OPTIMAL BOUNDS FOR THE TANGENT AND HYPERBOLIC SINE MEANS

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Abstract. We provide the optimal bounds for the tangent and hyperbolic sine mean in terms of various weighted means of the arithmetic and geometric means.

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1. Introduction, definitions and notation

The means

(tangent mean)
$$\mathsf{M}_{\tan}(x,y)=\begin{cases}\dfrac{x-y}{2\tan\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}$$

and

(hyperbolic sine mean)
$$\mathsf{M}_{\sinh}(x,y)=\begin{cases}\dfrac{x-y}{2\sinh\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases},$$

defined for all positive x, y, were introduced in [4], where one of the authors investigates means of the form

$$\mathsf{M}_f(x,y)=\begin{cases}\dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x\neq y\\ x & x=y\end{cases}. \tag{1}$$

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It was shown that every symmetric and homogeneous mean can be represented in the form (1) and that every function $f : (0,1) \to \mathbb{R}$ (called Seiffert function) satisfying

$$\frac{z}{1+z} \le f(z) \le \frac{z}{1-z}$$

produces a mean. The correspondence between means and Seiffert functions is given by the formula

$$f(z) = \frac{z}{M(1-z,1+z)}, \quad \text{where} \quad z = \frac{|x-y|}{x+y}.$$

The aim of this paper is to determine various optimal bounds for the $\mathsf{M}_{\tan}$ and $\mathsf{M}_{\sinh}$ with the arithmetic and geometric means (denoted here by A and G).

For two means M, N, the symbol $M < N$ denotes that for all positive $x \ne y$ the inequality $M(x,y) < N(x,y)$ holds.

Our main tool will be the obvious fact that if for two Seiffert means the inequality $f < g$ holds, then their corresponding means satisfy $M_f > M_g$. Thus every inequality between means can be expressed in terms of their Seiffert functions.

Remark 1.1. Note that the Seiffert function of the geometric mean $\mathsf{G}(x,y) = \sqrt{xy}$ is $\mathsf{g}(z) = \frac{z}{\sqrt{1-z^2}}$ and that of the arithmetic mean $\mathsf{A}(x,y) = \frac{x+y}{2}$ is the identity function $\mathsf{a}(z) = z$. Clearly, the Seiffert functions of $\mathsf{M}_{\tan}$ and $\mathsf{M}_{\sinh}$ are the functions tan and sinh, respectively.

Remark 1.2. Throughout this paper all means are defined on $(0,\infty)^2$.

For the reader's convenience in the following sections we place the main results with their proofs, while all lemmas and technical details can be found in the last section of this paper.

The motivation for our research are the inequalities $\mathsf{G} < \mathsf{L} < \mathsf{M}_{\tan} < \mathsf{M}_{\sinh} < \mathsf{A}$ proven in [4, Lemma 3.2]. The results obtained in this paper show what the distance is between the new and the classical means measured in different ways.

2. Linear bounds

Given three means $K < L < M$ one may try to find the best α, β satisfying the double inequality $(1-\alpha)K + \alpha M < L < (1-\beta)K + \beta M$ or equivalently $\alpha < \frac{L-K}{M-K} < \beta$. If k, l, m are respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l} - \frac{1}{k}}{\frac{1}{m} - \frac{1}{k}} < \beta. \tag{2}$$

Thus the problem reduces to finding upper and lower bounds for certain functions defined on the interval $(0,1)$.

Theorem 2.1. *The inequalities*

$$(1-\alpha)\,\mathsf{G} + \alpha\mathsf{A} < \mathsf{M}_{\tan} < (1-\beta)\,\mathsf{G} + \beta\mathsf{A}$$

hold if and only if $\alpha \le \frac{1}{3}$ *and* $\beta \ge \cot 1 \approx 0.6421$.

Proof. Taking Remark 1.1 and the formula (2) into account we should investigate the function

$$h(z) = \frac{\dfrac{1}{\tan z} - \frac{\sqrt{1-z^2}}{z}}{\frac{1}{z} - \frac{\sqrt{1-z^2}}{z}} = -\frac{\frac{z}{\tan z} - 1}{\sqrt{1-z^2} - 1} + 1.$$

We shall show that h increases or - which is equivalent - that $\frac{\frac{z}{\tan z}-1}{\sqrt{1-z^2}-1}$ decreases. By Lemma 7.3 it is enough to prove that the function $r(z)=\frac{(z/\tan z-1)'}{(\sqrt{1-z^2}-1)'}$ decreases.

We have

$$r(z)=\frac{\sqrt{1-z^2}(2z-\sin 2z)}{2z\sin^2 z}$$

and

$$r'(z)=-\frac{(8z^3-4z)\sin z+(-8z^4+8z^2-1)\cos z+\cos 3z}{4z^2\sqrt{1-z^2}\sin^3 z}.$$

The function $s(z)=(8z^3-4z)\sin z+(-8z^4+8z^2-1)\cos z+\cos 3z$ satisfies $s(0)=s'(0)=s''(0)=s'''(0)=0$ and

$$\begin{aligned} s^{(4)}(z) &= 60z(9-2z^2)\sin z+(-8z^4+488z^2-81)\cos z+81\cos 3z\\ &> (-8z^4+488z^2-81)\cos z+81\cos 3z\\ &= 2\cos z(-4z^4+244z^2+81\cos 2z-81)\\ &= 2\cos z(4(z^2-z^4)+162(z^2-\sin^2 z)+78z^2)>0. \end{aligned}$$

Thus s is positive and r' is negative, which shows that h increases. We complete the proof by noting that $\lim_{z\to 0} h(z)=1/3$ and $\lim_{z\to 1} h(z)=\cot 1$. □

Theorem 2.2. *The inequalities*

$$(1-\alpha)\,\mathsf{G}+\alpha\mathsf{A}<\mathsf{M}_{\sinh}<(1-\beta)\,\mathsf{G}+\beta\mathsf{A}$$

hold if and only if $\alpha\le\frac{2}{3}$ *and* $\beta\ge\frac{1}{\sinh 1}\approx 0.8509$.

Proof. We use formula (2) once more and investigate the function

$$h(z)=\frac{\dfrac{1}{\sinh z}-\frac{\sqrt{1-z^2}}{z}}{\frac{1}{z}-\frac{\sqrt{1-z^2}}{z}}=-\frac{\frac{z}{\sinh z}-1}{\sqrt{1-z^2}-1}+1.$$

We shall show that h increases or - which is equivalent - that $\frac{\frac{z}{\sinh z}-1}{\sqrt{1-z^2}-1}$ decreases. By Lemma 7.3 it is enough to prove that the function

$$r(z)=\frac{\left(\frac{z}{\sinh z}-1\right)'}{\left(\sqrt{1-z^2}-1\right)'}$$

decreases. A simple calculation reveals that

$$r(z)=\frac{\sqrt{1-z^2}(z\cosh z-\sinh z)}{z\sinh^2 z}$$

and

$$r'(z)=-\frac{-3z^4-z^4\cosh 2z+2z^3\sinh 2z+3z^2+z^2\cosh 2z-z\sinh 2z-\cosh 2z+1}{2z^2\sqrt{1-z^2}\sinh^3 z}.$$

Let $s(z)=1+3z^2-3z^4-(1-z^2+z^4)\cosh 2z-(z-2z^3)\sinh 2z$. Then

$$\begin{aligned} s'(z) &= 6z-12z^3+(-3+8z^2-2z^4)\sinh 2z\\ &= \sum_{\substack{n\ge 5\\ n\text{ odd}}} 2^{n-3}\left(-\frac{24}{n!}+\frac{16}{(n-2)!}-\frac{1}{(n-4)!}\right)z^n\\ &= \frac{88}{15}z^5-\frac{64}{105}z^7-\frac{316}{945}z^9-\sum_{\substack{n\ge 11\\ n\text{ odd}}} 2^{n-3}\frac{n^4-6n^3-5n^2+10n+24}{n!}z^n. \end{aligned}\tag{3}$$

Since for $n \geq 11$ we have $z^n < z^5$ and

$$n^4 - 6n^3 - 5n^2 + 10n + 24 > n^3(n-11) + 10n + 24 > 0,$$
$$n^4 - 6n^3 - 5n^2 + 10n + 24 = n^4 - 6n^3 - 5(n-1)^2 + 29 < n^4,$$
$$\frac{n^4}{(n-3)(n-2)(n-1)n} < \frac{n^4}{(n-3)^4} \leq \left(\frac{11}{8}\right)^4 < 4,$$

we can continue equation (3)

$$\begin{aligned} s'(z) &> z^5 \left(\frac{88}{15} - \frac{64}{105} - \frac{316}{945} - 8 \sum_{\substack{n \geq 11 \\ n \text{ odd}}} \frac{2^{n-4}}{(n-4)!} \right) \\ &= z^5 \left(\frac{4652}{945} - 8 \left(\sinh 2 - 2 - \frac{2^3}{3!} - \frac{2^5}{5!} \right) \right) > 4z^5 > 0. \end{aligned}$$

Since $s(0) = 0$, we conclude that s is positive in $(0,1)$, so r' is negative and r decreases and h increases. To complete the proof we note that $\lim_{z\to 0} h(z) = 2/3$. □

3. Harmonic bounds

In this section we look for optimal bounds for means $K < L < M$ of the form $\frac{1-\alpha}{M} + \frac{\alpha}{K} < \frac{1}{L} < \frac{1-\beta}{M} + \frac{\beta}{K}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l-m}{k-m} < \beta. \tag{4}$$

We shall use the above to prove two theorems.

Theorem 3.1. *The inequalities*

$$\frac{1-\alpha}{\mathsf{A}} + \frac{\alpha}{\mathsf{G}} < \frac{1}{\mathsf{M}_{\tan}} < \frac{1-\beta}{\mathsf{A}} + \frac{\beta}{\mathsf{G}}$$

hold if and only if $\alpha \leq 0$ *and* $\beta \geq \frac{2}{3}$.

Proof. By (4) we shall consider the function

$$h(z) = \frac{\tan z - z}{\frac{z}{\sqrt{1-z^2}} - z}.$$

We notice immediately that $\lim_{z\to 1} h(z) = 0$ and $\lim_{z\to 0} h(z) = 2/3$ so the only thing we have to show is that $2/3$ is the upper bound for h. Note that the inequality $h(z) < 2/3$ is equivalent to $3\tan z - z - \frac{2z}{\sqrt{1-z^2}} < 0$. Substituting $z = \sin t$ transforms this inequality into $p(t) = 3\tan(\sin t) - \sin t - 2\tan t < 0$. We have $p(0) = 0$ and

$$\begin{aligned} p'(t) &= \frac{3\cos t}{\cos^2(\sin t)} - \cos t - \frac{2}{\cos^2 t} < 3\frac{\cos t}{\cos^2 t} - \cos t - \frac{2}{\cos^2 t} \\ &= -\frac{(\cos t - 1)^2(\cos t + 2)}{\cos^2 t} < 0. \end{aligned}$$

Therefore p is negative in $(0, \pi/2)$, which completes the proof. □

And now it is time for the bound of $\mathsf{M}_{\sinh}$.

Theorem 3.2. *The inequalities*

$$\frac{1-\alpha}{\mathsf{A}}+\frac{\alpha}{\mathsf{G}}<\frac{1}{\mathsf{M}_{\sinh}}<\frac{1-\beta}{\mathsf{A}}+\frac{\beta}{\mathsf{G}}$$

hold if and only if $\alpha \le 0$ *and* $\beta \ge \frac{1}{3}$.

Proof. This time we investigate the function

$$h(z)=\frac{\sinh z-z}{\frac{z}{\sqrt{1-z^2}}-z}.$$

As in the proof of Theorem 3.1 we notice that $\lim_{z\to 1} h(z)=0$ and $\lim_{z\to 0} h(z)=1/3$. We shall show that $h(z)<1/3$ for all $0<z<1$. This inequality can be written as $3\sinh z-2z-\frac{z}{\sqrt{1-z^2}}<0$. Substituting $z=\sin t$ transforms this inequality into $p(t)=3\sinh(\sin t)-2\sin t-\tan t<0$. We have $p(0)=0$ and by Lemma 7.4 we obtain

$$\begin{aligned}p'(t)&=3\cosh(\sin t)\cos t-2\cos t-\frac{1}{\cos^2 t}<3\cosh t\cos t-2\cos t-\frac{1}{\cos^2 t}\\&<3\left(1-\frac{2\cos t+\cos^{-2} t}{3}\right)<0\end{aligned}$$

(the last inequality is valid by the AG inequality). So p is negative and we are done.

□

4. Quadratic bounds

Given three means $K<L<M$ one may try to find the best α,β satisfying the double inequality $\sqrt{(1-\alpha)K^2+\alpha M^2}<L<\sqrt{(1-\beta)K^2+\beta M^2}$ or equivalently $\alpha<\frac{L^2-K^2}{M^2-K^2}<\beta$. If k,l,m are respective Seiffert functions, then the latter can be written as

$$\alpha<\frac{\frac{1}{l^2}-\frac{1}{k^2}}{\frac{1}{m^2}-\frac{1}{k^2}}<\beta. \tag{5}$$

Thus the problem reduces to finding upper and lower bounds for certain functions defined on the interval $(0,1)$.

Theorem 4.1. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{G}^2+\alpha\mathsf{A}^2}<\mathsf{M}_{\tan}<\sqrt{(1-\beta)\,\mathsf{G}^2+\beta\mathsf{A}^2}$$

hold if and only if $\alpha\le\frac{1}{3}$ *and* $\beta\ge\frac{1}{\tan^2 1}\approx 0.4123$.

Proof. By formula (5) we should investigate the function

$$h(z)=\frac{\frac{1}{\tan^2 z}-\frac{1}{\frac{z^2}{1-z^2}}}{\frac{1}{z^2}-\frac{1}{\frac{z^2}{1-z^2}}}=\frac{1}{\tan^2 z}-\frac{1}{z^2}+1.$$

Since $h'(z)=\frac{2}{\sin^3 z}\left(\frac{\sin^3 z}{z^3}-\cos z\right)>0$ (by Lemma 7.1), the function h increases. We complete the proof by noting that $\lim\limits_{z\to 0} h(z)=1/3$. □

And here comes the hyperbolic sine version of the previous theorem.

Theorem 4.2. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{G}^2+\alpha\mathsf{A}^2}<\mathsf{M}_{\sinh}<\sqrt{(1-\beta)\,\mathsf{G}^2+\beta\mathsf{A}^2}$$

hold if and only if $\alpha \le \frac{2}{3}$ *and* $\beta \ge \frac{1}{\sinh^2 1} \approx 0.7241$.

Proof. The function to be considered here is

$$h(z)=\frac{\frac{1}{\sinh^2 z}-\frac{1}{\frac{z^2}{1-z^2}}}{\frac{1}{z^2}-\frac{1}{\frac{z^2}{1-z^2}}}=\frac{1}{\sinh^2 z}-\frac{1}{z^2}+1.$$

Its derivative equals $h'(z)=\frac{2}{\sinh^3 z}\left(\frac{\sinh^3 z}{z^3}-\cosh z\right)$. By Lemma 7.2 we have that $h'(z)>0$, so the function h increases. We complete the proof by noting that $\lim\limits_{z\to 0} h(z)=2/3$. □

5. Bounds with the weighted power mean of order −2

In this section we look for optimal bounds for means $K<L<M$ of the form $\sqrt{\frac{1-\alpha}{M^2}+\frac{\alpha}{K^2}}<\frac{1}{L}<\sqrt{\frac{1-\beta}{M^2}+\frac{\beta}{K^2}}$ or, in terms of their Seiffert functions,

$$\alpha<\frac{l^2-m^2}{k^2-m^2}<\beta.$$

Theorem 5.1. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{A}^2}+\frac{\alpha}{\mathsf{G}^2}}<\frac{1}{\mathsf{M}_{\tan}}<\sqrt{\frac{1-\beta}{\mathsf{A}^2}+\frac{\beta}{\mathsf{G}^2}}$$

hold if and only if $\alpha \le 0$ *and* $\beta \ge \frac{2}{3}$.

Proof. To prove the theorem we investigate the function

$$h(z)=\frac{\tan^2 z-z^2}{\frac{z^2}{1-z^2}-z^2}=(1-z^2)\frac{\tan^2 z-z^2}{z^4}.$$

Clearly $\lim_{z\to 1} h(z)=0$, which shows that the lower bound for h is $\alpha=0$. Using Taylor expansion one can easily check that $\lim_{z\to 0} h(z)=2/3$. We shall show that this is the best upper bound for h.
Elementary calculations show that the inequality $h(z)<2/3$ is equivalent to $3(1-z^2)<(3-z^4)\cos^2 z$. To prove this, note that

$$\begin{aligned}(3-z^4)\cos^2 z-3(1-z^2)&>(3-z^4)\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}\right)^2-3(1-z^2)\\&=\frac{z^6}{(6!)^2}(449280-167940z^2+22860z^4-1617z^6+60z^8-z^{10})>0.\end{aligned}$$

□

Theorem 5.2. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{A}^2}+\frac{\alpha}{\mathsf{G}^2}}<\frac{1}{\mathsf{M}_{\sinh}}<\sqrt{\frac{1-\beta}{\mathsf{A}^2}+\frac{\beta}{\mathsf{G}^2}}$$

hold if and only if $\alpha \le 0$ *and* $\beta \ge \frac{1}{3}$.

Proof. We follow the same line as in the previous proof. Let

$$h(z)=\frac{\sinh^2 z-z^2}{\frac{z^2}{1-z^2}-z^2}=(1-z^2)\frac{\sinh^2 z-z^2}{z^4}.$$

The lower bound of the function h is zero, because $\lim_{z\to 1} h(z)=0$. We shall demonstrate that $\lim_{z\to 0} h(z)=1/3$ is the best bound for h above.
The inequality $h(z)<1/3$ is equivalent to $(1-z^2)\cosh^2 z<1-2z^4/3$. Using the Taylor series expansion $\cosh^2 z=(\cosh 2z+1)/2=1+z^2+\sum_{n=2}^{\infty}\frac{2^{2n-1}}{(2n)!}z^{2n}$ we get

$$(1-z^2)\cosh^2 z=1-\frac{2}{3}z^4+\sum_{n=3}^{\infty}\frac{2^{2n-3}}{(2n-2)!}\left(\frac{2}{n(2n-1)}-1\right)z^{2n}<1-\frac{2}{3}z^4,$$

which shows that $h(z)<1/3$. □

6. Bounds with varying arguments.

If N is a mean, then the formula $N^{\{t\}}(x,y)=N\left(\frac{x+y}{2}+t\frac{x-y}{2},\frac{x+y}{2}-t\frac{x-y}{2}\right)$ defines a homotopy between that arithmetic mean $\mathsf{A}=N^{\{0\}}$ and $N=N^{\{1\}}$. Therefore if $N<M<\mathsf{A}$ it makes sense to ask what the optimal numbers α,β are satisfying $N^{\{\alpha\}}<M<N^{\{\beta\}}$. Theorem 6.1 from [4] gives a method for finding such numbers in terms of the Seiffert functions of the means involved. It says

Theorem 6.1. *For a Seiffert function k denote $\widehat{k}(z)=k(z)/z$. Let M and N be two means with Seiffert functions m and n, respectively. Suppose that $\widehat{n}(z)$ is strictly monotone and let $p_0=\inf\limits_z\frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$ and $q_0=\sup\limits_z\frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$.*
If $\mathsf{A}(x,y)<M(x,y)<N(x,y)$ for all $x\neq y$, then the inequalities

$$N^{\{p\}}(x,y)\leqslant M(x,y)\leqslant N^{\{q\}}(x,y)$$

hold if and only if $p\leqslant p_0$ and $q\geqslant q_0$.
If $N(x,y)<M(x,y)<\mathsf{A}(x,y)$ for all $x\neq y$, then the inequalities

$$N^{\{q\}}(x,y)\leqslant M(x,y)\leqslant N^{\{p\}}(x,y)$$

hold if and only if $p\leqslant p_0$ and $q\geqslant q_0$.

In the case of $N=\mathsf{G}$ we see that $\widehat{g}=\frac{1}{\sqrt{1-z^2}}$ and $\widehat{g}^{-1}(x)=\sqrt{1-x^{-2}}$.

Theorem 6.2. *The inequalities*

$$\mathsf{G}\left(\tfrac{x+y}{2}+\alpha\tfrac{x-y}{2},\tfrac{x+y}{2}-\alpha\tfrac{x-y}{2}\right)<\mathsf{M}_{\tan}<\mathsf{G}\left(\tfrac{x+y}{2}+\beta\tfrac{x-y}{2},\tfrac{x+y}{2}-\beta\tfrac{x-y}{2}\right)$$

hold if and only if $\alpha\geq\sqrt{\frac{2}{3}}\approx 0.8165$ and $\beta\leq\sqrt{1-\cot^2 1}\approx 0.7666$.

Proof. Using Theorem 6.1 we should find the range of the function

$$h(z)=\frac{\sqrt{1-\left(\frac{z}{\tan z}\right)^2}}{z}=\sqrt{\frac{1}{z^2}-\frac{1}{\tan^2 z}}.$$

A slight modification of the proof of Theorem 4.1 shows that h decreases from $\sqrt{2/3}$ to $\sqrt{1-\cot^2 1}$, which completes the proof.

□

Theorem 6.3. *The inequalities*

$$\mathsf{G}\left(\tfrac{x+y}{2}+\alpha\tfrac{x-y}{2},\tfrac{x+y}{2}-\alpha\tfrac{x-y}{2}\right)<\mathsf{M}_{\sinh}<\mathsf{G}\left(\tfrac{x+y}{2}+\beta\tfrac{x-y}{2},\tfrac{x+y}{2}-\beta\tfrac{x-y}{2}\right)$$

hold if and only if $\alpha\geq\sqrt{\frac{1}{3}}\approx 0.5773$ *and* $\beta\leq\sqrt{2-\coth^2 1}\approx 0.5253$.

Proof. Using Theorem 6.1 we should find the range of the function

$$h(z)=\frac{\sqrt{1-\left(\frac{z}{\sinh z}\right)^2}}{z}=\sqrt{\frac{1}{z^2}-\frac{1}{\sinh^2 z}}.$$

We refer to the proof of Theorem 4.2 to show that h decreases from $\sqrt{1/3}$ to $\sqrt{2-\coth^2 1}$, which completes the proof. □

7. Tools and lemmas

In this section we place all the technical details needed to prove our main results.

Lemma 7.1 (Mitrinović & Adamović [3])**.** *Consider the functions* $f_u:[0,\pi/2)\to\mathbb{R}$

$$f_u(x)=\cos^u x\sin x-x,\quad -1<u<0.$$

For $-1\leq u\leq-\frac{1}{3}$ *the functions* f_u *are positive. For* $-\frac{1}{3}<u<0$ *there exists* $0<x_u<\frac{\pi}{2}$ *such that* f_u *is negative in* $(0,x_u)$ *and positive in* (x_u,∞).

Proof. We have $f_u(0)=f_u'(0)=0$ and

$$f_u''(x)=u(u-1)\sin x\cos^u x\left[\tan^2 x-\frac{1+3u}{u(u-1)}\right].$$

If $-1\leq u<-1/3$, we have $\frac{3u+1}{u(u-1)}\leq 0$, so f_u is convex, thus positive.

For $-1/3<u<0$ the equation $\tan^2 x-\frac{1+3u}{u(u-1)}=0$ has exactly one solution ξ_u, so f_u is concave and negative on $(0,\xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u. □

Lemma 7.2 (Lazarević [2])**.** *Consider the functions* $g_u:[0,\infty)\to\mathbb{R}$

$$g_u(x)=\cosh^u x\sinh x-x,\quad -1<u<0.$$

For $-\frac{1}{3}\leq u<0$ *the functions* g_u *are positive. For* $-1<u<-\frac{1}{3}$ *there exists* $x_u>0$ *such that* g_u *is negative in* $(0,x_u)$ *and positive in* (x_u,∞).

Proof. We have $g_u(0)=g_u'(0)=0$ and

$$g_u''(x)=u(u-1)\sinh x\cosh^u x\left[\tanh^2 x+\frac{1+3u}{u(u-1)}\right].$$

If $-1/3\leq u<0$, we have $\frac{1+3u}{u(u-1)}\geq 0$, so g_u is convex thus positive. For $-1<u<-1/3$ the equation $\tanh^2 x+\frac{1+3u}{u(u-1)}=0$ has exactly one solution ξ_u, so g_u is concave and negative on $(0,\xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u. □

The next lemma comes from [1, Theorem 1.25].

Lemma 7.3. *Suppose* $f,g:(a,b)\to\mathbb{R}$ *are differentiable with* $g'(x)\neq 0$ *and such that* $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$ *or* $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$. *Then*

(1) if $\frac{f'}{g'}$ *is increasing on* (a,b)*, then* $\frac{f}{g}$ *is increasing on* (a,b)*,*

(2) if $\frac{f'}{g'}$ *is decreasing on* (a,b)*, then* $\frac{f}{g}$ *is decreasing on* (a,b)*.*

Lemma 7.4. *For* $0 < t < \frac{\pi}{2}$ *the inequality* $\cos t \cosh t < 1$ *holds.*

Proof. It follows immediately because

$$(\cos t \cosh t)' = \cos t \cosh t(\tanh t - \tan t) < 0.$$

□

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