On weighted fractional inequalities using generalized Katugampola fractional integral operator

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Abstract

In this paper, we obtain some new weighted fractional inequalities which is presented by M. Houas in the paper(Certain weighted integral inequalities involving fractional hypergeometric operator, Scientia, series A: Mathematical Science 27(2016),87-97), using generalized Katugampola fractional integral operator.

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1 Introduction

Fractional Calculus is traditionally associated to non-integers where the order of derivative or integral is considered to be non-integer. Fractional calculus have found significant important and application science and technology. The fractional integral inequalities gained more attention due to it's application in continuation solution, uniqueness of solution of fractional differential solutions. During the past few years,, many researchers have investigated well known fractional integral inequalities and its application using Riemann-Liouville, Hadamard, Saigo, Erdélyi-Kober, Katugampola, k-fractional integral, k-Hadamard integral and generalized k-fractional integral, see [2, 3, 4, 5, 6, 9, 10, 12, 13, 15, 16, 18, 19, 22]. In [8], Curiel and Galue introduced the Gauss hypergeometric function operator. Recently, V. L. Chinchane, et al. [4], Baleanu et al. [3] proposed fractional integral inequalities using the generalized k-fractional integral operator in terms of the Gauss hypergeometric functions. In [11], M. Houas obtained certain weighted integral inequalities involving the fractional hypergeometric operators.

Recently, A. B. Nale, et al. [17] investigated new fractional integral inequalities for convex function using generalized Katugampola fractional integral. T. A Aljaaidi, et al. [1] and J. V. Sousa et al. [23] have established Grüss-type inequalities using generalized Katugampola fractional integral. E. Set, el al.[20] established several Chebyshev type inequalities using generalized Katugampola fractional integral operator. Motivated from above work, the aim of this paper is to obtain some new weighted fractional integral inequalities using generalized Katugampola fractional integral operators. The paper has been organized as follows. In Section 2, we define basic definitions and proposition related to generalized Katugampola fractional derivatives and integrals. In Section 3, we give weighted fractional integral inequalities by employing generalized Katugampola fractional integral operator. In section 4, we give concluding remarks.

2 Preliminaries

In this section, we give some basic definition and mathematical preliminaries of Generalized Katugampola fractional integral, see [1, 13, 20, 23].

Definition 2.1 *Consider the space $X_c^p(a,b)(c \in \mathbb{R}, 1 \leq p \leq \infty)$, of those complex valued Lebesgue measurable functions f on (a,b) for which the norm $||f||_{X_c^p} < \infty$, such that*

$$||f||_{X_c^p} = \left(\int_x^b |x^c f|^p \frac{dx}{x}\right)^{\frac{1}{p}}, (1 \leq p < \infty),$$

and

$$||f||_{X_c^p} = supess_{x\in(a,b)}[x^c|f|].$$

In particular, when $c = \frac{1}{p}$, the space $X_c^p(a,b)$ coincides with the space $L^p(a,b)$.

Definition 2.2 *The left and right sided fractional integrals of a function f where $f \in x_c^p(a,b)$, $\alpha > 0$ and $\beta, \rho, \eta, k \in \mathbb{R}$, are defined respectively by*

$${}^{\rho}\mathcal{J}_{a+;\eta,k}^{\alpha,\beta} f(x) = \frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau)d\tau, \;\; 0 \leq a < x < b \leq \infty, \quad (2.1)$$

and

$${}^{\rho}\mathcal{J}_{b-;\eta,k}^{\alpha,\beta} f(x) = \frac{\rho^{1-\beta}x^{k\rho}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{k+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} f(\tau)d\tau, \;\; 0 \leq a < x < b \leq < b \leq \infty, \quad (2.2)$$

if the integral exist.
To represent and discuss our new results in this paper we use the left sided fractional integrals, the right sided fractional can be proved similarly, also we consider $a = 0$, in (2.1), to obtain

$$ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta} f(x) = \frac{\rho^{1-\beta} x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f(\tau) d\tau. \tag{2.3} $$

The above fractional integrals has the following composition (index) formulae

$$ \begin{aligned} {}^{\rho}\mathcal{J}_{a+;\eta_1,k_1}^{\alpha_1,\beta_1}\, {}^{\rho}\mathcal{J}_{a+;\eta_2,-\rho\eta_1}^{\alpha_2,\beta_2} f(x) &= {}^{\rho}\mathcal{J}_{a+;\eta_2,k_1}^{\alpha_1+\alpha_2,\beta_1+\beta_2} f(x), \\ {}^{\rho}\mathcal{J}_{b-;\eta_1,-\rho\eta_2}^{\alpha_1,\beta_1}\, {}^{\rho}\mathcal{J}_{b-;\eta_2,k_2}^{\alpha_2,\beta_2} f(x) &= {}^{\rho}\mathcal{J}_{a+;\eta_1,k_2}^{\alpha_1+\alpha_2,\beta_1+\beta_2} f(x). \end{aligned} \tag{2.4} $$

Here, we recall definition of beta function as:

Definition 2.3 *The beta function $B(\alpha, \beta)$ is defined ([21], section 1.1)*

$$ B(\alpha,\beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt & (\mathbb{R}(\alpha) > 0; \mathbb{R}(\beta) > 0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha, \beta \in \mathbb{C} / \mathbb{Z}_0^-) \end{cases} \tag{2.5} $$

For the convenience of establishing our results we define the following function as in [20, 23], using (2.5) and let $x > 0, \alpha > 0, \rho, k, \beta, \eta \in \mathbb{R}$, then

$$ \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) = \frac{\rho^{1-\beta} x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} \rho^{-\beta} x^{k+\rho(\eta+\alpha)}. \tag{2.6} $$

Remark 2.1 *The fractional integral (2.1) contain five well-known fractional integral as its particular cases, see [1, 12, 13, 17, 20, 23]*

1. *Setting $k = 0, \eta = 0, a = 0$ and taking the limit $\rho \to 1$ in (2.1), the integral operator (2.1) reduces to the Riemann-Liouville fractional integral.*

2. *Setting $k = 0, \eta = 0$ and taking the limit $\rho \to 1$ in (2.1), the integral operator (2.1) reduces to the Liouville fractional integral.*

3. *Setting $\beta = \alpha, k = 0, \eta = 0$ and taking the limit $\rho \to 0^+$ with L' Hospital rule in (2.1), the integral operator (2.1) reduces to the Hadamard fractional integral see [14].*

4. *Setting $\beta = 0, k = -\rho(\alpha+\eta)$ in (2.1), the integral operator (2.1) reduces to the Erdelyi-Kober fractional integral.*

5. *Setting $\beta = \alpha, k = 0$ and $\eta = 0$ in (2.1), the integral operator (2.1) reduces to the Katugampola fractional integral.*

3 Main Result

Here, we obtain new fractional integral inequalities using generalized Katugampola fractional integral operators.

Theorem 3.1 *Let* f *be positive and continuous functions on* $[0,\infty)$*, such that*

$$(\sigma^{\xi}\mathrm{f}^{\varpi}(\tau)-\tau^{\varpi}\mathrm{f}^{\varpi}(\sigma))(\mathrm{f}^{\varpi-\lambda}(\tau)-\mathrm{f}^{\varpi-\lambda}(\sigma))\geq 0, \tag{3.1}$$

and $w:[0,\infty)\to\mathbb{R}^{+}$ *be positive continuous function. Then for all* $\alpha\geq 0$, $x>0$, $\beta,\rho,\eta,k\in\mathbb{R}$ *we have,*

$$\begin{aligned}&{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[w(x)\mathrm{f}^{\xi+\lambda}(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[w(x)x^{\xi}\mathrm{f}^{\varpi}(x)]\\ &\leq {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[w(x)\mathrm{f}^{\xi+\varpi}(x)]\,{}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[w(x)x^{\varpi}\mathrm{f}^{\lambda}(x)]\end{aligned} \tag{3.2}$$

Proof:- Since f be positive and continuous functions on $[0,\infty)$, then for all $\xi>0,\varpi\geq 0,\lambda>0$, $\tau,\sigma\in(0,x),x>0$, then From (3.2),

$$\begin{aligned}&\sigma^{\xi}\mathrm{f}^{\varpi-\lambda}(\sigma)\mathrm{f}^{\varpi}(\tau)+\tau^{\varpi}\mathrm{f}^{\varpi-\lambda}(\tau)\mathrm{f}^{\varpi}(\sigma)\\ &\leq\sigma^{\xi}\mathrm{f}^{\varpi+\xi-\lambda}(\tau)+\tau^{\varpi}\mathrm{f}^{\varpi+\xi-\lambda}(\sigma).\end{aligned} \tag{3.3}$$

Multiplying both side of equation (3.3) by $\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}w(\tau)\mathrm{f}^{\lambda}(\tau),\tau\in(0,x),x>0$ which is positive, and integrating the obtain result with respective to τ from 0 to x, we get

$$\begin{aligned}&\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\sigma^{\xi}\mathrm{f}^{\varpi-\lambda}(\sigma)w(\tau)\mathrm{f}^{\lambda+\varpi}(\tau)d\tau\\ &+\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\tau^{\varpi}\mathrm{f}^{\varpi-\lambda}(\tau)\mathrm{f}^{\varpi}(\sigma)w(\tau)\mathrm{f}^{\lambda}(\tau)d\tau\\ &\leq\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}x)\sigma^{\xi}\mathrm{f}^{\varpi+\xi-\lambda}(\tau)w(\tau)\mathrm{f}^{\lambda}(\tau)d\tau\\ &+\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\tau^{\varpi}\mathrm{f}^{\varpi+\xi-\lambda}(\sigma)w(\tau)\mathrm{f}^{\lambda}(\tau)d\tau,\end{aligned} \tag{3.4}$$

consequently,

$$\begin{aligned}&\sigma^{\xi}\mathrm{f}^{\varpi-\lambda}(\sigma)\ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[w(x)\mathrm{f}^{\lambda+\varpi}(x)]+\mathrm{f}^{\varpi}(\sigma)\ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[w(x)\tau^{\varpi}\mathrm{f}^{\varpi}(x)]\\ &\geq\sigma^{\xi}\ \ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[w(x)\sigma^{\xi}\mathrm{f}^{\xi+\varpi}(x)]+\mathrm{f}^{\xi+\varpi-\lambda}(\sigma)\ {}^{\rho}\mathcal{J}^{\alpha,\beta}_{\eta,k}[wx\mathrm{f}^{\lambda}(x)].\end{aligned} \tag{3.5}$$

Multiplying both side of equation (3.5) by $\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}w(\sigma)\mathrm{f}^{\lambda}(\sigma),\sigma\in(0,x),x>0$ which is positive, and integrating the obtain result with respective to σ from 0 to x, we get

$$
\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi+\lambda}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\mathrm{f}^{\varpi}(x)]\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\mathrm{f}^{\varpi}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi+\lambda}(x)]\\
&\geq\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi+\lambda}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\mathrm{f}^{\lambda}(x)]\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\mathrm{f}^{\lambda}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi+\lambda}(x)].
\end{aligned}
\tag{3.6}
$$

This completes the proof of inequality 3.1.
Now, we give our main result.

Theorem 3.2 *Let* f *be two positive and continuous functions on* $[0,\infty)$ *and satisfies (3.1). Let* $w:[0,\infty)\to\mathbb{R}^{+}$ *be positive continuous function. Then for all* $\alpha,\theta\geq 0$, $x>0$, $\beta,\pi,\rho,\eta,k\in\mathbb{R}$ *we have,*

$$
\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)x^{\xi}\mathrm{f}^{\varpi}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi+\lambda}(x)]\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\mathrm{f}^{\varpi}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}^{\xi+\lambda}(x)]\\
&\leq\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)x^{\xi}\mathrm{f}^{\lambda}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi+\varpi}(x)]\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\mathrm{f}^{\lambda}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}^{\xi+\varpi}(x)].
\end{aligned}
\tag{3.7}
$$

Proof:- Now multiplying both sides of (3.3) by $\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}w(\sigma)\mathrm{f}^{\lambda}(\sigma),\sigma\in(0,x),x>0$, this function remains positive under the conditions stated with the theorem. Integrating the obtain result with respective to σ from 0 to x, we get

$$
\begin{aligned}
&\frac{\mathrm{f}^{\varpi}(\tau)\,\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\pi}}w(\sigma)\sigma^{\xi}\mathrm{f}^{\varpi}(\sigma)d\sigma\\
&+\frac{\tau^{\xi}\mathrm{f}^{\varpi-\lambda}(\tau)\,\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\pi}}\tau^{\varpi}w(\sigma)\mathrm{f}^{\xi+\lambda}(\sigma)d\sigma\\
&\leq\frac{\mathrm{f}^{\varpi+\xi-\lambda}(\tau)\,\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\pi}}w(\sigma)\sigma^{\xi}\mathrm{f}^{\lambda}d\sigma\\
&+\frac{\tau^{\varpi}\,\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\pi}}w(\sigma)\mathrm{f}^{\varpi+\xi}(\sigma)d\sigma,
\end{aligned}
\tag{3.8}
$$

consequently

$$
\begin{aligned}
&\mathrm{f}^{\varpi}(\tau)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)x^{\xi}\mathrm{f}^{\varpi}(x)]+\sigma^{\xi}\mathrm{f}^{\varpi-\lambda}(x)\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}^{\xi+\lambda}(x)]\\
&\geq\mathrm{f}^{\xi+\varpi-\lambda}(\tau)\ \ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)x^{\xi}\mathrm{f}^{\varpi}(x)]+x^{\xi}\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}^{\xi+\varpi}(x)].
\end{aligned}
\tag{3.9}
$$

Multiplying both side of equation (3.9) by $\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}w(\tau)\mathrm{f}^{\lambda}(\tau),\tau\in(0,x),x>0$ which is positive, and integrating the obtain result with respective to σ from 0 to x, we get

$$
\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)x^{\xi}\mathrm{f}^{\varpi}(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\tau^{\xi}w(\tau)\mathrm{f}^{\xi+\lambda}(\tau)d\tau\\
&+^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}^{\xi+\lambda}(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}w(\tau)\tau^{\xi}\mathrm{f}^{\varpi}(\tau)d\tau\\
&\leq^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)x^{\xi}\mathrm{f}^{\lambda}(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}w(\tau)\mathrm{f}^{\varpi+\xi}(\tau)d\tau\\
&+^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[wx\mathrm{f}^{\xi+\varpi}(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}w(\tau)\tau^{\xi}\mathrm{f}^{\lambda}(\tau)d\tau,
\end{aligned}
\tag{3.10}
$$

This complete the proof of theorem 3.2.

Theorem 3.3 *Let* f *and* h *be positive and continuous functions on* $[0,\infty)$, *such that*

$$
(\mathrm{h}^{\xi}(\sigma)\mathrm{f}^{\xi}(\tau)-\mathrm{h}^{\xi}(\tau)\mathrm{f}^{\xi}(\sigma))(\mathrm{f}^{\varpi-\lambda}(\tau)-\mathrm{f}^{\varpi-\lambda}(\sigma))\geq 0, \tag{3.11}
$$

and let $w:[0,\infty)\to\mathbb{R}^{+}$ *be positive continuous function. Then for all* $\alpha\geq 0$, $x>0$, $\beta,\rho,\eta,k\in\mathbb{R}$ *we have,*

$$
\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x)]\\
&\leq\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\varpi+\xi}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\lambda}(x)].
\end{aligned}
\tag{3.12}
$$

Proof:- Let $(\tau,\sigma)\in(0,\sigma)$,$x>0$, for any $\varpi>\lambda>0$, $\xi>0$. Then from (3.11)

$$
(\mathrm{h}^{\xi}(\sigma)\mathrm{f}^{\varpi-\lambda}(\sigma)\mathrm{f}^{\xi}(\tau))+\mathrm{h}^{\xi}(\tau)\mathrm{f}^{\xi}(\sigma)\mathrm{f}^{\varpi-\lambda}(\tau)\leq(\mathrm{h}^{\xi}(\sigma)\mathrm{f}^{\varpi+\xi-\lambda}(\tau)+\mathrm{h}^{\xi}(\tau)\mathrm{f}^{\varpi+\xi-\lambda}(\sigma). \tag{3.13}
$$

Multiplying of both sides of (3.13) by $\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}w(\tau)\mathrm{f}^{\lambda}(\tau)$, then integrating the resulting inequality with respect to τ over $(0,x)$, we obtain

$$
\begin{aligned}
&\mathrm{h}^{\xi}(\sigma)\mathrm{f}^{\varpi-\lambda}(\sigma){}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi}(x)]\ +\mathrm{f}^{\xi}(\sigma){}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x)]\\
&\leq\ \mathrm{h}^{\xi}(\sigma){}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\varpi+\xi}(x)]+\mathrm{f}^{\xi+\varpi-\lambda}(\sigma){}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\lambda}(x)].
\end{aligned}
\tag{3.14}
$$

Multiplying of both sides of (3.14) by $\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}w(\sigma)\mathrm{f}^{\lambda}(\sigma)$, then integrating the resulting inequality with respect to σ over $(0, x)$, we obtain

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi}(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}w(\sigma)\mathrm{h}^{\xi}(\sigma)\mathrm{f}^{\varpi}(\sigma)d\sigma \\
&+^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}\mathrm{f}^{\xi}(\sigma)w(\sigma)d\sigma \\
&\leq {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\varpi+\xi}(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}x)\mathrm{f}^{\lambda}(\sigma)w(\sigma)\mathrm{h}^{\xi}(\sigma)d\sigma \\
&+^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\lambda}(x)]\frac{\rho^{1-\beta}x^{k}}{\Gamma(\alpha)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\alpha}}w(\sigma)\mathrm{f}^{\varpi+\xi}(\sigma)d\sigma,
\end{aligned}\tag{3.15}$$

which implies that

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x) \\
&+^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[\mathrm{f}^{\xi}(x)w(x)] \\
&\leq {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\varpi+\xi}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[\mathrm{f}^{\lambda}(x)w(x)\mathrm{h}^{\xi}(x)] \\
&+^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\lambda}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\varpi+\xi}(x)],
\end{aligned}\tag{3.16}$$

this complete proof of inequality 3.11.

Theorem 3.4 *Let* f *and* h *be two positive and continuous functions on* $[0, \infty)$ *and satisfying (3.11). Let* $w : [0, \infty) \to \mathbb{R}^{+}$ *be positive continuous function. Then for all* $\alpha, \theta \geq 0$, $x > 0$, $\beta, \pi, \rho, \eta, k \in \mathbb{R}$ *we have,*

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi}(x)] \\
&+^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}^{\xi}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x)] \\
&\leq {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}\mathrm{f}^{\lambda}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\varpi+\xi}(x)] \\
&+^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}^{\xi+\varpi-\lambda}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\lambda}(x)].
\end{aligned}\tag{3.17}$$

Proof:- Multiplying the inequality (3.14) by $\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}w(\sigma)\mathrm{f}^{\lambda}(\sigma), \sigma \in (0, x), x > 0$, this function remains positive under the conditions stated with the theorem. Integrating the obtain result with respective to σ from 0 to x, we get

$$
\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi}(x)]\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}w(\sigma)\mathrm{h}^{\xi}(\sigma)\mathrm{f}^{\varpi}(\sigma)d\sigma\\
&+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x)]\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}\mathrm{f}^{\xi}(\sigma)w(\sigma)d\sigma\\
&\leq {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\varpi+\xi}(x)]\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}x)\mathrm{f}^{\lambda}(\sigma)w(\sigma)\mathrm{h}^{\xi}(\sigma)d\sigma\\
&+{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\lambda}(x)]\frac{\rho^{1-\pi}x^{k}}{\Gamma(\theta)}\int_{0}^{x}\frac{\sigma^{\rho(\eta+1)-1}}{(x^{\rho}-\sigma^{\rho})^{1-\theta}}w(\sigma)\mathrm{f}^{\varpi+\xi}(\sigma)d\sigma,
\end{aligned}
\tag{3.18}
$$

which implies that,

$$
\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\xi}(x)]\\
&+{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}^{\xi}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\varpi}(x)]\\
&\leq {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}\mathrm{f}^{\lambda}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}^{\varpi+\xi}(x)]\\
&+{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}^{\xi+\varpi-\lambda}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}^{\lambda}(x)],
\end{aligned}
\tag{3.19}
$$

hence result is proved.
Next, we shall propose a new generalization of weighted fractional integral inequalities using a family of n positive functions defined on $[0,\infty)$.

Theorem 3.5 *Let* $\mathrm{f_i}, \mathrm{i}=1,...,\mathrm{n}$ *be* n *positive and continuous functions on* $[0,\infty)$ *such that*

$$
(\sigma^{\xi}\mathrm{f}_{r}^{\xi}(\tau)-\tau^{\xi}\mathrm{f}_{r}^{\xi}(\sigma))(\mathrm{f}_{r}^{\varpi-\lambda_{r}}(\tau)-\mathrm{f}^{\varpi-\lambda_{r}}(\sigma))\geq 0. \tag{3.20}
$$

Let $w:[0,\infty)\to\mathbb{R}^{+}$*. Then for all* $\alpha\geq 0$*,* $x>0$*,* $\beta,\rho,\eta,k\in\mathbb{R}$*,* $\xi>0$*,* $\varpi\geq\lambda_{r}>0, r\in\{1,...,n\}$*, the following fractional inequality*

$$
\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_{r}^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_{i}^{\lambda_{i}}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\mathrm{f}_{r}^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_{i}^{\lambda_{i}}(x)]\\
&\leq {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\Pi_{i=1}^{n}\mathrm{f}_{i}^{\lambda_{i}}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_{r}^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_{i}^{\lambda_{i}}(x)].
\end{aligned}
\tag{3.21}
$$

is valid.

proof:- Suppose $\mathrm{f_i}, \mathrm{i}=1,...,\mathrm{n}$ be n positive and continuous functions on $[0,\infty)$, then for any fixed $r\in\{1,...,n\}$ and for any $\xi>0,\varpi\geq\lambda_{r}>0$, $\tau,\sigma\in(0,x)$, $x>0$. From (3.20), we obtain

$$
\begin{aligned}
&\sigma^{\xi}\mathrm{f}_{r}^{\varpi-\lambda_{r}}(\sigma)\mathrm{f}_{r}^{\xi}(\tau)+\tau^{\xi}\mathrm{f}_{r}^{\xi}(\sigma)\mathrm{f}_{r}^{\varpi-\lambda_{r}}(\tau)\\
&\leq \sigma^{\xi}\mathrm{f}_{r}^{\varpi+\xi-\lambda_{r}}(\tau)+\tau^{\xi}\mathrm{f}_{r}^{\varpi+\xi-\lambda_{r}}(\sigma),
\end{aligned}
\tag{3.22}
$$

Now multiplying both sides of (3.22) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}w(\tau)\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(\tau)$, then integrating the resulting inequality with respect to τ over $(0,x)$, we obtain

$$\begin{aligned}
&\sigma^\xi \mathrm{f}_r^{\varpi-\lambda_r}(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}[w(\tau)\mathrm{f}_r^\xi(\tau)\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(\tau)]d\tau \\
&+\mathrm{f}^\xi(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}[w(\tau)\tau^\xi\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(\tau)]d\tau \\
&\leq \sigma^\xi\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}[w(\tau)\mathrm{f}_r^{\varpi+\xi}(\tau)\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(\tau)]d\tau \\
&+\mathrm{f}_r^{\varpi+\xi-\lambda_r}(\sigma)\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}[w(\tau)\tau^\xi\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(\tau)]d\tau,
\end{aligned}\tag{3.23}$$

consequently

$$\begin{aligned}
&\sigma^\xi \mathrm{f}_r^{\varpi-\lambda_r}(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^\xi(x)\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(x)] \\
&+\mathrm{f}^\xi(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^\xi\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(x)] \\
&\leq \sigma^\xi\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(x)] \\
&+\mathrm{f}_r^{\varpi+\xi-\lambda_r}(\sigma)\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^\xi\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(x)].
\end{aligned}\tag{3.24}$$

Again, multiplying the inequality (3.24) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}w(\sigma)\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(\sigma), \sigma\in (0,x), x>0$, this function remains positive under the conditions stated with the theorem. Integrating the obtain result with respective to σ from 0 to x, we get

$$\begin{aligned}
&{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^\xi(x)\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(x)]\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}w(\sigma)\sigma^\xi\mathrm{f}_r^\varpi(\sigma)\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(\sigma)d\sigma \\
&+\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^\xi\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(x)]\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}w(\sigma)\mathrm{f}_r^\xi(\sigma)\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(\sigma)d\sigma \\
&\leq\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(x)]\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}w(\sigma)\sigma^\xi\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(\sigma)d\sigma \\
&+\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^\xi\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(x)]\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\int_0^x \frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}w(\sigma)\mathrm{f}_r^{\varpi+\xi}(\sigma)\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(\sigma)d\sigma,
\end{aligned}\tag{3.25}$$

therefore,

$$\begin{aligned}
&{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^\xi(x)\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(x)]\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[x^\xi\mathrm{f}_r^\varpi(x)\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(x)] \\
&+\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^\xi\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(x)]\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^\xi(x)\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(x)] \\
&\leq\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(x)]\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^\xi\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(x)] \\
&+\ {}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^\xi\Pi_{i=1}^n \mathrm{f}_i^{\lambda_i}(x)]\,{}^\rho\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^n \mathrm{f}_i^{\lambda_i}(x)].
\end{aligned}\tag{3.26}$$

This ends the inequality 3.20.

Theorem 3.6 *Let* $\mathrm{f_i}, \mathrm{i} = 1, ..., \mathrm{n}$ *be* n *positive and continuous functions on* $[0, \infty)$*and satisfying (3.20). Let* $w : [0, \infty) \to \mathbb{R}^+$*. Then for all* $\alpha, \theta \geq 0$, $x > 0$,$\beta, \pi, \rho, \eta, k \in \mathbb{R}$, $\xi > 0$, $\varpi \geq \lambda_r > 0, r \in \{1, ..., n\}$, *then we have inequality*

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)x^{\xi}\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\leq\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)x^{\xi}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)].
\end{aligned} \tag{3.27}$$

is valid.

Proof:- We multiplying the inequality (3.24) by $\frac{\rho^{1-\pi}x^k}{\Gamma(\theta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\theta}}w(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma), \sigma \in (0, x), x > 0$, this function remains positive under the conditions stated with the theorem. Integrating the obtain result with respective to σ from 0 to x, we get

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\frac{\rho^{1-\pi}x^k}{\Gamma(\theta)}\int_0^x\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\theta}}w(\sigma)\sigma^{\xi}\mathrm{f}_r^{\varpi}(\sigma)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\frac{\rho^{1-\pi}x^k}{\Gamma(\theta)}\int_0^x\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\theta}}w(\sigma)\mathrm{f}_r^{\xi}(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma\\
&\leq\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\frac{\rho^{1-\pi}x^k}{\Gamma(\theta)}\int_0^x\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\theta}}w(\sigma)\sigma^{\xi}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma\\
&+\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)x^{\xi}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\frac{\rho^{1-\pi}x^k}{\Gamma(\theta)}\int_0^x\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\theta}}w(\sigma)\mathrm{f}_r^{\varpi+\xi}(\sigma)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma,
\end{aligned} \tag{3.28}$$

which gives the inequality (3.27).

Theorem 3.7 *Let* $\mathrm{f_i}, \mathrm{i} = 1, ..., \mathrm{n}$ *be* h *positive and continuous functions on* $[0, \infty)$*. such that*

$$(\mathrm{h}^{\xi}(\sigma)\mathrm{f}_r^{\xi}(\tau) - \mathrm{h}^{\xi}(\tau)\mathrm{f}_r^{\xi}(\sigma))(\mathrm{f}_r^{\varpi-\lambda_r}(\tau) - \mathrm{f}_r^{\varpi-\lambda_r}(\sigma)) \geq 0. \tag{3.29}$$

and let $w : [0, \infty) \to \mathbb{R}^+$*. Then for all* $\alpha \geq 0$, $x > 0$, $\beta, \rho, \eta, k \in \mathbb{R}$, $\xi > 0$, $\varpi \geq \lambda_r > 0, r \in \{1, ..., n\}$*, the following fractional inequality*

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\leq\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)].
\end{aligned} \tag{3.30}$$

is valid.

Proof:- Let $\tau, \sigma \in (0,x), x>0$, for any $\xi>0, \varpi \geq \lambda_i>0, r\in\{1,2,...n\}$ then from (3.29), we have

$$\begin{aligned} &\mathrm{h}^{\xi}(\sigma)\mathrm{f}_r^{\varpi-\lambda_r}(\sigma)\mathrm{f}_r^{\xi}(\tau)+\mathrm{f}_r^{\xi}(\sigma)\mathrm{h}_r^{\xi}(\tau)\mathrm{f}_r^{\varpi-\lambda_r}(\tau)\\ &\leq \mathrm{h}^{\xi}(\sigma)\mathrm{f}_r^{\varpi+\xi-\lambda_r}(\tau)+\mathrm{f}_r^{\varpi+\xi-\lambda_r}(\sigma)\mathrm{h}^{\xi}(\tau), \end{aligned} \tag{3.31}$$

multiplying both sides of (3.31) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}w(\tau)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\tau)$, then integrating the resulting inequality with respect to τ over $(0,x)$, we obtain

$$\begin{aligned} &\mathrm{h}^{\xi}(\sigma)\mathrm{f}_r^{\varpi-\lambda_r}(\sigma)^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &+\mathrm{f}_r^{\xi}(\sigma)\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &\leq \mathrm{h}^{\xi}(\sigma)\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &+\mathrm{f}_r^{\varpi+\xi-\lambda_r}(\sigma)^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]. \end{aligned} \tag{3.32}$$

Multiplying both sides of (3.32) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\alpha}}w(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)$, then integrating the resulting inequality with respect to σ over $(0,x)$, we have

$$\begin{aligned} &2\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &\leq 2\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]. \end{aligned} \tag{3.33}$$

This complete the proof of theorem 3.7.

Theorem 3.8 *Let* $\mathrm{f_i}, \mathrm{i}=1,...,\mathrm{n}$ *be* h *positive and continuous functions on* $[0,\infty)$. *such that*

$$(\mathrm{h}^{\xi}(\sigma)\mathrm{f}_r^{\xi}(\tau)-\mathrm{h}^{\xi}(\tau)\mathrm{f}_r^{\xi}(\sigma))(\mathrm{f}_r^{\varpi-\lambda_r}(\tau)-\mathrm{f}_r^{\varpi-\lambda_r}(\sigma))\geq 0, \tag{3.34}$$

and let $w:[0,\infty)\to\mathbb{R}^+$. *Then for all* $\alpha,\theta\geq 0$, $x>0$,$\beta,\pi,\rho,\eta,k\in\mathbb{R}$, $\xi>0$, $\varpi\geq\lambda_r>0, r\in\{1,...,n\}$, *the following fractional inequality*

$$\begin{aligned} &{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]+\\ &{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &\leq\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)], \end{aligned} \tag{3.35}$$

is valid.

Proof:- Multiplying both sides of (3.31) by $\frac{\rho^{1-\pi}x^k}{\Gamma(\theta)}\frac{\sigma^{\rho(\eta+1)-1}}{(x^\rho-\sigma^\rho)^{1-\theta}}w(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)$, then integrating the resulting inequality with respect to σ over $(0,x)$, we have

$$\begin{aligned} &\mathrm{f}_r^{\xi}(\tau)^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\ +\mathrm{h}^{\xi}(x)\mathrm{f}_r^{\varpi-\lambda_r}(\tau)^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &\leq\ \mathrm{f}_r^{\varpi+\xi-\lambda_r}(\tau)^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]+\mathrm{h}^{\xi}(\tau)^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]. \end{aligned} \tag{3.36}$$

Multiplying both sides of (3.36) by $\frac{\rho^{1-\beta}x^k}{\Gamma(\alpha)}\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}w(\tau)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\tau)$, then integrating the resulting inequality with respect to τ over $(0,x)$, we have

$$\begin{aligned}
&{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]+\\
&{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\ {}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}_r^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\leq\ {}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{h}^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&{}^{\rho}\mathcal{J}_{\eta,k}^{\theta,\pi}[w(x)\mathrm{f}_r^{\varpi+\xi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]{}^{\rho}\mathcal{J}_{\eta,k}^{\alpha,\beta}[w(x)\mathrm{h}^{\xi}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)],
\end{aligned}\tag{3.37}$$

The proof is completed.

4 Concluding Remarks

Several fractional integral inequalities have been investigated by employing the different fractional integral operators. In this paper, we established weighted fractional integral inequalities using generalized Katugampola fractional integral operators. This works give some contribution to the theory fractional integral inequalities and fractional calculus.

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