REVERSES OF DAVIS-CHOI-JENSEN'S INEQUALITY FOR THE MONOTONIC INTEGRAL TRANSFORM

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Abstract. Some applications for operator monotone and operator convex functions are also provided.

1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [2] (see also [6, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e., if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is normalised if it preserves the identity operator, i.e., $\Phi(1) = 1$. We write $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1 \leq A \leq \beta 1$, then $\alpha 1 \leq \Phi(A) \leq \beta 1$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1)\Psi\Psi^{-1/2}(1)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda)f(A) + \lambda f(B)$$

for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

The following Jensen's type result is well known [2]:

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Theorem 1 (Davis-Choi-Jensen's Inequality)**.** *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I and $\Phi \in \mathfrak{P}_N [\mathcal{B}(H), \mathcal{B}(K)]$, then for any selfadjoint operator A whose spectrum is contained in I we have*

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1.1}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1)\Psi\Psi^{-1/2}(1)$ in (1.1) we get

$$f\left(\Psi^{-1/2}(1)\Psi(A)\Psi^{-1/2}(1)\right) \leq \Psi^{-1/2}(1)\Psi(f(A))\Psi^{-1/2}(1).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps:*

$$\Psi^{1/2}(1) f\left(\Psi^{-1/2}(1)\Psi(A)\Psi^{-1/2}(1)\right)\Psi^{1/2}(1) \leq \Psi(f(A)). \tag{1.2}$$

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.3}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t-1} + \frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.4}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.5) exists for all $t > 0$.

For μ, the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{1.6}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t), \ t > 0. \tag{1.7}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t - 1)\mathcal{D}(w_{\ln})(t), \ t > 0. \tag{1.8}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.9}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.10}$$

for $T > 0$.

A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

Theorem 2. *A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda), \tag{1.11}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty. \tag{1.12}$$

If f is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.11).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1 - \lambda)A + \lambda B) \leq (\geq)(1 - \lambda)f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 3. *A function $f : (0, \infty) \to \mathbb{R}$ is operator convex in $(0, \infty)$ if and only if it has the representation*

$$f(t) = a + bt + ct^2 + \int_0^\infty \frac{t^2\lambda}{t + \lambda} d\mu(\lambda), \tag{1.13}$$

where a, $b \in \mathbb{R}$, $c \geq 0$ and a positive measure μ on $(0, \infty)$ such that (1.4) holds. If f is operator convex in $[0, \infty)$, then $a = f(0)$ and $b = f'_+(0)$, the right derivative, in (1.13).

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t), \ t > 0. \tag{1.14}$$

For $t>0$ we have

$$\mathcal{M}(w,\mu)(t) := t\mathcal{D}(w,\mu)(t) = \int_0^\infty w(\lambda)\, t\,(t+\lambda)^{-1}\, d\mu(\lambda) \tag{1.15}$$
$$= \int_0^\infty w(\lambda)(t+\lambda-\lambda)(t+\lambda)^{-1}\, d\mu(\lambda)$$
$$= \int_0^\infty w(\lambda)\left[1-\lambda(t+\lambda)^{-1}\right] d\mu(\lambda).$$

If $\int_0^\infty w(\lambda)\, d\mu(\lambda) < \infty$, then

$$\mathcal{M}(w,\mu)(t) = \int_0^\infty w(\lambda)\, d\mu(\lambda) - \mathcal{D}(\ell w,\mu)(t), \tag{1.16}$$

where $\ell(t) = t$, $t>0$.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a>0$. Then after some calculations, we get

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda} d\lambda = E_1(at)\exp(at),\ t\geq 0$$

and

$$\int_0^\infty w(\lambda)\, d\lambda = \int_0^\infty \exp(-a\lambda)\, d\lambda = \frac{1}{a},$$

where

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

This gives that

$$\mathcal{M}(e_{-a})(t) = t\mathcal{D}(w,\mu)(t) = tE_1(at)\exp(at),\ t\geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda\exp(-a\lambda)}{t+\lambda} d\lambda = \frac{1}{a} - tE_1(at)\exp(at)$$

for $t>0$.

One observes that

$$\mathcal{M}(e_{-a})(t) = \int_0^\infty w(\lambda)\, d\lambda - \mathcal{D}(\ell e_{-a},\mu)(t),\ t>0$$

and the equality (1.16) is verified in this case.

If we take $w_r(\lambda) = \lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\, d\lambda = \infty$ and the equality (1.16) does not hold in this case.

For all $T>0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w,\mu)(T) = T\mathcal{D}(w,\mu)(T) = \int_0^\infty w(\lambda)\left[1-\lambda(T+\lambda)^{-1}\right] d\mu(\lambda). \tag{1.17}$$

This gives the representation

$$T^r = \frac{\sin(r\pi)}{\pi}\mathcal{M}(w_r,\mu)(T), \tag{1.18}$$

where $w_r(\lambda) = \lambda^{r-1}$, $r\in(0,1]$ and μ is the usual Lebesgue norm. Also, from (1.8) we have the representation

$$T\ln T = (T-1)\,\mathcal{M}(w_{\ln})(T),\ T>0, \tag{1.19}$$

where $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$.

In this paper we show among others that, if the linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive, preserves the operator order and is normalised while A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$ for some scalars $m < M$, then

$$\begin{aligned} 0 &\leq \mathcal{M}(w,\mu)(\Phi(A)) - \Phi[\mathcal{M}(w,\mu)(A)] \\ &\leq \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)], \end{aligned}$$

$$\begin{aligned} 0 &\leq \mathcal{M}(w,\mu)(\Phi(A)) - \Phi[\mathcal{M}(w,\mu)(A)] \\ &\leq \frac{1}{4}(M-m)[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)][\Phi(A)]^{-1} \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \mathcal{M}(w,\mu)(\Phi(A)) - \Phi[\mathcal{M}(w,\mu)(A)] \\ &\leq \mathcal{M}(w,\mu)\left(\frac{m+M}{2}\right) - \frac{\mathcal{M}(w,\mu)(m) + \mathcal{M}(w,\mu)(M)}{2}. \end{aligned}$$

2. Main Results

We recall the following reverse inequalities [6, p. 29]:

Lemma 1. *Let $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and A a positive operator on H satisfying the condition $M \geq A \geq m > 0$ for some scalars $m < M$. Then*

$$0 \leq \Phi\left(A^{-1}\right) - [\Phi(A)]^{-1} \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM} \tag{2.1}$$

and

$$\Phi\left(A^{-1}\right) \leq \frac{(M+m)^2}{4mM}[\Phi(A)]^{-1}, \tag{2.2}$$

or, equivalently

$$0 \leq \Phi\left(A^{-1}\right) - [\Phi(A)]^{-1} \leq \frac{(M-m)^2}{4mM}[\Phi(A)]^{-1}. \tag{2.3}$$

We have the following main result:

Theorem 4. *Let $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and A a positive operator on H satisfying the condition $M \geq A \geq m > 0$. Then*

$$\begin{aligned} 0 &\leq \mathcal{M}(w,\mu)(\Phi(A)) - \Phi[\mathcal{M}(w,\mu)(A)] \\ &\leq \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)] \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} 0 &\leq \mathcal{M}(w,\mu)(\Phi(A)) - \Phi[\mathcal{M}(w,\mu)(A)] \\ &\leq \frac{1}{4}(M-m)[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)][\Phi(A)]^{-1} \\ &\leq \frac{1}{4}\left(\frac{M}{m}-1\right)[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)]. \end{aligned} \tag{2.5}$$

Proof. We have by the properties of $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and of Bochner integral that

$$\Phi[\mathcal{M}(w,\mu)(A)] = \int_0^\infty w(\lambda)\left[1 - \lambda\Phi\left[(A+\lambda)^{-1}\right]\right] d\mu(\lambda)$$

and

$$\mathcal{M}(w,\mu)(\Phi(A)) = \int_0^\infty w(\lambda)\left[1 - \lambda\left[(\Phi(A)+\lambda)^{-1}\right]\right] d\mu(\lambda),$$

which implies that

$$\begin{aligned}(2.6)\qquad & \mathcal{M}(w,\mu)(\Phi(A)) - \Phi[\mathcal{M}(w,\mu)(A)] \\ &= \int_0^\infty w(\lambda)\left[1 - \lambda\left[(\Phi(A)+\lambda)^{-1}\right]\right] d\mu(\lambda) \\ &- \int_0^\infty w(\lambda)\left[1 - \lambda\Phi\left[(A+\lambda)^{-1}\right]\right] d\mu(\lambda) \\ &= \int_0^\infty \lambda w(\lambda)\left(\Phi\left[(\lambda+A)^{-1}\right] - (\lambda+\Phi(A))^{-1}\right) d\mu(\lambda).\end{aligned}$$

Since the function $f(t) = t^{-1}$ is operator convex, then by (1.1) we have

$$\Phi\left[(\lambda+A)^{-1}\right] - (\lambda+\Phi(A))^{-1} \geq 0$$

for all $\lambda \geq 0$, which by multiplication with $\lambda w(\lambda) \geq 0$ and integration gives, by (2.6), the first inequality in (2.4).

Since $M+\lambda \geq A+\lambda \geq m+\lambda > 0$ for all $\lambda \geq 0$, then by (2.1) we get

$$\begin{aligned}(2.7)\qquad 0 &\leq \Phi\left((\lambda+A)^{-1}\right) - [\Phi(\lambda+A)]^{-1} \leq \frac{\left(\sqrt{M+\lambda} - \sqrt{m+\lambda}\right)^2}{(m+\lambda)(M+\lambda)} \\ &= \frac{(M-m)^2}{\left(\sqrt{M+\lambda} + \sqrt{m+\lambda}\right)^2 (m+\lambda)(M+\lambda)} \\ &\leq \frac{(M-m)^2}{\left(\sqrt{M} + \sqrt{m}\right)^2 (m+\lambda)(M+\lambda)}\end{aligned}$$

for all $\lambda \geq 0$.

If we multiply by $\lambda w(\lambda) \geq 0$ and integrate, then we get by (2.6) that

$$\begin{aligned}(2.8)\qquad 0 &\leq \mathcal{M}(w,\mu)(\Phi(A)) - \Phi[\mathcal{M}(w,\mu)(A)] \\ &\leq \frac{(M-m)^2}{\left(\sqrt{M}+\sqrt{m}\right)^2} \int_0^\infty \frac{\lambda w(\lambda)\, d\mu(\lambda)}{(m+\lambda)(M+\lambda)}.\end{aligned}$$

Observe that

$$\begin{aligned}& \mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m) \\ &= \int_0^\infty w(\lambda)\left[1 - \lambda(M+\lambda)^{-1}\right] d\mu(\lambda) - \int_0^\infty w(\lambda)\left[1 - \lambda(m+\lambda)^{-1}\right] d\mu(\lambda) \\ &= \int_0^\infty \lambda w(\lambda)\left(\frac{1}{m+\lambda} - \frac{1}{M+\lambda}\right) d\mu(\lambda) = (M-m)\int_0^\infty \frac{\lambda w(\lambda)\, d\mu(\lambda)}{(m+\lambda)(M+\lambda)}\end{aligned}$$

and by (2.8) we derive (2.4).

From (2.3) we get

$$0 \le \Phi\left((A+\lambda)^{-1}\right) - [\Phi(A+\lambda)]^{-1} \le \frac{(M-m)^2}{4(m+\lambda)(M+\lambda)}[\Phi(A+\lambda)]^{-1} \le \frac{(M-m)^2}{4(m+\lambda)(M+\lambda)}[\Phi(A)]^{-1} \tag{2.9}$$

for all $\lambda \ge 0$, which by multiplication with $\lambda w(\lambda) \ge 0$ and integration gives

$$\begin{aligned} 0 &\le \int_0^\infty w(\lambda)\left(\Phi\left[(\lambda+T)^{-1}\right] - (\lambda+\Phi(A))^{-1}\right) d\mu(\lambda) \\ &\le \int_0^\infty \frac{\lambda w(\lambda)(M-m)^2}{4(m+\lambda)(M+\lambda)}[\Phi(A)]^{-1} \\ &= \frac{(M-m)^2}{4}\int_0^\infty \frac{\lambda w(\lambda)}{(m+\lambda)(M+\lambda)}[\Phi(A)]^{-1} \\ &= \frac{M-m}{4}[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)][\Phi(A)]^{-1} \\ &\le \frac{M-m}{4m}[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)]. \end{aligned} \tag{2.10}$$

By utilising (2.6) and (2.10) we get (2.5). □

Corollary 1. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ that has the representation (1.11). If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M \ge A \ge m > 0$, then*

$$0 \le f(\Phi(A)) - \Phi[f(A)] \le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}[f(M)-f(m)-b(M-m)] \le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}[f(M)-f(m)] \tag{2.11}$$

and

$$\begin{aligned} 0 &\le f(\Phi(A)) - \Phi[f(A)] \\ &\le \frac{1}{4}(M-m)[f(M)-f(m)-b(M-m)][\Phi(A)]^{-1} \\ &\le \frac{1}{4}(M-m)[f(M)-f(m)][\Phi(A)]^{-1} \le \frac{1}{4}\left(\frac{M}{m}-1\right)[f(M)-f(m)]. \end{aligned} \tag{2.12}$$

Proof. From (1.11) we have

$$\mathcal{M}(\ell,\mu)(t) = f(t) - a - bt,$$

where $\ell(\lambda)=\lambda$, $a \in R$, $b \ge 0$ and μ is a positive measure.

Since

$$\mathcal{M}(w,\mu)(\Phi(A)) = f(\Phi(A)) - a - b\Phi(A),$$
$$\Phi[\mathcal{M}(w,\mu)(A)] = \Phi[f(A)] - a - b\Phi(A)$$

and

$$\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m) = f(M) - f(m) - b(M-m),$$

hence by (2.4) and (2.5) we get (2.11) and (2.12). □

Remark 1. *If we write the inequalities (2.11) and (2.12) for the operator monotone function* $f(t) = t^r$, $r \in (0,1]$, *then we get*

$$0 \le \Phi^r(A) - \Phi(A^r) \le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}(M^r - m^r) \tag{2.13}$$

and

$$\begin{aligned} 0 \le \Phi^r(A) - \Phi(A^r) &\le \frac{1}{4}(M-m)(M^r - m^r)[\Phi(A)]^{-1} \\ &\le \frac{1}{4}\left(\frac{M}{m}-1\right)(M^r - m^r), \end{aligned} \tag{2.14}$$

where $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ *and* A *is a positive operator on* H *satisfying the condition* $M \ge A \ge m > 0$.

If we write the inequalities (2.11) and (2.12) for the operator monotone function $f(t) = \ln t$, *then we get*

$$0 \le \ln\Phi(A) - \Phi(\ln A) \le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\ln\left(\frac{M}{m}\right) \tag{2.15}$$

and

$$\begin{aligned} 0 \le \ln\Phi(A) - \Phi(\ln A) &\le \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right)[\Phi(A)]^{-1} \\ &\le \frac{1}{4}\left(\frac{M}{m}-1\right)\ln\left(\frac{M}{m}\right), \end{aligned} \tag{2.16}$$

where $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ *and* A *is a positive operator on* H *satisfying the condition* $M \ge A \ge m > 0$.

Corollary 2. *Assume that* $f : [0,\infty) \to \mathbb{R}$ *is operator convex in* $[0,\infty)$ *and has the representation (1.13). If* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ *and* A *is a positive operator on* H *satisfying the condition* $M \ge A \ge m > 0$, *then*

$$\begin{aligned} 0 &\le f(\Phi(A))[\Phi(A)]^{-1} - \Phi\left(f(A)A^{-1}\right) + f(0)\left(\Phi\left(A^{-1}\right) - [\Phi(A)]^{-1}\right) \\ &\le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\frac{f(M)}{M} - \frac{f(m)}{m} + f(0)\left(\frac{M-m}{Mm}\right) - c(M-m)\right] \\ &\le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\frac{f(M)}{M} - \frac{f(m)}{m} + f(0)\left(\frac{M-m}{Mm}\right)\right] \end{aligned} \tag{2.17}$$

and

$$
\begin{aligned}
(2.18)\qquad & 0 \le f(\Phi(A))[\Phi(A)]^{-1} - \Phi\left(f(A)A^{-1}\right) + f(0)\left(\Phi\left(A^{-1}\right) - [\Phi(A)]^{-1}\right) \\
& \le \frac{1}{4}(M-m)\left[\frac{f(M)}{M} - \frac{f(m)}{m} + f(0)\left(\frac{M-m}{Mm}\right) - c(M-m)\right] \\
& \times [\Phi(A)]^{-1} \\
& \le \frac{1}{4}(M-m)\left[\frac{f(M)}{M} - \frac{f(m)}{m} + f(0)\left(\frac{M-m}{Mm}\right)\right][\Phi(A)]^{-1} \\
& \le \frac{1}{4}\left(\frac{M}{m}-1\right)\left[\frac{f(M)}{M} - \frac{f(m)}{m} + f(0)\left(\frac{M-m}{Mm}\right)\right].
\end{aligned}
$$

Proof. We have by (1.13) that

$$\mathcal{M}(\ell,\mu) = \frac{f(t)-f(0)}{t} - f'_+(0) - ct$$

for some positive measure μ, where $\ell(t) = t$, $t > 0$ and $c \ge 0$.

Since

$$\mathcal{M}(w,\mu)(\Phi(A)) = [f(\Phi(A)) - f(0)][\Phi(A)]^{-1} - f'_+(0) - c\Phi(A),$$

$$\Phi[\mathcal{M}(w,\mu)(A)] = \Phi\left(f(A)A^{-1}\right) - f(0)\Phi\left(A^{-1}\right) - f'_+(0) - c\Phi(A)$$

and

$$
\begin{aligned}
& \mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m) \\
& = \frac{f(M)-f(0)}{M} - \frac{f(m)-f(0)}{m} - c(M-m) \\
& = \frac{f(M)}{M} - \frac{f(m)}{m} + f(0)\left(\frac{M-m}{Mm}\right) - c(M-m)
\end{aligned}
$$

hence by (2.4) and (2.5) we get (2.17) and (2.18). □

Remark 2. *With the assumptions of Corollary 2 and if $f(0) = 0$, then we get from (2.17) and (2.18) that*

(2.19)

$$0 \le f(\Phi(A))[\Phi(A)]^{-1} - \Phi\left(f(A)A^{-1}\right) \le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\frac{f(M)}{M} - \frac{f(m)}{m}\right]$$

and

$$
\begin{aligned}
(2.20)\qquad & 0 \le f(\Phi(A))[\Phi(A)]^{-1} - \Phi\left(f(A)A^{-1}\right) \\
& \le \frac{1}{4}(M-m)\left[\frac{f(M)}{M} - \frac{f(m)}{m}\right][\Phi(A)]^{-1} \\
& \le \frac{1}{4}\left(\frac{M}{m}-1\right)\left[\frac{f(M)}{M} - \frac{f(m)}{m}\right].
\end{aligned}
$$

If we take in (2.19) and (2.20) the operator convex function $f(t) = -\ln(t+1)$, *then we get*

$$0 \le \Phi\left(A^{-1}\ln(A+1)\right) - \ln(\Phi(A)+1)[\Phi(A)]^{-1} \le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\frac{\ln(m+1)}{m} - \frac{\ln(M+1)}{M}\right] \tag{2.21}$$

and

$$\begin{aligned} 0 &\le \Phi\left(A^{-1}\ln(A+1)\right) - \ln(\Phi(A)+1)[\Phi(A)]^{-1} \\ &\le \frac{1}{4}(M-m)\left[\frac{\ln(m+1)}{m} - \frac{\ln(M+1)}{M}\right][\Phi(A)]^{-1} \\ &\le \frac{1}{4}\left(\frac{M}{m}-1\right)\left[\frac{\ln(m+1)}{m} - \frac{\ln(M+1)}{M}\right] \end{aligned} \tag{2.22}$$

if $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ *and* A *is a positive operator on* H *satisfying the condition* $M \ge A \ge m > 0$.

From a different perspective we also have the upper bound in terms of *Jensen's difference*:

Theorem 5. *Let* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ *and* A *a positive operator on* H *satisfying the condition* $M \ge A \ge m > 0$. *Then*

$$\begin{aligned} 0 &\le \mathcal{M}(w,\mu)(\Phi(A)) - \Phi[\mathcal{M}(w,\mu)(A)] \\ &\le \mathcal{M}(w,\mu)\left(\frac{m+M}{2}\right) - \frac{\mathcal{M}(w,\mu)(m) + \mathcal{M}(w,\mu)(M)}{2}. \end{aligned} \tag{2.23}$$

Proof. As in the proof of Theorem 4, see (2.7),

$$\begin{aligned} 0 \le \Phi\left((\lambda+A)^{-1}\right) - [\Phi(\lambda+A)]^{-1} &\le \frac{\left(\sqrt{M+\lambda}-\sqrt{m+\lambda}\right)^2}{(m+\lambda)(M+\lambda)} \\ &= \frac{(M-m)^2}{\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2(m+\lambda)(M+\lambda)} \end{aligned} \tag{2.24}$$

for $\lambda \ge 0$.

Using the elementary inequality $\sqrt{a}+\sqrt{b} \ge \sqrt{a+b}$, a, $b \ge 0$ we deduce that

$$\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2 \ge M+m+2\lambda$$

for $\lambda \ge 0$, which implies that

$$\frac{(M-m)^2}{\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2(m+\lambda)(M+\lambda)} \le \frac{(M-m)^2}{(M+m+2\lambda)(m+\lambda)(M+\lambda)}.$$

We observe that, by performing the calculations, one has the equality

$$\begin{aligned} &\frac{1}{\left(\frac{M+m}{2}+\lambda\right)(m+\lambda)(M+\lambda)} \\ &= \frac{1}{(M-m)^2}\left(\frac{1}{m+\lambda} + \frac{1}{M+\lambda} - \frac{2}{\lambda+\frac{m+M}{2}}\right), \end{aligned}$$

for $\lambda \ge 0$.

Therefore

$$\frac{(M-m)^2}{\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2(m+\lambda)(M+\lambda)} \tag{2.25}$$
$$\leq \frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}-\frac{2}{\lambda+\frac{m+M}{2}}\right)$$
$$= \frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}\right)-\frac{1}{\lambda+\frac{m+M}{2}}.$$

for $\lambda \geq 0$.

If we use (2.6) then by (2.24) and (2.25) we get

$$\mathcal{M}(w,\mu)(\Phi(A)) - \Phi[\mathcal{M}(w,\mu)(A)] \tag{2.26}$$
$$= \int_0^\infty \lambda w(\lambda)\left(\Phi\left[(\lambda+T)^{-1}\right]-(\lambda+\Phi(A))^{-1}\right)d\mu(\lambda)$$
$$\leq \int_0^\infty \lambda w(\lambda)\left(\frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}\right)-\frac{1}{\lambda+\frac{m+M}{2}}\right)d\mu(\lambda).$$

Observe also that

$$\mathcal{M}(w,\mu)\left(\frac{m+M}{2}\right)-\frac{\mathcal{M}(w,\mu)(m)+\mathcal{M}(w,\mu)(M)}{2}$$
$$= \int_0^\infty w(\lambda)\left[1-\lambda\left(\frac{m+M}{2}+\lambda\right)^{-1}\right]d\mu(\lambda)$$
$$-\frac{1}{2}\int_0^\infty w(\lambda)\left[1-\lambda(m+\lambda)^{-1}\right]d\mu(\lambda)$$
$$-\frac{1}{2}\int_0^\infty w(\lambda)\left[1-\lambda(M+\lambda)^{-1}\right]d\mu(\lambda)$$
$$= \int_0^\infty \lambda w(\lambda)\left(\frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}\right)-\frac{1}{\lambda+\frac{m+M}{2}}\right)d\mu(\lambda)$$

and the inequality (2.23) is thus obtained. □

The case of operator monotone functions is as follows:

Corollary 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M\geq A\geq m>0$, then*

$$0\leq f(\Phi(A))-\Phi[f(A)]\leq f\left(\frac{m+M}{2}\right)-\frac{f(m)+f(M)}{2}. \tag{2.27}$$

Proof. From (1.11) we have

$$\mathcal{M}(\ell,\mu)(t)=f(t)-a-bt,$$

where $\ell(\lambda)=\lambda$, $a\in R$, $b\geq 0$ and μ is a positive measure.

Since

$$\begin{aligned}&\mathcal{M}(w,\mu)\left(\frac{m+M}{2}\right)-\frac{\mathcal{M}(w,\mu)(m)+\mathcal{M}(w,\mu)(M)}{2}\\&=f\left(\frac{m+M}{2}\right)-a-b\frac{m+M}{2}-\frac{1}{2}\left(f(m)-a-bm+f(M)-a-bM\right)\\&=f\left(\frac{m+M}{2}\right)-\frac{f(m)+f(M)}{2},\end{aligned}$$

hence by (2.23) we derive (2.27). □

Remark 3. *If we write the inequalities (2.27) for the operator monotone function $f(t)=t^r$, $r\in(0,1]$, then we get*

$$0\le\Phi^r(A)-\Phi(A^r)\le\left(\frac{m+M}{2}\right)^r-\frac{m^r+M^r}{2}\tag{2.28}$$

if $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M\ge A\ge m>0$.

If we write the inequalities (2.27) for the operator monotone function $f(t)=\ln t$, then we get the logarithmic inequalities

$$0\le\ln\Phi(A)-\Phi(\ln A)\le\ln\left(\frac{m+M}{2\sqrt{mM}}\right).\tag{2.29}$$

The case of operator convex functions is as follows:

Corollary 4. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$. If $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M\ge A\ge m>0$ then*

$$\begin{aligned}0&\le f(\Phi(A))[\Phi(A)]^{-1}-\Phi\left(f(A)A^{-1}\right)+f(0)\left(\Phi\left(A^{-1}\right)-[\Phi(A)]^{-1}\right)\\&\le f\left(\frac{m+M}{2}\right)\left(\frac{m+M}{2}\right)^{-1}-\frac{f(m)m^{-1}+f(M)M^{-1}}{2}\\&+f(0)\frac{(M-m)^2}{2mM(m+M)}.\end{aligned}\tag{2.30}$$

Proof. We have by (1.13) that

$$\mathcal{M}(\ell,\mu)=\frac{f(t)-f(0)}{t}-f'_+(0)-ct$$

for some positive measure μ, where $\ell(t)=t$, $t>0$ and $c\ge0$.

Since

$$\begin{aligned}
&\mathcal{M}(w,\mu)\left(\frac{m+M}{2}\right)-\frac{\mathcal{M}(w,\mu)(m)+\mathcal{M}(w,\mu)(M)}{2}\\
&=\frac{f\left(\frac{m+M}{2}\right)-f(0)}{\frac{m+M}{2}}-f_{+}'(0)-c\frac{m+M}{2}\\
&-\frac{1}{2}\left(\frac{f(m)-f(0)}{m}-f_{+}'(0)-cm\right)\\
&-\frac{1}{2}\left(\frac{f(M)-f(0)}{M}-f_{+}'(0)-cM\right)\\
&=f\left(\frac{m+M}{2}\right)\left(\frac{m+M}{2}\right)^{-1}-\frac{f(m)m^{-1}+f(M)M^{-1}}{2}\\
&-f(0)\left[\frac{2}{m+M}-\frac{1}{2}\left(\frac{1}{m}+\frac{1}{M}\right)\right]\\
&=f\left(\frac{m+M}{2}\right)\left(\frac{m+M}{2}\right)^{-1}-\frac{f(m)m^{-1}+f(M)M^{-1}}{2}\\
&+f(0)\frac{(M-m)^2}{2mM(m+M)},
\end{aligned}$$

hence by (2.23) we derive (2.30). □

Remark 4. *With the assumptions of Corollary 4 and if $f(0)=0$, then we get from (2.30) that*

$$\begin{aligned}
(2.31)\qquad 0&\le f(\Phi(A))[\Phi(A)]^{-1}-\Phi\left(f(A)A^{-1}\right)\\
&\le f\left(\frac{m+M}{2}\right)\left(\frac{m+M}{2}\right)^{-1}-\frac{f(m)m^{-1}+f(M)M^{-1}}{2}.
\end{aligned}$$

If we take in (2.31) the operator convex function $f(t)=-\ln(t+1)$, then we get

$$\begin{aligned}
(2.32)\qquad 0&\le \Phi\left(A^{-1}\ln(A+1)\right)-\ln(\Phi(A)+1)[\Phi(A)]^{-1}\\
&\le\frac{m^{-1}\ln(m+1)+M^{-1}\ln(M+1)}{2}\\
&-\left(\frac{m+M}{2}\right)^{-1}\ln\left(\frac{m+M}{2}+1\right),
\end{aligned}$$

if $\Phi\in\mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M\ge A\ge m>0$.

3. Some Examples

Let $P_j\in\mathcal{B}(H)$, $j=1,...,k$ be contractions with

$$(3.1)\qquad \sum_{j=1}^{k}P_j^{*}P_j=1_H.$$

The map $\Phi:\mathcal{B}(H)\to\mathcal{B}(H)$ defined by [6]

$$\Phi(A):=\sum_{j=1}^{k}P_j^{*}AP_j$$

is a *normalized positive linear map* on $\mathcal{B}(H)$.

Assume that A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$ for some scalars $m < M$. Then by Theorem 4 we get the following reverses of operator Jensen's inequality

$$0 \leq \mathcal{M}(w,\mu)\left(\sum_{j=1}^{k} P_j^* A P_j\right) - \sum_{j=1}^{k} P_j^* \mathcal{M}(w,\mu)(A) P_j \leq \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)\right] \tag{3.2}$$

and

$$\begin{aligned} 0 &\leq \mathcal{M}(w,\mu)\left(\sum_{j=1}^{k} P_j^* A P_j\right) - \sum_{j=1}^{k} P_j^* \mathcal{M}(w,\mu)(A) P_j \\ &\leq \frac{1}{4}(M-m)\left[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)\right]\left(\sum_{j=1}^{k} P_j^* A P_j\right)^{-1} \\ &\leq \frac{1}{4}\left(\frac{M}{m}-1\right)\left[\mathcal{M}(w,\mu)(M) - \mathcal{M}(w,\mu)(m)\right], \end{aligned} \tag{3.3}$$

while by Theorem 5 we get

$$\begin{aligned} 0 &\leq \mathcal{M}(w,\mu)\left(\sum_{j=1}^{k} P_j^* A P_j\right) - \sum_{j=1}^{k} P_j^* \mathcal{M}(w,\mu)(A) P_j \\ &\leq \mathcal{M}(w,\mu)\left(\frac{m+M}{2}\right) - \frac{\mathcal{M}(w,\mu)(m) + \mathcal{M}(w,\mu)(M)}{2}. \end{aligned} \tag{3.4}$$

Further, if we assume that the function $f:(0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ and A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$, then by Corollary 1 we get

$$0 \leq f\left(\sum_{j=1}^{k} P_j^* A P_j\right) - \sum_{j=1}^{k} P_j^* f(A) P_j \leq \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[f(M) - f(m)\right] \tag{3.5}$$

and

$$\begin{aligned} 0 &\leq f\left(\sum_{j=1}^{k} P_j^* A P_j\right) - \sum_{j=1}^{k} P_j^* f(A) P_j \\ &\leq \frac{1}{4}(M-m)\left[f(M) - f(m)\right]\left(\sum_{j=1}^{k} P_j^* A P_j\right)^{-1} \\ &\leq \frac{1}{4}\left(\frac{M}{m}-1\right)\left[f(M) - f(m)\right]. \end{aligned} \tag{3.6}$$

From Corollary 3 we obtain

$$0 \le f\left(\sum_{j=1}^{k} P_j^* A P_j\right) - \sum_{j=1}^{k} P_j^* f(A) P_j \le f\left(\frac{m+M}{2}\right) - \frac{f(m)+f(M)}{2}.$$

Assume that $f : [0,\infty) \to \mathbb{R}$ is operator convex in $[0,\infty)$ with $f(0) = 0$ and A is a positive operator on H satisfying the condition $M \ge A \ge m > 0$, then by (2.19) and (2.20) we have

$$\begin{aligned} 0 &\le f\left(\sum_{j=1}^{k} P_j^* A P_j\right)\left(\sum_{j=1}^{k} P_j^* A P_j\right)^{-1} - \sum_{j=1}^{k} P_j^* f(A) A^{-1} P_j \\ &\le \frac{M-m}{\left(\sqrt{M}+\sqrt{m}\right)^2}\left[\frac{f(M)}{M} - \frac{f(m)}{m}\right] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} 0 &\le f\left(\sum_{j=1}^{k} P_j^* A P_j\right)\left(\sum_{j=1}^{k} P_j^* A P_j\right)^{-1} - \sum_{j=1}^{k} P_j^* f(A) A^{-1} P_j \\ &\le \frac{1}{4}(M-m)\left[\frac{f(M)}{M} - \frac{f(m)}{m}\right]\left(\sum_{j=1}^{k} P_j^* A P_j\right)^{-1} \\ &\le \frac{1}{4}\left(\frac{M}{m} - 1\right)\left[\frac{f(M)}{M} - \frac{f(m)}{m}\right]. \end{aligned} \tag{3.8}$$

From (2.31) we get

$$\begin{aligned} 0 &\le f\left(\sum_{j=1}^{k} P_j^* A P_j\right)\left(\sum_{j=1}^{k} P_j^* A P_j\right)^{-1} - \sum_{j=1}^{k} P_j^* f(A) A^{-1} P_j \\ &\le f\left(\frac{m+M}{2}\right)\left(\frac{m+M}{2}\right)^{-1} - \frac{f(m)m^{-1} + f(M)M^{-1}}{2}. \end{aligned} \tag{3.9}$$

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