) \ln \left( \frac{M}{m} \right),
\]

where \( \Phi \in \mathcal{P}_N [\mathcal{B}(H), \mathcal{B}(K)] \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \).

Corollary 2. Assume that \( f : [0, \infty) \to \mathbb{R} \) is operator convex in \([0, \infty)\) and has the representation (1.13). If \( \Phi \in \mathcal{P}_N [\mathcal{B}(H), \mathcal{B}(K)] \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \), then

\[
0 \leq f(\Phi(A)) [\Phi(A)]^{-1} - \Phi(f(A) A^{-1}) + f(0) \left( \Phi(A^{-1}) - [\Phi(A)]^{-1} \right)
\]

\[
\leq \frac{M - m}{(\sqrt{M} + \sqrt{m})^2} \left[ \frac{f(M)}{M} - \frac{f(m)}{m} + f(0) \left( \frac{M - m}{M m} \right) - c(M - m) \right]
\]

\[
\leq \frac{M - m}{(\sqrt{M} + \sqrt{m})^2} \left[ \frac{f(M)}{M} - \frac{f(m)}{m} + f(0) \left( \frac{M - m}{M m} \right) \right]
\]
and

\begin{equation}
0 \leq f(\Phi (A)) [\Phi (A)]^{-1} - \Phi (f (A) A^{-1}) + f (0) \left( \Phi (A^{-1}) - [\Phi (A)]^{-1} \right)
\end{equation}

\begin{align*}
&\leq \frac{1}{4} (M - m) \left[ \frac{f(M)}{M} - \frac{f(m)}{m} + f (0) \left( \frac{M - m}{M m} \right) - c(M - m) \right] \\
&\times [\Phi (A)]^{-1} \\
&\leq \frac{1}{4} (M - m) \left[ \frac{f(M)}{M} - \frac{f(m)}{m} + f (0) \left( \frac{M - m}{M m} \right) \right] [\Phi (A)]^{-1} \\
&\leq \frac{1}{4} \left( \frac{M - m}{m - 1} \right) \left[ \frac{f(M)}{M} - \frac{f(m)}{m} + f (0) \left( \frac{M - m}{M m} \right) \right].
\end{align*}

Proof. We have by (1.13) that

\[ \mathcal{M}(\ell, \mu) = \frac{f(t) - f(0)}{t} - f_+^\prime (0) - ct \]

for some positive measure \( \mu \), where \( \ell(t) = t, t > 0 \) and \( c \geq 0 \).

Since

\[ \mathcal{M}(w, \mu) (\Phi (A)) = \left[ f (\Phi (A)) - f (0) \right] [\Phi (A)]^{-1} - f_+^\prime (0) - c\Phi (A), \]

\[ \Phi [\mathcal{M}(w, \mu) (A)] = \Phi (f (A) A^{-1}) - f (0) \Phi (A^{-1}) - f_+^\prime (0) - c\Phi (A) \]

and

\[ \mathcal{M}(w, \mu) (M) - \mathcal{M}(w, \mu) (m) \]

\[ = \frac{f(M) - f(0)}{M} - \frac{f(m) - f(0)}{m} - c(M - m) \]

\[ = \frac{f(M)}{M} - \frac{f(m)}{m} + f (0) \left( \frac{M - m}{M m} \right) - c(M - m) \]

hence by (2.4) and (2.5) we get (2.17) and (2.18). \( \square \)

Remark 2. With the assumptions of Corollary 2 and if \( f (0) = 0 \), then we get from (2.17) and (2.18) that

\begin{equation}
0 \leq f(\Phi (A)) [\Phi (A)]^{-1} - \Phi (f (A) A^{-1}) \leq \frac{M - m}{(\sqrt{M} + \sqrt{m})^2} \left[ \frac{f(M)}{M} - \frac{f(m)}{m} \right]
\end{equation}

and

\begin{equation}
0 \leq f(\Phi (A)) [\Phi (A)]^{-1} - \Phi (f (A) A^{-1}) \leq \frac{1}{4} (M - m) \left[ \frac{f(M)}{M} - \frac{f(m)}{m} \right] [\Phi (A)]^{-1} \\
\leq \frac{1}{4} \left( \frac{M - m}{m - 1} \right) \left[ \frac{f(M)}{M} - \frac{f(m)}{m} \right].
\end{equation}
If we take in (2.19) and (2.20) the operator convex function \( f(t) = \ln(t + 1) \), then we get
\[
0 \leq \Phi \left( A^{-1} \ln(A + 1) \right) - \ln(\Phi(A + 1)\Phi(A))^{-1} \leq \frac{M - m}{(\sqrt{M} + \sqrt{m})^2} \left[ \frac{\ln(m + 1)}{m} - \frac{\ln(M + 1)}{M} \right]
\]
and
\[
0 \leq \Phi \left( A^{-1} \ln(A + 1) \right) - \ln(\Phi(A + 1)\Phi(A))^{-1} \leq \frac{1}{4} (M - m) \left[ \frac{\ln(m + 1)}{m} - \frac{\ln(M + 1)}{M} \right] \Phi(A)^{-1} \leq \frac{1}{4} \left( \frac{M}{m} - 1 \right) \left[ \frac{\ln(m + 1)}{m} - \frac{\ln(M + 1)}{M} \right]
\]
if \( \Phi \in \mathcal{P}_N \left[ \mathcal{B}(H), \mathcal{B}(K) \right] \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \).

From a different perspective we also have the upper bound in terms of Jensen’s difference:

**Theorem 5.** Let \( \Phi \in \mathcal{P}_N \left[ \mathcal{B}(H), \mathcal{B}(K) \right] \) and \( A \) a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \). Then
\[
0 \leq \mathcal{M}(w, \mu)(\Phi(A)) - \Phi[\mathcal{M}(w, \mu)(A)] \leq \mathcal{M}(w, \mu) \left( \frac{m + M}{2} \right) - \mathcal{M}(w, \mu)(m) + \mathcal{M}(w, \mu)(M) \frac{M}{2}.
\]

**Proof.** As in the proof of Theorem 4, see (2.7),
\[
0 \leq \Phi \left( (\lambda + A)^{-1} \right) - \Phi[\lambda + A]^{-1} \leq \frac{(M - m)^2}{(\sqrt{M + \lambda} + \sqrt{m + \lambda})^2 (m + \lambda)(M + \lambda)}
\]
for \( \lambda \geq 0 \).

Using the elementary inequality \( \sqrt{a} + \sqrt{b} \geq \sqrt{a + b}, a, b \geq 0 \) we deduce that
\[
(\sqrt{M + \lambda} + \sqrt{m + \lambda})^2 \geq M + m + 2\lambda
\]
for \( \lambda \geq 0 \), which implies that
\[
\frac{(M - m)^2}{(\sqrt{M + \lambda} + \sqrt{m + \lambda})^2 (m + \lambda)(M + \lambda)} \leq \frac{(M - m)^2}{(M + m + 2\lambda)(m + \lambda)(M + \lambda)}.
\]
We observe that, by performing the calculations, one has the equality
\[
\frac{1}{(M - m)^2} \left( \frac{1}{m + \lambda} + \frac{1}{M + \lambda} - \frac{2}{\lambda + \frac{m + M}{2}} \right),
\]
for \( \lambda \geq 0 \).
Therefore

\[
\frac{(M - m)^2}{(\sqrt{M + \lambda} + \sqrt{m + \lambda})^2 (m + \lambda) (M + \lambda)} \leq \frac{1}{2} \left( \frac{1}{m + \lambda} + \frac{1}{M + \lambda} - \frac{2}{\lambda + \frac{m + M}{2}} \right) = \frac{1}{2} \left( \frac{1}{m + \lambda} + \frac{1}{M + \lambda} \right) - \frac{1}{\lambda + \frac{m + M}{2}}.
\]

for \( \lambda \geq 0 \).

If we use (2.6) then by (2.24) and (2.25) we get

\[
\mathcal{M}(w, \mu)(\Phi(A)) - \Phi[\mathcal{M}(w, \mu)(A)] = \int_{0}^{\infty} \lambda w(\lambda) \left( \Phi[(\lambda + T)^{-1}] - (\lambda + \Phi(A))^{-1} \right) d\mu(\lambda)
\]

\[
\leq \int_{0}^{\infty} \lambda w(\lambda) \left( \frac{1}{2} \left( \frac{1}{m + \lambda} + \frac{1}{M + \lambda} \right) - \frac{1}{\lambda + \frac{m + M}{2}} \right) d\mu(\lambda).
\]

Observe also that

\[
\mathcal{M}(w, \mu)\left( \frac{m + M}{2} \right) - \frac{\mathcal{M}(w, \mu)(m) + \mathcal{M}(w, \mu)(M)}{2}
\]

\[
= \int_{0}^{\infty} w(\lambda) \left[ 1 - \lambda \left( \frac{m + M}{2} + \lambda \right)^{-1} \right] d\mu(\lambda)
\]

\[- \frac{1}{2} \int_{0}^{\infty} w(\lambda) \left[ 1 - \lambda (m + \lambda)^{-1} \right] d\mu(\lambda)
\]

\[- \frac{1}{2} \int_{0}^{\infty} w(\lambda) \left[ 1 - \lambda (M + \lambda)^{-1} \right] d\mu(\lambda)
\]

\[= \int_{0}^{\infty} \lambda w(\lambda) \left( \frac{1}{2} \left( \frac{1}{m + \lambda} + \frac{1}{M + \lambda} \right) - \frac{1}{\lambda + \frac{m + M}{2}} \right) d\mu(\lambda)
\]

and the inequality (2.23) is thus obtained. \( \square \)

The case of operator monotone functions is as follows:

**Corollary 3.** Assume that the function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \( (0, \infty) \). If \( \Phi \in \Psi_N[\mathcal{B}(H), \mathcal{B}(K)] \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \), then

\[
0 \leq f(\Phi(A)) - \Phi[f(A)] \leq f\left( \frac{m + M}{2} \right) - \frac{f(m) + f(M)}{2}.
\]

**Proof.** From (1.11) we have

\[\mathcal{M}(\ell, \mu)(t) = f(t) - a - bt,\]

where \( \ell(\lambda) = \lambda, a \in \mathbb{R}, b \geq 0 \) and \( \mu \) is a positive measure.
Since
\[ M(w, \mu) \left( \frac{m + M}{2} \right) = \frac{M(w, \mu)(m) + M(w, \mu)(M)}{2} \]
\[ = f \left( \frac{m + M}{2} \right) - a - b m + f(m) - a - b M \]
\[ = f \left( \frac{m + M}{2} \right) - \frac{f(m) + f(M)}{2}, \]
hence by (2.23) we derive (2.27).

**Remark 3.** If we write the inequalities (2.27) for the operator monotone function \( f(t) = t^r, r \in (0, 1] \), then we get

\[ 0 \leq \Phi^r(A) - \Phi(A^r) \leq \left( \frac{m + M}{2} \right)^r - \frac{m^r + M^r}{2} \]

if \( \Phi \in \Phi_N [\mathcal{B}(H), \mathcal{B}(K)] \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \).

If we write the inequalities (2.27) for the operator monotone function \( f(t) = \ln t \), then we get the logarithmic inequalities

\[ 0 \leq \ln \Phi(A) - \Phi(\ln A) \leq \ln \left( \frac{m + M}{2\sqrt{mM}} \right). \]

The case of operator convex functions is as follows:

**Corollary 4.** Assume that \( f : [0, \infty) \to \mathbb{R} \) is operator convex in \([0, \infty)\). If \( \Phi \in \Phi_N [\mathcal{B}(H), \mathcal{B}(K)] \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \) then

\[ 0 \leq f(\Phi(A)) \Phi(A)^{-1} - \Phi(f(A)A^{-1}) + f(0) \left( \Phi(A^{-1}) - [\Phi(A)]^{-1} \right) \]
\[ \leq f \left( \frac{m + M}{2} \right) \left( \frac{m + M}{2} \right)^{-1} - \frac{f(m)m^{-1} + f(M)M^{-1}}{2} \]
\[ + f(0) \frac{(M - m)^2}{2mM(m + M)}. \]

**Proof.** We have by (1.13) that

\[ \mathcal{M}(\ell, \mu) = \frac{f(t) - f(0)}{t} - f_+(0) - ct \]

for some positive measure \( \mu \), where \( \ell(t) = t, t > 0 \) and \( c \geq 0 \).
Since
\[
\mathcal{M}(w, \mu) \left( \frac{m + M}{2} \right) = \mathcal{M}(w, \mu) \left( \frac{m}{2} \right) + \mathcal{M}(w, \mu) \left( \frac{M}{2} \right)
\]
\[
= f \left( \frac{m + M}{2} \right) - f(0) - \frac{cm + M}{2}
\]
\[
- \frac{1}{2} \left( \frac{f(m) - f(0)}{m} - f'_-(0) - cm \right)
\]
\[
- \frac{1}{2} \left( \frac{f(M) - f(0)}{M} - f'_+(0) - cM \right)
\]
\[
f \left( \frac{m + M}{2} \right) \left( \frac{m + M}{2} \right)^{-1} - \frac{f(m) m^{-1} + f(M) M^{-1}}{2}
\]
\[
- f(0) \left[ \frac{2}{m + M} - \frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right) \right]
\]
\[
f \left( \frac{m + M}{2} \right) \left( \frac{m + M}{2} \right)^{-1} - \frac{f(m) m^{-1} + f(M) M^{-1}}{2}
\]
\[
+ f(0) \frac{(M - m)^2}{2mM (m + M)},
\]
hence by (2.23) we derive (2.30).

Remark 4. With the assumptions of Corollary 4 and if \( f(0) = 0 \), then we get from (2.30) that
\[
0 \leq f \left( \Phi(A) \right) \left[ \Phi(A) \right]^{-1} - \Phi \left( f(A) A^{-1} \right)
\]
\[
\leq f \left( \frac{m + M}{2} \right) \left( \frac{m + M}{2} \right)^{-1} - \frac{f(m) m^{-1} + f(M) M^{-1}}{2}.
\]

If we take in (2.31) the operator convex function \( f(t) = -\ln(t + 1) \), then we get
\[
0 \leq \Phi \left( A^{-1} \ln(A + 1) \right) - \ln(\Phi(A) + 1) \left[ \Phi(A) \right]^{-1}
\]
\[
\leq \frac{m^{-1} \ln(m + 1) + M^{-1} \ln(M + 1)}{2}
\]
\[
- \left( \frac{m + M}{2} \right)^{-1} \ln \left( \frac{m + M}{2} + 1 \right),
\]
if \( \Phi \in \mathfrak{P}_N \left[ \mathcal{B}(H), \mathcal{B}(K) \right] \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \).

3. Some Examples

Let \( P_j \in \mathcal{B}(H) \), \( j = 1, \ldots, k \) be contractions with
\[
\sum_{j=1}^{k} P_j^* P_j = 1_H.
\]
The map \( \Phi : \mathcal{B}(H) \to \mathcal{B}(H) \) defined by [6]
\[
\Phi(A) := \sum_{j=1}^{k} P_j^* AP_j
\]
is a normalized positive linear map on $\mathcal{B}(H)$.

Assume that $A$ is a positive operator on $H$ satisfying the condition $M \geq A \geq m > 0$ for some scalars $m < M$. Then by Theorem 4 we get the following reverses of operator Jensen’s inequality

\begin{equation}
0 \leq \mathcal{M}(w, \mu) \left( \sum_{j=1}^{k} P_j^* A P_j \right) - \sum_{j=1}^{k} P_j^* \mathcal{M}(w, \mu) (A) P_j \\
\leq \frac{M - m}{\sqrt{M + \sqrt{m}}} \left[ \mathcal{M}(w, \mu) (M) - \mathcal{M}(w, \mu) (m) \right]
\end{equation}

and

\begin{equation}
0 \leq \mathcal{M}(w, \mu) \left( \sum_{j=1}^{k} P_j^* A P_j \right) - \sum_{j=1}^{k} P_j^* \mathcal{M}(w, \mu) (A) P_j \\
\leq \frac{1}{4} (M - m) \left[ \mathcal{M}(w, \mu) (M) - \mathcal{M}(w, \mu) (m) \right] \left( \sum_{j=1}^{k} P_j^* A P_j \right)^{-1} \\
\leq \frac{1}{4} \left( \frac{M}{m} - 1 \right) \left[ \mathcal{M}(w, \mu) (M) - \mathcal{M}(w, \mu) (m) \right],
\end{equation}

while by Theorem 5 we get

\begin{equation}
0 \leq \mathcal{M}(w, \mu) \left( \sum_{j=1}^{k} P_j^* A P_j \right) - \sum_{j=1}^{k} P_j^* \mathcal{M}(w, \mu) (A) P_j \\
\leq \mathcal{M}(w, \mu) \left( \frac{m + M}{2} \right) - \mathcal{M}(w, \mu) (m) + \mathcal{M}(w, \mu) (M) - \frac{M (m + M)}{2}.
\end{equation}

Further, if we assume that the function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and $A$ is a positive operator on $H$ satisfying the condition $M \geq A \geq m > 0$, then by Corollary 1 we get

\begin{equation}
0 \leq f \left( \sum_{j=1}^{k} P_j^* A P_j \right) - \sum_{j=1}^{k} P_j^* f (A) P_j \\
\leq \frac{M - m}{\sqrt{M + \sqrt{m}}} \left[ f(M) - f(m) \right]
\end{equation}

and

\begin{equation}
0 \leq f \left( \sum_{j=1}^{k} P_j^* A P_j \right) - \sum_{j=1}^{k} P_j^* f (A) P_j \\
\leq \frac{1}{4} (M - m) \left[ f(M) - f(m) \right] \left( \sum_{j=1}^{k} P_j^* A P_j \right)^{-1} \\
\leq \frac{1}{4} \left( \frac{M}{m} - 1 \right) \left[ f(M) - f(m) \right].
\end{equation}
From Corollary 3 we obtain

$$0 \leq f \left( \sum_{j=1}^{k} P_j^* A P_j \right) - \sum_{j=1}^{k} P_j^* f(A) P_j \leq f \left( \frac{m + M}{2} \right) - \frac{f(m) + f(M)}{2}.$$ 

Assume that \( f : [0, \infty) \to \mathbb{R} \) is operator convex in \([0, \infty)\) with \( f(0) = 0 \) and \( A \) is a positive operator on \( H \) satisfying the condition \( M \geq A \geq m > 0 \), then by (2.19) and (2.20) we have

\[
(3.7) \quad 0 \leq f \left( \sum_{j=1}^{k} P_j^* A P_j \right) \left( \sum_{j=1}^{k} P_j^* A P_j \right)^{-1} - \sum_{j=1}^{k} P_j^* f(A) A^{-1} P_j \\
\leq \frac{M - m}{\left( \sqrt{M} + \sqrt{m} \right)^2} \left[ \frac{f(M)}{M} - \frac{f(m)}{m} \right]
\]

and

\[
(3.8) \quad 0 \leq f \left( \sum_{j=1}^{k} P_j^* A P_j \right) \left( \sum_{j=1}^{k} P_j^* A P_j \right)^{-1} - \sum_{j=1}^{k} P_j^* f(A) A^{-1} P_j \\
\leq \frac{1}{4} \left( \frac{M - m}{m - 1} \right) \left[ \frac{f(M)}{M} - \frac{f(m)}{m} \right].
\]

From (2.31) we get

\[
(3.9) \quad 0 \leq f \left( \sum_{j=1}^{k} P_j^* A P_j \right) \left( \sum_{j=1}^{k} P_j^* A P_j \right)^{-1} - \sum_{j=1}^{k} P_j^* f(A) A^{-1} P_j \\
\leq f \left( \frac{m + M}{2} \right) \left( \frac{m + M}{2} \right)^{-1} - \frac{f(m) + f(M)}{2}.
\]

References


[5] T. Furuta, Precise lower bound of \( f(A) - f(B) \) for \( A > B > 0 \) and non-constant operator monotone function \( f \) on \([0, \infty)\). J. Math. Inequal. 9 (2015), no. 1, 47–52.


1Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

2DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa.