OPTIMAL BOUNDS FOR THE TANGENT AND HYPERBOLIC SINE MEANS III

MONIKA NOWICKA AND ALFRED WITKOWSKI

Abstract. We provide optimal bounds for the tangent and hyperbolic sine means in terms of various weighted means of the arithmetic and minimum means.

Contents

1. Introduction, definitions and notation

The means

(tangent mean)
$$\mathsf{M}_{\tan}(x,y)=\begin{cases}\dfrac{x-y}{2\tan\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}$$

and

(hyperbolic sine mean)
$$\mathsf{M}_{\sinh}(x,y)=\begin{cases}\dfrac{x-y}{2\sinh\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases},$$

defined for all positive x,y, were have been introduced in [4], where one of the authors investigates means of the form

(1)
$$\mathsf{M}_f(x,y)=\begin{cases}\dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x\neq y\\ x & x=y\end{cases}.$$

2020 *Mathematics Subject Classification.* Primary 26D150.
Key words and phrases. Seiffert-like mean; Seiffert function; convex function.

RGMIA Res. Rep. Coll. 23 (2020), Art. 88, 8 pp. Received 23/07/20

It was shown that every symmetric and homogeneous mean of positive arguments can be represented in the form (1) and that every function $f : (0, 1) \to \mathbb{R}$ (called Seiffert function) satisfying

$$\frac{z}{1+z} \le f(z) \le \frac{z}{1-z}$$

produces a mean. The correspondence between means and Seiffert functions is given by the formula

$$f(z) = \frac{z}{M(1-z, 1+z)}, \quad \text{where} \quad z = \frac{|x-y|}{x+y}.$$

The aim of this paper is to determine various optimal bounds for the $\mathsf{M}_{\tan}$ and $\mathsf{M}_{\sinh}$ with the arithmetic and minimum means (denoted here by A and MIN).

For two means M, N, the symbol $M < N$ denotes that for all positive $x \ne y$ the inequality $M(x, y) < N(x, y)$ holds.

Our main tool will be the obvious fact that if for two Seiffert functions the inequality $f < g$ holds, then their corresponding means satisfy $M_f > M_g$. Thus every inequality between means can be expressed in terms of their Seiffert functions.

Remark 1.1. Note that the Seiffert function of the minimum mean $\mathsf{MIN}(x, y)$ is $\mathsf{min}(z) = \frac{z}{1-z}$ and that of the arithmetic mean $\mathsf{A}(x, y) = \frac{x+y}{2}$ is the identity function $\mathsf{a}(z) = z$. Clearly, the Seiffert functions of $\mathsf{M}_{\tan}$ and $\mathsf{M}_{\sinh}$ are the functions tan and sinh, respectively.

Remark 1.2. Throughout this paper all means are defined on $(0, \infty)^2$.

For the reader's convenience in the following sections we place the main results with their proofs, while all lemmas and technical details can be found in the last section of this paper.

The motivation for our research are the inequalities $\mathsf{MIN} < \mathsf{M}_{\tan} < \mathsf{M}_{\sinh} < \mathsf{A}$ proven in [4, Lemma 3.2]. The results obtained in this paper show what the distance is between the new and the classical means measured in different ways.

Note that optima bounds for $\mathsf{M}_{\tan}$ and $\mathsf{M}_{\sinh}$ by the arithmetic mean and the geometric and harmonic means were obtained by the authors in [2, 3].

2. Linear bounds

Given three means $K < L < M$ one may try to find the best α, β satisfying the double inequality $(1-\alpha)K + \alpha M < L < (1-\beta)K + \beta M$ or equivalently $\alpha < \frac{L-K}{M-K} < \beta$. If k, l, m are respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l} - \frac{1}{k}}{\frac{1}{m} - \frac{1}{k}} < \beta. \tag{2}$$

Thus the problem reduces to finding upper and lower bounds for certain functions defined on the interval $(0, 1)$.

Theorem 2.1. *The inequalities*

$$(1-\alpha)\,\mathsf{MIN} + \alpha\mathsf{A} < \mathsf{M}_{\tan} < (1-\beta)\,\mathsf{MIN} + \beta\mathsf{A}$$

hold if and only if $\alpha \le \frac{1}{\tan 1} \approx 0.6421$ *and* $\beta \ge 1$.

Proof. Taking Remark 1.1 and formula (2) into account we should investigate the function

$$h(z) = \frac{\frac{1}{\tan z} - \frac{1-z}{z}}{\frac{1}{z} - \frac{1-z}{z}} = \frac{1}{\tan z} - \frac{1}{z} + 1.$$

Since

$$h'(z) = \frac{1}{z^2} - \frac{1}{\sin^2 z} = \frac{\sin^2 z - z^2}{z^2 \sin^2 z} < 0,$$

so h decreases from $\lim_{z\to 0} h(z) = 1$ to $h(1) = \frac{1}{\tan 1} \approx 0.6421$. □

Theorem 2.2. *The inequalities*

$$(1-\alpha)\,\mathsf{MIN} + \alpha\mathsf{A} < \mathsf{M}_{\sinh} < (1-\beta)\,\mathsf{MIN} + \beta\mathsf{A}$$

hold if and only if $\alpha \le \frac{1}{\sinh 1} \approx 0.8509$ *and* $\beta \ge 1$.

Proof. We use formula (2) once more and investigate the function

$$h(z) = \frac{\frac{1}{\sinh z} - \frac{1-z}{z}}{\frac{1}{z} - \frac{1-z}{z}} = \frac{1}{\sinh z} - \frac{1}{z} + 1.$$

We shall show that h decreases. We have

$$\begin{aligned} h'(z) &= \frac{\sinh^2 z - z^2 \cosh z}{z^2 \sinh^2 z} \\ &= \frac{\cosh z}{z^2 \sinh^2 z}\left(\sinh z \cosh^{-\frac{1}{2}} z + z\right)\left(\sinh z \cosh^{-\frac{1}{2}} z - z\right) \\ &=: \frac{\cosh z}{z^2 \sinh^2 z}\left(\sinh z \cosh^{-\frac{1}{2}} z + z\right) g(z). \end{aligned}$$

Observe that $g(1) = \sinh 1 \cosh^{-\frac{1}{2}} 1 - 1 \approx -0.0539$, so by Lemma 7.1 the function $g(z)$ is negative in $(0,1)$. Consequently $h'(z) < 0$. To complete the proof we note that $\lim_{z\to 0} h(z) = 1$. □

3. Harmonic bounds

In this section we look for optimal bounds for means $K < L < M$ of the form $\frac{1-\alpha}{M} + \frac{\alpha}{K} < \frac{1}{L} < \frac{1-\beta}{M} + \frac{\beta}{K}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l-m}{k-m} < \beta. \tag{3}$$

We shall use the above to prove two theorems.

Theorem 3.1. *The inequalities*

$$\frac{1-\alpha}{\mathsf{A}} + \frac{\alpha}{\mathsf{MIN}} < \frac{1}{\mathsf{M}_{\tan}} < \frac{1-\beta}{\mathsf{A}} + \frac{\beta}{\mathsf{MIN}}$$

hold if and only if $\alpha \le 0$ *and* $\beta \ge 0.0939\ldots$. *See the proof for detailed explanation.*

Proof. By (3) we shall consider the function

$$h(z) = \frac{\tan z - z}{\frac{z}{1-z} - z} = \frac{(1-z)(\tan z - z)}{z^2}, \quad 0 < z < 1.$$

This function is nonnegative and $h(0+) = h(1) = 0$. Our goal then is to evaluate the maximal value of 'h' We have

$$h'(z) = \frac{(1-z)z + (z-2)\sin z\cos z + z\cos^2 z}{z^3\cos^2 z} =: \frac{g(z)}{z^3\cos^2 z}.$$

We shall show that g has only one zero in $(0,1)$. Note that $g(0) = g'(0) = 0$ and $g(1) < 0$. To achieve our goal we investigate the convexity pattern of g. We have

$$g''(z) = -2(1-(1-z)(\sin 2z + \cos 2z))$$

and

$$\begin{aligned} g'''(z) &= -2((3-2z)\sin 2z + (2z-1)\cos 2z) \\ &= -2(1-2z)\sin 2z\left(\frac{3-2z}{1-2z} - \cot 2z\right). \end{aligned}$$

We see that $g'''(\frac{1}{2}) \neq 0$. The homographic function strictly increases in $(0, \frac{1}{2})$ and in $(\frac{1}{2}, 1)$, while cot strictly decreases, so their difference strictly increases in both halves of the unit interval. Moreover, from

$$\lim_{z\to 0^+} \frac{3-2z}{1-2z} - \cot 2z = -\infty \quad \text{and} \quad \lim_{z\to\frac{1}{2}^-} \frac{3-2z}{1-2z} - \cot 2z = \infty,$$

and

$$\lim_{z\to\frac{1}{2}^+} \frac{3-2z}{1-2z} - \cot 2z = -\infty \quad \text{and} \quad \lim_{z\to 1} \frac{3-2z}{1-2z} - \cot 2z = -1 - \cot 2 < 0,$$

we conclude that g''' has exactly one zero in $(0,1)$. Additionally $g'''(0) = 2$ which means that there is z_0 such that g'' increases in $(0, z_0)$ and decreases in $(z_0, 1)$. Since $g''(0) = 0$ and $g''(1) = -2$ this means that g is convex and increasing on an interval $(0, z_1)$ then becomes concave on $(z_1, 1)$ and eventually decreases to attain negative value of $g(1)$. Thus g vanishes at exactly one point $\xi \approx 0.5637$.

Let us get back to the function h. We have $h(0+) = h(1) = 0$, so it has a maximum at ξ and $h(\xi) \approx 0.0939$.

□

And now it is time for the bound of $\mathsf{M}_{\sinh}$.

Theorem 3.2. *The inequalities*

$$\frac{1-\alpha}{\mathsf{A}} + \frac{\alpha}{\mathsf{MIN}} < \frac{1}{\mathsf{M}_{\sinh}} < \frac{1-\beta}{\mathsf{A}} + \frac{\beta}{\mathsf{MIN}}$$

hold if and only if $\alpha \leq 0$ and $\beta \geq 0.0422...$. See the proof for detailed explanation.

Proof. This time we investigate the function

$$h(z) = \frac{\sinh z - z}{\frac{z}{1-z} - z} = \frac{(1-z)(\sinh z - z)}{z^2}, \quad 0 < z < 1.$$

We have

$$h'(z) = \frac{z + (z-2)\sinh z - z(z-1)\cosh z}{z^3} =: \frac{g(z)}{z^3}.$$

This time

$$\begin{aligned} g'''(z) &= -(z^2 - z + 3)\sinh z + (1 - 5z)\cosh z \\ &= -(z^2 - z + 3)\cosh z \left(\tanh z - \frac{1 - 5z}{z^2 - z + 3}\right). \end{aligned}$$

The function $\frac{1-5z}{z^2-z+3}$ strictly decreases from $\frac{1}{3}$ to $-\frac{4}{3}$, while the hyperbolic tangent increases from 0 to $\tanh 1$, which shows that g''' has exactly one zero in the interval $(0, 1)$. Further reasoning is the same as in the proof of previous theorem.

The function h has then a single critical point $\xi \approx 0.5063$ which satisfies $g(\xi) = 0$ and its maximal value $h(\xi) \approx 0.0422$.

□

4. Quadratic bounds

Given three means $K < L < M$ one may try to find the best α, β satisfying the double inequality $\sqrt{(1-\alpha)K^2 + \alpha M^2} < L < \sqrt{(1-\beta)K^2 + \beta M^2}$ or equivalently $\alpha < \frac{L^2-K^2}{M^2-K^2} < \beta$. If k, l, m are respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l^2} - \frac{1}{k^2}}{\frac{1}{m^2} - \frac{1}{k^2}} < \beta. \tag{4}$$

Thus the problem reduces to finding upper and lower bounds for certain functions defined on the interval $(0, 1)$.

Theorem 4.1. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{MIN}^2 + \alpha \mathsf{A}^2} < \mathsf{M}_{\tan} < \sqrt{(1-\beta)\,\mathsf{MIN}^2 + \beta \mathsf{A}^2}$$

hold if and only if $\alpha \le \frac{1}{\tan^2 1} \approx 0.4123$ *and* $\beta \ge 1$.

Proof. By formula (4) we should investigate the function

$$h(z) = \frac{\dfrac{1}{\tan^2 z} - \frac{1}{\frac{z^2}{(1-z)^2}}}{\frac{1}{z^2} - \frac{1}{\frac{z^2}{(1-z)^2}}} = \frac{(z-1)^2 - \frac{z^2}{\tan^2 z}}{(z-2)z}.$$

We shall show that h decreases in the interval $(0, 1)$. We have

$$h'(z) = \frac{2\left((1-z)\sin^3 z + z^3(z-2)\cos z + z^2\cos^2 z \sin z\right)}{z^2(z-2)^2\sin^3 z}.$$

Using the known inequalities $\sin x < x - x^3/3! + x^5/5!$ and $1 - x^2/2! < \cos x < 1 - x^2/2! + x^4/4!$ we get

$$\begin{aligned} &(1-z)\sin^3 z + z^3(z-2)\cos z + z^2\cos^2 z \sin z \\ &< (1-z)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!}\right)^3 + z^3(z-2)\left(1 - \frac{z^2}{2!}\right) + z^2\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!}\right)^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!}\right) \\ &= \frac{z^5}{1728000}\big(-z^{11} + 26z^{10} + 60z^9 - 1160z^8 - 1560z^7 + 21360z^6 + 22400z^5 - 204800z^4 \\ &\quad -187200z^3 + 1065600z^2 - 1152000\big) < 0. \end{aligned}$$

Hence h' is negative which shows that h decreases. To complete the proof we note that $\lim_{z\to 0} h(z) = 1$. □

And here comes the hyperbolic sine version of the previous theorem.

Theorem 4.2. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{MIN}^2+\alpha\mathsf{A}^2}<\mathsf{M}_{\sinh}<\sqrt{(1-\beta)\,\mathsf{MIN}^2+\beta\mathsf{A}^2}$$

hold if and only if $\alpha \le \frac{1}{\sinh^2 1}\approx 0.7241$ *and* $\beta \ge 1$.

Proof. The function to be considered here is

$$h(z)=\frac{\frac{1}{\sinh^2 z}-\frac{1}{\frac{z^2}{(1-z)^2}}}{\frac{1}{z^2}-\frac{1}{\frac{z^2}{(1-z)^2}}}=\frac{(z-1)^2-\frac{z^2}{\sinh^2 z}}{(z-2)z}.$$

We shall show that h decreases. We have

$$h'(z)=\frac{2\left((1-z)\sinh^3 z+z^3(z-2)\cosh z+z^2\sinh z\right)}{z^2(z-2)^2\sinh^3 z}.$$

From the inequality $1+x^2/2!<\cosh x$ and Lemma 7.2 we obtain

$$\begin{aligned}
&(1-z)\sinh^3 z+z^3(z-2)\cosh z+z^2\sinh z\\
&<(1-z)\left(z+\frac{z^3}{3!}+2\frac{z^5}{5!}\right)^3+z^3(z-2)\left(1+\frac{z^2}{2!}\right)+z^2\left(z+\frac{z^3}{3!}+2\frac{z^5}{5!}\right)\\
&=\frac{z^5}{216000}\left(-z^{11}+z^{10}-30z^9+30z^8-480z^7+480z^6\right.\\
&\quad\left.-4600z^5+4600z^4-28800z^3+32400z^2-72000\right)<0.
\end{aligned}$$

Thus h' is negative which shows that h decreases. To complete the proof we note that $\lim_{z\to 0} h(z)=1$. □

5. Bounds with the weighted power mean of order −2

In this section we look for optimal bounds for means $K<L<M$ of the form $\sqrt{\frac{1-\alpha}{M^2}+\frac{\alpha}{K^2}}<\frac{1}{L}<\sqrt{\frac{1-\beta}{M^2}+\frac{\beta}{K^2}}$ or, in terms of their Seiffert functions,

$$\alpha<\frac{l^2-m^2}{k^2-m^2}<\beta.$$

Theorem 5.1. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{A}^2}+\frac{\alpha}{\mathsf{MIN}^2}}<\frac{1}{\mathsf{M}_{\tan}}<\sqrt{\frac{1-\beta}{\mathsf{A}^2}+\frac{\beta}{\mathsf{MIN}^2}}$$

hold if and only if $\alpha\le 0$ *and* $\beta\ge 0.0662\ldots$.

Proof. To prove the theorem we investigate the function

$$h(z)=\frac{\tan^2 z-z^2}{\frac{z^2}{(1-z)^2}-z^2}=\frac{(1-z)^2(\tan^2 z-z^2)}{(2-z)z^3}.$$

This function is obviously non-negative and $h(0+)=h(1)=0$. As in Section 3 the maximum of h cannot be evaluated in an algebraic form, so we just note that its approximate maximal value equals 0.0662... at $z\approx 0.4287$. □

Theorem 5.2. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{A}^2}+\frac{\alpha}{\mathsf{MIN}^2}}<\frac{1}{\mathsf{M}_{\sinh}}<\sqrt{\frac{1-\beta}{\mathsf{A}^2}+\frac{\beta}{\mathsf{MIN}^2}}$$

hold if and only if $\alpha \le 0$ *and* $\beta \ge 0.0307$.

Proof. We follow the same line as in the previous proof. Let

$$h(z)=\frac{\sinh^2 z - z^2}{\frac{z^2}{(1-z)^2}-z^2}=\frac{(1-z)^2(\sinh^2 z-z^2)}{(2-z)z^3}.$$

This function is obviously non-negative and $h(0+)=h(1)=0$. As in Section 3 the maximum of h cannot be evaluated in an algebraic form, so we just note that its approximate maximal value equals 0.0307... at $z \approx 0.3909$. □

6. Bounds with varying arguments.

If N is a mean, then the formula $N^{\{t\}}(x,y)=N\left(\frac{x+y}{2}+t\frac{x-y}{2},\frac{x+y}{2}-t\frac{x-y}{2}\right)$ defines a homotopy between that arithmetic mean $\mathsf{A}=N^{\{0\}}$ and $N=N^{\{1\}}$. Therefore if $N<M<\mathsf{A}$ it makes sense to ask what the optimal numbers α,β are satisfying $N^{\{\alpha\}}<M<N^{\{\beta\}}$. Theorem 6.1 from [4] gives a method for finding such numbers in terms of the Seiffert functions of the means involved. It says

Theorem 6.1. *For a Seiffert function k denote $\widehat{k}(z)=k(z)/z$. Let M and N be two means with Seiffert functions m and n, respectively. Suppose that $\widehat{n}(z)$ is strictly monotone and let $p_0=\inf\limits_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$ and $q_0=\sup\limits_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$.*
If $\mathsf{A}(x,y)<M(x,y)<N(x,y)$ for all $x\neq y$, then the inequalities

$$N^{\{p\}}(x,y)\leqslant M(x,y)\leqslant N^{\{q\}}(x,y)$$

hold if and only if $p\leqslant p_0$ and $q\geqslant q_0$.
If $N(x,y)<M(x,y)<\mathsf{A}(x,y)$ for all $x\neq y$, then the inequalities

$$N^{\{q\}}(x,y)\leqslant M(x,y)\leqslant N^{\{p\}}(x,y)$$

hold if and only if $p\leqslant p_0$ and $q\geqslant q_0$.

In the case of $N=\mathsf{MIN}$ we see that $\widehat{\mathsf{min}}(z)=\frac{1}{1-z}$ and $\widehat{\mathsf{min}}^{-1}(x)=1-x^{-1}$.

Theorem 6.2. *The inequalities*

$$\mathsf{MIN}\left(\tfrac{x+y}{2}+\alpha\tfrac{x-y}{2},\tfrac{x+y}{2}-\alpha\tfrac{x-y}{2}\right)<\mathsf{M}_{\tan}<\mathsf{MIN}\left(\tfrac{x+y}{2}+\beta\tfrac{x-y}{2},\tfrac{x+y}{2}-\beta\tfrac{x-y}{2}\right)$$

hold if and only if $\alpha\ge 1-\frac{1}{\tan 1}\approx 0.3579$ *and* $\beta\le 0$.

Proof. Using Theorem 6.1 we should find the range of the function

$$h(z)=\frac{\widehat{\mathsf{min}}^{-1}\left(\frac{\tan z}{z}\right)}{z}=\frac{1}{z}-\frac{1}{\tan z}.$$

From the proof of Theorem 2.1 we know that the function $g(z)=\frac{1}{\tan z}-\frac{1}{z}<0$ decreases. Thus $h(z)=-g(z)$ increases, which completes the proof. □

Theorem 6.3. *The inequalities*

$$\mathsf{MIN}\left(\tfrac{x+y}{2}+\alpha\tfrac{x-y}{2},\tfrac{x+y}{2}-\alpha\tfrac{x-y}{2}\right)<\mathsf{M}_{\sinh}<\mathsf{MIN}\left(\tfrac{x+y}{2}+\beta\tfrac{x-y}{2},\tfrac{x+y}{2}-\beta\tfrac{x-y}{2}\right)$$

hold if and only if $\alpha\ge 1-\frac{1}{\sinh 1}\approx 0.1491$ *and* $\beta\le 0$.

Proof. Using Theorem 6.1 we should find the range of the function

$$h(z) = \frac{\widehat{\mathsf{min}}^{-1}\left(\frac{\sinh z}{z}\right)}{z} = \frac{1}{z} - \frac{1}{\sinh z}.$$

From the proof of Theorem 2.2 we know that the function $g(z) = \frac{1}{\sinh z} - \frac{1}{z} < 0$ decreases. So $h(z) = -g(z)$ increases, which completes the proof. □

7. Tools and lemmas

In this section we place all the technical details needed to prove our main results.

Lemma 7.1 (Lazarević [1])**.** *Consider the functions* $g_u : [0, \infty) \to \mathbb{R}$

$$g_u(x) = \cosh^u x \sinh x - x, \quad -1 < u < 0.$$

For $-\frac{1}{3} \le u < 0$ *the functions* g_u *are positive. For* $-1 < u < -\frac{1}{3}$ *there exists* $x_u > 0$ *such that* g_u *is negative in* $(0, x_u)$ *and positive in* (x_u, ∞)*.*

Proof. We have $g_u(0) = g_u'(0) = 0$ and

$$g_u''(x) = u(u-1)\sinh x \cosh^u x \left[\tanh^2 x + \frac{1+3u}{u(u-1)}\right].$$

If $-1/3 \le u < 0$, we have $\frac{1+3u}{u(u-1)} \ge 0$, so g_u is convex thus positive. For $-1 < u < -1/3$ the equation $\tanh^2 x + \frac{1+3u}{u(u-1)} = 0$ has exactly one solution ξ_u, so g_u is concave and negative on $(0, \xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u. □

Lemma 7.2. *For* $0 < x < 1$ *the following inequality holds*

$$\sinh x < x + \frac{x^3}{3!} + 2\frac{x^5}{5!}.$$

Proof.

$$\sinh x - x - \frac{x^3}{3!} - \frac{x^5}{5!} = \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$
$$< \frac{x^5}{5!}\left(\frac{1}{6\cdot 7} + \frac{1}{6\cdot 7\cdot 8\cdot 9} + \dots\right) < \frac{x^5}{5!}.$$ □

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Institute of Mathematics and Physics, UTP University of Science and Technology, al. prof. Kaliskiego 7, 85-796 Bydgoszcz, Poland

Email address: monika.nowicka@utp.edu.pl

AUDoMAT, ul. Mielczarskiego 4/29, 85-796 Bydgoszcz, Poland

Email address: audomat.aw@gmail.com