OPTIMAL BOUNDS FOR THE TANGENT AND HYPERBOLIC SINE MEANS II

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Abstract. We provide the optimal bounds for the tangent and hyperbolic sine means in terms of various weighted means of the arithmetic and harmonic means.

Contents

1. Introduction, definitions and notation

The means

(tangent mean)
$$\mathsf{M}_{\tan}(x,y)=\begin{cases}\dfrac{x-y}{2\tan\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases},$$

and

(hyperbolic sine mean)
$$\mathsf{M}_{\sinh}(x,y)=\begin{cases}\dfrac{x-y}{2\sinh\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}$$

have been introduced in [4], where one of the authors investigates the means of the form

$$\mathsf{M}_f(x,y)=\begin{cases}\dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x\neq y\\ x & x=y\end{cases}. \tag{1}$$

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It was shown that every symmetric and homogeneous mean can be represented in the form (1) and that every function $f:(0,1)\to\mathbb{R}$ (called Seiffert function) satisfying

$$\frac{z}{1+z}\le f(z)\le\frac{z}{1-z}$$

produces a mean. The correspondence between means and Seiffert functions is given by the formula

$$f(z)=\frac{z}{M(1-z,1+z)},\quad\text{where}\quad z=\frac{|x-y|}{x+y}.$$

The aim of this paper is to determine various optimal bounds for the $\mathsf{M}_{\tan}$ and $\mathsf{M}_{\sinh}$ by the arithmetic and harmonic means (denoted here by A and H). Note that the optimal bounds of the above in terms of the geometric and arithmetic means can be found in [3].

For two means M,N the symbol $M<N$ means that the inequality $M(x,y)<N(x,y)$ holds for all $x\ne y$.

Our main tool will be the obvious fact that if for two Seiffert means the inequality $f<g$ holds, then their corresponding means satisfy $M_f>M_g$. Thus every inequality between means can be expressed in terms of their Seiffert functions.

Remark 1.1. Note that the Seiffert function of the harmonic mean $\mathsf{H}(x,y)=\frac{2xy}{x+y}$ is $\mathsf{h}(z)=\frac{z}{1-z^2}$ and that of the arithmetic mean $\mathsf{A}(x,y)=\frac{x+y}{2}$ is the identity function $\mathsf{a}(z)=z$. Clearly, the Seiffert functions of $\mathsf{M}_{\tan}$ and $\mathsf{M}_{\sinh}$ are the functions tan and sinh respectively.

For the reader's convenience in the following sections we place the main results with their proofs, while all lemmas and technical details can be found in the last section of this paper.

The motivation for our research are the inequalities $\mathsf{H}<\mathsf{L}<\mathsf{M}_{\tan}<\mathsf{M}_{\sinh}<\mathsf{A}$ proven in [4, Lemma 3.2].

2. Linear bounds

Given three means $K<L<M$ one may try to find the best α,β satisfying double inequality $(1-\alpha)K+\alpha M<L<(1-\beta)K+\beta M$ or equivalently $\alpha<\frac{L-K}{M-K}<\beta$. If k,l,m are respective Seiffert functions, then the latter can be written as

$$\alpha<\frac{\frac{1}{l}-\frac{1}{k}}{\frac{1}{m}-\frac{1}{k}}<\beta. \tag{2}$$

Thus the problem reduces to finding the upper and lower bound for certain function defined on the interval $(0,1)$.

Theorem 2.1. *The inequalities*

$$(1-\alpha)\,\mathsf{H}+\alpha\mathsf{A}<\mathsf{M}_{\tan}<(1-\beta)\,\mathsf{H}+\beta\mathsf{A}$$

hold if and only if $\alpha\le\frac{1}{\tan 1}\approx 0.6421$ *and* $\beta\ge\frac{2}{3}$.

Proof. By the formula (2) and Remark 1.1 we investigate the function

$$h(z)=\frac{\frac{1}{\tan z}-\frac{1-z^2}{z}}{\frac{1}{z}-\frac{1-z^2}{z}}=\frac{1}{z}\left(\frac{1}{\tan z}-\frac{1}{z}\right)+1:=\frac{1}{z}p(z)+1.$$

The function p satisfies $\lim_{z\to 0} p(z) = 0$ and $p''(z) = \frac{2}{\sin^3 z}\left(\cos z - \frac{\sin^3 z}{z^3}\right) < 0$ (by Lemma 7.1), so by Property 7.2 the function $p(z)/z$ decreases and so does the function h. We complete the proof by noting that $\lim\limits_{z\to 0} h(z) = \frac{2}{3}$. □

Theorem 2.2. *The inequality*

$$(1-\alpha)\mathsf{H} + \alpha\mathsf{A} < \mathsf{M}_{\sinh} < (1-\beta)\mathsf{H} + \beta\mathsf{A}$$

holds if and only if $\alpha \le \frac{5}{6}$ *and* $\beta \ge \frac{1}{\sinh 1} \approx 0.8509$.

Proof. According to the formula (2) we should investigate the function

$$h(z) = \frac{\frac{1}{\sinh z} - \frac{1-z^2}{z}}{\frac{1}{z} - \frac{1-z^2}{z}} = \frac{1}{z}\left(\frac{1}{\sinh z} - \frac{1}{z}\right) + 1.$$

We shall show that h increases. Now

$$h'(z) = \frac{1}{z^3 \sinh^2 z}(2\sinh^2 z - z\sinh z - z^2 \cosh z) > 0$$

because using the inequality $\sinh x > x + x^3/3! + x^5/5!$ and Lemma 7.4 and 7.5

$$\begin{aligned}
&2\sinh^2 z - z\sinh z - z^2\cosh z \\
&> 2\left(z + \frac{z^3}{3!} + \frac{z^5}{5!}\right)^2 - z\left(z + \frac{z^3}{3!} + 2\frac{z^5}{5!}\right) - z^2\left(1 + \frac{z^2}{2} + \frac{z^4}{4!} + 2\frac{z^6}{6!}\right) \\
&= \frac{z^{10}}{7200} + \frac{z^8}{360} + 11\frac{z^6}{360} > 0.
\end{aligned}$$

We complete the proof by noting that $\lim_{z\to 0} h(z) = \frac{5}{6}$. □

3. Harmonic bounds

In this section we look for the optimal bounds for means $K < L < M$ of the form $\frac{1-\alpha}{M} + \frac{\alpha}{K} < \frac{1}{L} < \frac{1-\beta}{M} + \frac{\beta}{K}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l-m}{k-m} < \beta. \tag{3}$$

We shall use the above to prove two theorems.

Theorem 3.1. *The inequalities*

$$\frac{1-\alpha}{\mathsf{A}} + \frac{\alpha}{\mathsf{H}} < \frac{1}{\mathsf{M}_{\tan}} < \frac{1-\beta}{\mathsf{A}} + \frac{\beta}{\mathsf{H}}$$

hold if and only if $\alpha \le 0$ *and* $\beta \ge \frac{1}{3}$.

Proof. Taking into account the formula (3) we should investigate the function

$$h(z) = \frac{\tan z - z}{\frac{z}{1-z^2} - z} = (1-z^2)\frac{\tan z - z}{z^3}.$$

We shall show that h decreases. Observe that

$$h'(z) = \frac{2z\cos^2 z + z - z^3 + (z^2-3)\cos z \sin z}{z^4 \cos^2 z}.$$

The function $p(z) = 2z\cos^2 z + z - z^3 + (z^2-3)\cos z\sin z$ satisfies $p(0) = p'(0) = 0$ and

$$p''(z) = (3-2z^2)\sin 2z - 6z < (3-2z^2)2z - 6z = -4z^3 < 0.$$

Thus p is negative and so is h'. Consequently, h decreases. We complete the proof by noting that $\lim_{z\to 0} h(z) = \frac{1}{3}$. □

And now it is time for the bound of $\mathsf{M}_{\sinh}$.

Theorem 3.2. *The inequalities*

$$\frac{1-\alpha}{\mathsf{A}} + \frac{\alpha}{\mathsf{H}} < \frac{1}{\mathsf{M}_{\sinh}} < \frac{1-\beta}{\mathsf{A}} + \frac{\beta}{\mathsf{H}}$$

hold if and only if $\alpha \le 0$ *and* $\beta \ge \frac{1}{6}$.

Proof. We use once more the formula (3) and investigate the function

$$h(z) = \frac{\sinh z - z}{\frac{z}{1-z^2} - z} = (1-z^2)\frac{\sinh z - z}{z^3}.$$

We shall show that h is decreasing. We have

$$h'(z) = \frac{(z-z^3)\cosh z + (z^2-3)\sinh z + 2z}{z^4}.$$

The function $p(z) = (z-z^3)\cosh z + (z^2-3)\sinh z + 2z$ satisfies $p(0) = p'(0) = p''(0) = 0$ and

$$p'''(z) = -z\left[(z^2+11)\sinh z + 8z\cosh z\right] < 0.$$

So p is negative and so is h'. Consequently, h decreases. A simple calculation shows that $\lim_{z\to 0} h(z) = \frac{1}{6}$. □

4. Quadratic bounds

Given three means $K < L < M$ one may try to find the best α, β satisfying double inequality $\sqrt{(1-\alpha)K^2 + \alpha M^2} < L < \sqrt{(1-\beta)K^2 + \beta M^2}$ or equivalently $\alpha < \frac{L^2-K^2}{M^2-K^2} < \beta$. If k, l, m are respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l^2} - \frac{1}{k^2}}{\frac{1}{m^2} - \frac{1}{k^2}} < \beta. \tag{4}$$

Thus the problem reduces to finding the upper and lower bound for certain function defined on the interval $(0,1)$.

Theorem 4.1. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{H}^2 + \alpha\mathsf{A}^2} < \mathsf{M}_{\tan} < \sqrt{(1-\beta)\,\mathsf{H}^2 + \beta\mathsf{A}^2}$$

hold if and only if $\alpha \le \frac{1}{\tan^2 1} \approx 0.4123$ *and* $\beta \ge \frac{2}{3}$.

Proof. Using the formula (4) we investigate the function

$$h(z) = \frac{\frac{1}{\tan^2 z} - \frac{(1-z^2)^2}{z^2}}{\frac{1}{z^2} - \frac{(1-z^2)^2}{z^2}} = \frac{((z^2-1)\sin z - z\cos z)((z^2-1)\sin z + z\cos z)}{z^2(z^2-2)\sin^2 z}.$$

We show that the function h decreases. We have

$$h'(z) = \frac{2}{z^3(z^2-2)^2\sin^3 z}\left(z^4 \sin z\cos^2 z + 2(1-z^2)\sin^3 z + (z^2-2)z^3\cos z\right) < 0$$

by Lemma 7.3.

We complete the proof noting that $\lim_{z\to 0} h(z) = \frac{2}{3}$. □

And here comes the hyperbolic sine version of the previous theorem.

Theorem 4.2. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{H}^2+\alpha\mathsf{A}^2}<\mathsf{M}_{\sinh}<\sqrt{(1-\beta)\,\mathsf{H}^2+\beta\mathsf{A}^2}$$

hold if and only if $\alpha\le\frac{1}{\sinh^2 1}\approx 0.7241$ *and* $\beta\ge\frac{5}{6}$.

Proof. The function to be investigated this time is

$$h(z)=\frac{\frac{1}{\sinh^2 z}-\frac{(1-z^2)^2}{z^2}}{\frac{1}{z^2}-\frac{(1-z^2)^2}{z^2}}=-\frac{\frac{z^2}{\sinh^2 z}-1}{(1-z^2)^2-1}+1:=-g(z)+1.$$

We investigate the monotonicity of g using the monotone form of de l'Hospital's Rule (Lemma 7.2). We have

$$\frac{d}{dz}\frac{\frac{d}{dz}\left(\frac{z^2}{\sinh^2 z}-1\right)}{\frac{d}{dz}\left((1-z^2)^2-1\right)}=\frac{4z^3-2z-2z(2-z^2)\cosh 2z+(3-z^2)\sinh 2z}{4(z-1)^2(z+1)^2\sinh^4 z}>0$$

because using Lemma 7.4

$$\begin{aligned}&4z^3-2z-2z(2-z^2)\cosh 2z+(3-z^2)\sinh 2z\\&\quad>4z^3-2z-2z(2-z^2)\left(1+\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}+2\frac{(2z)^6}{6!}\right)+(3-z^2)\left(2z+\frac{(2z)^3}{3!}\right)\\&\quad=\frac{4}{45}z^7(4z^2+7)>0.\end{aligned}$$

Thus g increases and h decreases from $\lim_{z\to0}h(z)=\frac{5}{6}$. □

5. Bounds by weighted power mean of order −2

In this section we look for the optimal bounds for means $K<L<M$ of the form $\sqrt{\frac{1-\alpha}{M^2}+\frac{\alpha}{K^2}}<\frac{1}{L}<\sqrt{\frac{1-\beta}{M^2}+\frac{\beta}{K^2}}$ or, in terms of their Seiffert functions,

$$\alpha<\frac{l^2-m^2}{k^2-m^2}<\beta. \tag{5}$$

Theorem 5.1. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{A}^2}+\frac{\alpha}{\mathsf{H}^2}}<\frac{1}{\mathsf{M}_{\tan}}<\sqrt{\frac{1-\beta}{\mathsf{A}^2}+\frac{\beta}{\mathsf{H}^2}}$$

hold if and only if $\alpha\le 0$ *and* $\beta\ge\frac{1}{3}$.

Proof. Using formula (5) we get

$$h(z)=\frac{\tan^2 z-z^2}{\frac{z^2}{(1-z^2)^2}-z^2}=\frac{(1-z^2)^2(\tan^2 z-z^2)}{z^4(2-z^2)}.$$

The function h satisfies $\lim_{z\to1}h(z)=0$ and $\lim_{z\to0}h(z)=1/3$ so to complete the proof it is enough to show that $h(z)<1/3$ for $0<z<1$. Since $\cos x>$

$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$ we can write

$$\begin{aligned}\frac{1}{3}-h(z)&=\frac{z^4(2-z^2)-3(1-z^2)^2(\tan^2 z-z^2)}{3z^4(2-z^2)}\\&=\frac{((3-z^4)(1-z^2)+z^6)\cos^2 z-3(1-z^2)^2}{3z^4(2-z^2)\cos^2 z}\\&>\frac{((3-z^4)(1-z^2)+z^6)\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}\right)^2-3(1-z^2)^2}{3z^4(2-z^2)\cos^2 z}>0.\end{aligned}$$

The last inequality is valid because

$$\begin{aligned}&\frac{518400}{z^6}\left(((3-z^4)(1-z^2)+z^6)\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}\right)^2-3(1-z^2)^2\right)\\&=967680-1135620z^2+363600z^4-47517z^6+3297z^8-121z^{10}+2z^{12}\\&=(810-701z^2)^2+(311580-127801z^4-47517z^6)+z^8(3297-121z^2+2z^4)>0.\end{aligned}$$

□

Theorem 5.2. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{A}^2}+\frac{\alpha}{\mathsf{H}^2}}<\frac{1}{\mathsf{M}_{\sinh}}<\sqrt{\frac{1-\beta}{\mathsf{A}^2}+\frac{\beta}{\mathsf{H}^2}}$$

hold if and only if $\alpha\le 0$ *and* $\beta\ge\frac{1}{6}$.

Proof. The function to be considered here is

$$h(z)=\frac{\sinh^2 z-z^2}{\frac{z^2}{(1-z^2)^2}-z^2}=\frac{(1-z^2)^2(\sinh^2 z-z^2)}{z^4(2-z^2)}.$$

Since $\lim_{z\to 1}h(z)=0$ and $\lim_{z\to 0}h(z)=1/6$ the only thing we have to show is $h(z)\le 1/6$ for all $0<z<1$. Using Lemma 7.4 we obtain

$$\begin{aligned}\frac{1}{6}-h(z)&=\frac{3-7z^4+5z^6-3(1-z^2)^2\cosh 2z}{6z^4(2-z^2)}\\&>\frac{3-7z^4+5z^6-3(1-z^2)^2\left(1+\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}+2\frac{(2z)^6}{6!}\right)}{6z^4(2-z^2)}\\&=\frac{z^2(37-14z^2-8z^4)}{90(2-z^2)}>0.\end{aligned}$$

□

6. Bounds with varying arguments

If N is a mean that the formula $N^{\{t\}}(x,y)=N\left(\frac{x+y}{2}+t\frac{x-y}{2},\frac{x+y}{2}-t\frac{x-y}{2}\right)$ defines a homotopy between that arithmetic mean $\mathsf{A}=N^{\{0\}}$ and $N=N^{\{1\}}$. Therefore if $N<M<\mathsf{A}$ it make sense to ask what are the optimal numbers α,β satisfying $N^{\{\alpha\}}<M<N^{\{\beta\}}$. Theorem 6.1 from [4] gives a method for finding such numbers in terms of the Seiffert functions of the means involved. It says

Theorem 6.1. *For a Seiffert function k denote $\widehat{k}(z)=k(z)/z$. Let M and N be two means with Seiffert functions m and n, respectively. Suppose that $\widehat{n}(z)$ is*

strictly monotone and let $p_0 = \inf_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$ *and* $q_0 = \sup_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$.
If $\mathsf{A}(x,y) < M(x,y) < N(x,y)$ *for all* $x \neq y$ *then the inequalities*

$$N^{\{p\}}(x,y) \leqslant M(x,y) \leqslant N^{\{q\}}(x,y)$$

hold if and only if $p \leqslant p_0$ *and* $q \geqslant q_0$.
If $N(x,y) < M(x,y) < \mathsf{A}(x,y)$ *for all* $x \neq y$ *then the inequalities*

$$N^{\{q\}}(x,y) \leqslant M(x,y) \leqslant N^{\{p\}}(x,y)$$

hold if and only if $p \leqslant p_0$ *and* $q \geqslant q_0$.

In case of $N = \mathsf{H}$ we see that $\widehat{\mathsf{h}}(z) = \frac{1}{1-z^2}$ and $\widehat{\mathsf{h}}^{-1}(x) = \sqrt{1-x^{-1}}$.

Theorem 6.2. *The inequalities*

$$\mathsf{H}\left(\tfrac{x+y}{2} + \alpha\tfrac{x-y}{2}, \tfrac{x+y}{2} - \alpha\tfrac{x-y}{2}\right) < \mathsf{M}_{\tan} < \mathsf{H}\left(\tfrac{x+y}{2} + \beta\tfrac{x-y}{2}, \tfrac{x+y}{2} - \beta\tfrac{x-y}{2}\right)$$

hold if and only if $\alpha \geq \sqrt{1-\cot 1} \approx 0.5983$ *and* $\beta \leq \sqrt{\frac{1}{3}} \approx 0.5774$.

Proof. By Theorem 6.1 the function to investigate is

$$h(z) = \sqrt{\frac{1}{z^2} - \frac{1}{z\tan z}}.$$

From the proof of Theorem 2.1 we know that the function $g(z) = \frac{1}{z}\left(\frac{1}{\tan z} - \frac{1}{z}\right) < 0$ decreases. Thus $h(z) = \sqrt{-g(z)}$ increases, which completes the proof. □

Theorem 6.3. *The inequalities*

$$\mathsf{H}\left(\tfrac{x+y}{2} + \alpha\tfrac{x-y}{2}, \tfrac{x+y}{2} - \alpha\tfrac{x-y}{2}\right) < \mathsf{M}_{\sinh} < \mathsf{H}\left(\tfrac{x+y}{2} + \beta\tfrac{x-y}{2}, \tfrac{x+y}{2} - \beta\tfrac{x-y}{2}\right)$$

hold if and only if $\alpha \geq \sqrt{\frac{1}{6}} \approx 0.4082$ *and* $\beta \leq \sqrt{1-\frac{1}{\sinh 1}} \approx 0.3861$.

Proof. This time we consider the function

$$h(z) = \sqrt{\frac{1}{z^2} - \frac{1}{z\sinh z}}.$$

Monotonicity of the function h^2 follows from the proof of Theorem 2.2, so evaluation of the values of h at the endpoints completes the proof. □

7. Tools and lemmas

In this section we place the all technical details needed to prove our main results.

Property 7.1. A function $f : (a,b) \to \mathbb{R}$ is convex if and only if for every $a < \theta < b$ its divided difference $\frac{f(x)-f(\theta)}{x-\theta}$ increases for $x \neq \theta$.

Simple consequence of Property 7.1 is

Property 7.2. If a function $f : (a,b) \to \mathbb{R}$ is convex and $\lim_{x\to a} f(x) = \Theta$, then the function $\frac{f(x)-\Theta}{x-a}$ increases.

Lemma 7.1 (Mitrinović & Adamović [2]). *Consider the functions* $f_u : [0, \pi/2) \to \mathbb{R}$

$$f_u(x) = \cos^u x \sin x - x, \quad -1 < u < 0.$$

For $-1 \leq u \leq -\frac{1}{3}$ *the functions* f_u *are positive. For* $-\frac{1}{3} < u < 0$ *there exists* $0 < x_u < \frac{\pi}{2}$ *such that* f_u *is negative in* $(0, x_u)$ *and positive in* (x_u, ∞).

Proof. We have $f_u(0) = f_u'(0) = 0$ and

$$f_u''(x) = u(u-1)\sin x \cos^u x \left[\tan^2 x - \frac{1+3u}{u(u-1)}\right].$$

If $-1 \le u < -1/3$ we have $\frac{3u+1}{u(u-1)} \le 0$, so f_u is convex, thus positive.

For $-1/3 < u < 0$ the equation $\tan^2 x - \frac{1+3u}{u(u-1)} = 0$ has exactly one solution ξ_u, so f_u is concave and negative on $(0, \xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u. □

Lemma 7.2 ([1])**.** *Suppose $f, g : (a,b) \to \mathbb{R}$ are differentiable with $g'(x) \ne 0$ and such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0$. Then*

(1) if $\frac{f'}{g'}$ is increasing on (a,b), then $\frac{f}{g}$ is increasing on (a,b),
(2) if $\frac{f'}{g'}$ is decreasing on (a,b), then $\frac{f}{g}$ is decreasing on (a,b).

Lemma 7.3. *For $0 < x < 1$ we have*

$$x^4 \sin x \cos^2 x + 2(1-x^2)\sin^3 x + (x^2-2)x^3 \cos x < 0.$$

Proof. Given that $\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ and $1 - \frac{x^2}{2!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ we have

$$\begin{aligned}
&x^4 \sin x \cos^2 x + 2(1-x^2)\sin^3 x + (x^2-2)x^3 \cos x \\
&\quad < x^4\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)^2 + 2(1-x^2)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)^3 \\
&\qquad + (x^2-2)x^3\left(1 - \frac{x^2}{2!}\right) \\
&\quad = \frac{x^7\left(x^8(23x^2 - 978) + x^4(16560x^2 - 134480) + 459200x^2 - 777600\right)}{1728000} < 0.
\end{aligned}$$

□

Lemma 7.4. *For $0 < x < 1$ the following inequality holds*

$$\cosh 2x < 1 + \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + 2\frac{(2x)^6}{6!}.$$

Proof.

$$\begin{aligned}
\cosh 2x - 1 - \frac{(2x)^2}{2!} - \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} &= \frac{(2x)^8}{8!} + \frac{(2x)^{10}}{10!} + \dots \\
&< \frac{(2x)^6}{6!}\left(\frac{2^2}{7\cdot 8} + \frac{2^4}{7\cdot 8\cdot 9\cdot 10} + \dots\right) < \frac{(2x)^6}{6!}.
\end{aligned}$$

□

Lemma 7.5. *For $0 < x < 1$ the following inequality holds*

$$\sinh x < x + \frac{x^3}{3!} + 2\frac{x^5}{5!}.$$

Proof.

$$\begin{aligned}
\sinh x - x - \frac{x^3}{3!} - \frac{x^5}{5!} &= \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \\
&< \frac{x^5}{5!}\left(\frac{1}{6\cdot 7} + \frac{1}{6\cdot 7\cdot 8\cdot 9} + \dots\right) < \frac{x^5}{5!}.
\end{aligned}$$

□

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