OPTIMAL BOUNDS FOR THE SINE AND HYPERBOLIC TANGENT MEANS

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Abstract. We provide the optimal bounds for the sine and hyperbolic tangent sine means in terms of various weighted means of the arithmetic and quadratic means.

Contents

The means

(sine mean)
$$\mathsf{M}_{\sin}(x,y)=\begin{cases}\dfrac{x-y}{2\sin\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases},$$

and

(hyperbolic tangent mean)
$$\mathsf{M}_{\tanh}(x,y)=\begin{cases}\dfrac{x-y}{2\tanh\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}$$

defined for positive x,y, have been introduced in [6], where one of the authors investigates means of the form

$$\mathsf{M}_f(x,y)=\begin{cases}\dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x\neq y\\ x & x=y\end{cases}. \tag{1}$$

It was shown that every mean $M(x,y),\ x,y\in\mathbb{R}_+$ that is symmetric ($M(x,y)=M(y,x)$) and homogeneous ($M(\lambda x,\lambda y)=\lambda M(x,y),\ \lambda>0$) can be represented

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in the form (1) and that every function $f:(0,1)\to\mathbb{R}$ (called Seiffert function) satisfying

$$\frac{z}{1+z}\le f(z)\le\frac{z}{1-z}$$

produces a mean. Looking at the calculations below

$$\begin{aligned} M(x,y) &= \frac{x+y}{2}M\left(\frac{x+y-|y-x|}{x+y},\frac{x+y+|y-x|}{x+y}\right)\\ &= \frac{|y-x|}{2\dfrac{z}{M(1-z,1+z)}}=\frac{|y-x|}{2f(z)},\quad\text{where}\quad z=\frac{|x-y|}{x+y}, \end{aligned}\tag{2}$$

we see that the Seiffert function corresponding to M is given by the formula

$$f(z)=\frac{z}{M(1-z,1+z)}.$$

For two means M,N, the symbol $M<N$ means that the inequality $M(x,y)<N(x,y)$ holds for all $x\neq y$.

Our main tool will be the obvious fact that if for two Seiffert means the inequality $f<g$ holds, then their corresponding means satisfy $M_f>M_g$. Consequently, every inequality between means can be replaced by the inequality between their Seiffert functions.

In 2011, Chu, Wang and Gong in [2] investigated the relations between the second Seiffert mean $\mathsf{T}=\mathsf{M}_{\mathrm{arctan}}$, the arithmetic mean and the root mean square (denoted here by A and RMS) and discovered that the double inequality

$$\alpha\mathsf{RMS}+(1-\alpha)\mathsf{A}<\mathsf{T}<\beta\mathsf{RMS}+(1-\beta)\mathsf{A}$$

holds if and only if $\alpha\le(4-\pi)/[(\sqrt{2}-1)\pi]$ and $\beta\ge 2/3$, while the inequalities

$$\mathsf{RMS}^{\alpha}\mathsf{A}^{1-\alpha}<\mathsf{T}<\mathsf{RMS}^{\beta}\mathsf{A}^{1-\beta}$$

hold if and only if $\alpha\le 2/3$ and $\beta\ge 4-2\log\pi/\log 2$.

Similar results for the Neuman-Sándor mean $\mathsf{NS}=\mathsf{M}_{\mathrm{arsinh}}$ were obtained by Neuman in [5]:

$$\alpha\mathsf{RMS}+(1-\alpha)\mathsf{A}<\mathsf{NS}<\beta\mathsf{RMS}+(1-\beta)\mathsf{A}$$

holds true if and only if $\alpha\le[1-\log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})]$ and $\beta\ge 1/3$ and

$$\mathsf{RMS}^{\alpha}\mathsf{A}^{1-\alpha}<\mathsf{NS}<\mathsf{RMS}^{\beta}\mathsf{A}^{1-\beta}$$

holds if and only if $\alpha\le 1/3$, $\beta\ge\log((2+\sqrt{2})/3)/\log\sqrt{2}$.

The aim of this paper is to determine various optimal (i.e. the best possible in the class of inequalities considered) bounds for $\mathsf{M}_{\tan}$ and $\mathsf{M}_{\sinh}$ by the arithmetic and root mean square means.

Remark 0.1. Note that the root mean square mean

$$\mathsf{RMS}(x,y)=\sqrt{\frac{x^2+y^2}{2}}\quad\text{has Seiffert function}\quad\mathsf{rms}(z)=\frac{z}{\sqrt{1+z^2}},$$

and the arithmetic mean

$$\mathsf{A}(x,y)=\frac{x+y}{2}\quad\text{has Seiffert function}\quad\mathsf{a}(z)=z.$$

For the reader's convenience, in the following sections we place the main results with their proofs, while all lemmas and technical details can be found in the last section of this paper.

The motivation for our research are the inequalities

$$\mathsf{A} < \mathsf{M}_{\sin} < \mathsf{M}_{\tanh} < \mathsf{RMS},$$

proven in [6, Lemma 3.2] and Lemma 6.2.

1. Linear bounds

Given three means $K < L < M$, one may try to find the best α, β satisfying the double inequality $(1-\alpha)K + \alpha M < L < (1-\beta)K + \beta M$, or equivalently

$$\alpha < \frac{L-K}{M-K} < \beta.$$

If k, l, m are the respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l} - \frac{1}{k}}{\frac{1}{m} - \frac{1}{k}} < \beta. \tag{3}$$

Thus the problem reduces to finding the upper and lower bound for certain function defined on the interval $(0,1)$.

Theorem 1.1. *The inequalities*

$$(1-\alpha)\,\mathsf{A} + \alpha\mathsf{RMS} < \mathsf{M}_{\sin} < (1-\beta)\,\mathsf{A} + \beta\mathsf{RMS}$$

hold if, and only if, $\alpha \le \frac{1}{3}$ *and* $\beta \ge \frac{1-\sin 1}{(\sqrt{2}-1)\sin 1} \approx 0.4548$*.*

Proof. By formula (3) and Remark 0.1, we should investigate the function

$$h(z) = \frac{\frac{1}{\sin z} - \frac{1}{z}}{\frac{\sqrt{1+z^2}}{z} - \frac{1}{z}} = \frac{\dfrac{z}{\sin z} - 1}{\sqrt{1+z^2} - 1}, \quad z \in (0,1).$$

We shall show that h increases. By Lemma 6.5, it is enough to prove that the function $r(z) = (z/\sin z - 1)' / (\sqrt{1+z^2} - 1)'$ increases.

We have

$$r(z) = \frac{\sqrt{z^2+1}(\sin z - z\cos z)}{z \sin^2 z}$$

and

$$r'(z) = \frac{3z^4 + 3z^2 - (2z^3 + z)\sin 2z + (z^4 + z^2 + 1)\cos 2z - 1}{2z^2\sqrt{z^2+1}\sin^3 z}.$$

The numerator function $s(z) = 3z^4 + 3z^2 - (2z^3 + z)\sin 2z + (z^4 + z^2 + 1)\cos 2z - 1$ satisfies $s(0) = 0$ and

$$\begin{aligned} s'(z) &= 6(2z^3 + z) - (2z^4 + 8z^2 + 3)\sin 2z \\ &> 6(2z^3 + z) - (2z^4 + 8z^2 + 3)\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!}\right) \\ &= \frac{8}{15} z^5 (11 + z^2(1 - z^2)) > 0 \quad \text{as } z \in (0,1). \end{aligned}$$

The first inequality follows from $\sin 2x < 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!}$. Hence, s is positive and so is r', which shows that h increases. To complete the proof we use the Maclaurin series expansion and get

$$h(z) = \frac{z - \sin z}{(\sqrt{1+z^2}-1)\sin z} = \frac{\frac{z^3}{3!} + O(z^5)}{(\frac{z^2}{2} + O(z^4))(z + O(z^3))} = \frac{\frac{z^3}{3!} + O(z^5)}{\frac{z^3}{2} + O(z^5)} \xrightarrow[z\to 0]{} \frac{1}{3},$$

so h assumes values between $\frac{1}{3}$ and $h(1)$. □

Theorem 1.2. *The inequalities*

$$(1-\alpha)\,\mathsf{A} + \alpha\mathsf{RMS} < \mathsf{M}_{\tanh} < (1-\beta)\,\mathsf{A} + \beta\mathsf{RMS}$$

hold if, and only if, $\alpha \le \frac{2}{3}$ *and* $\beta \ge \frac{\coth 1 - 1}{\sqrt{2}-1} \approx 0.7557$.

Proof. We use once more formula (3) and investigate the function

$$h(z) = \frac{\frac{1}{\tanh z} - \frac{1}{z}}{\frac{\sqrt{1+z^2}}{z} - \frac{1}{z}} = \frac{\frac{z}{\tanh z} - 1}{\sqrt{1+z^2}-1}, \quad z \in (0,1).$$

We shall show that h increases. By Lemma 6.5, it is enough to prove that the function $r(z) = (z/\tanh z - 1)' / (\sqrt{1+z^2}-1)'$ increases. A simple calculation reveals that

$$r(z) = \frac{\sqrt{z^2+1}(\sinh 2z - 2z)}{2z\sinh^2 z}$$

and

$$r'(z) = \frac{(8z^4 + 8z^2 + 1)\cosh z - (8z^3 + 4z)\sinh z - \cosh 3z}{4z^2\sqrt{z^2+1}\sinh^3 z}.$$

Using the estimates from Lemma 6.1, we see that

$$\begin{aligned}
&(8z^4 + 8z^2 + 1)\cosh z - (8z^3 + 4z)\sinh z - \cosh 3z \\
&\quad > (8z^4 + 8z^2 + 1)\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!}\right) \\
&\quad\quad - (8z^3 + 4z)\left(z + \frac{z^3}{3!} + 2\frac{z^5}{5!}\right) - \left(1 + \frac{(3z)^2}{2!} + \frac{(3z)^4}{4!} + 2\frac{(3z)^6}{6!}\right) \\
&\quad = \frac{z^6(24z^2 + 109)}{120} > 0,
\end{aligned}$$

thus r' is positive and both r and h increase. Using Maclaurin's series we get

$$h(z) = \frac{z - \tanh z}{(\sqrt{1+z^2}-1)\tanh z} = \frac{\frac{z^3}{3} + O(z^5)}{(\frac{z^2}{2} + O(z^4))(z + O(z^3))} = \frac{\frac{z^3}{3} + O(z^5)}{\frac{z^3}{2} + O(z^5)} \xrightarrow[z\to 0]{} \frac{2}{3},$$

so h assumes values between $2/3$ and $h(1)$. □

2. Harmonic bounds

In this section, we look for the optimal bounds for means $K < L < M$ of the form

$$\frac{1-\alpha}{M} + \frac{\alpha}{K} < \frac{1}{L} < \frac{1-\beta}{M} + \frac{\beta}{K},$$

which can be written as

$$\alpha < \frac{\frac{1}{L} - \frac{1}{M}}{\frac{1}{K} - \frac{1}{M}} < \beta,$$

or — in terms of their Seiffert functions,

$$\alpha < \frac{l-m}{k-m} < \beta. \tag{4}$$

We shall use the above to prove two theorems.

Theorem 2.1. *The inequalities*

$$\frac{1-\alpha}{\mathsf{RMS}} + \frac{\alpha}{\mathsf{A}} < \frac{1}{\mathsf{M}_{\sin}} < \frac{1-\beta}{\mathsf{RMS}} + \frac{\beta}{\mathsf{A}}$$

hold if, and only if, $\alpha \le \frac{2\sin 1 - \sqrt{2}}{2-\sqrt{2}} \approx 0.4587$ *and* $\beta \ge \frac{2}{3}$.

Proof. According to formula (4), we investigate the function

$$h(z) = \frac{\sin z - \frac{z}{\sqrt{1+z^2}}}{z - \frac{z}{\sqrt{1+z^2}}} = \frac{\frac{\sqrt{1+z^2}\sin z}{z} - 1}{\sqrt{1+z^2}-1}, \quad z \in (0,1).$$

To show that h decreases we use Lemma 6.5. A simple calculation shows that

$$r(z) = \frac{\left(\frac{\sqrt{1+z^2}\sin z}{z} - 1\right)'}{(\sqrt{1+z^2}-1)'} = \frac{(z^3+z)\cos z - \sin z}{z^3}$$

and

$$r'(z) = \frac{(3-z^2-z^4)\sin z - 3z\cos z}{z^4}.$$

Using the known inequalities

$$\sin x < x - x^3/3! + x^5/5! \quad \text{and} \quad \cos x > 1 - x^2/2!$$

we get

$$\begin{aligned} r'(z) &< \frac{(3-z^2-z^4)\left(z - \frac{z^3}{3!} + \frac{z^5}{5!}\right) - 3z\left(1 - \frac{z^2}{2!}\right)}{z^4} \\ &= \frac{1}{120} z(-z^4 + 19z^2 - 97) < 0. \end{aligned}$$

Using Maclaurin's series we get

$$\begin{aligned} h(z) &= \frac{\sqrt{1+z^2}\sin z - z}{z(\sqrt{1+z^2}-1)} = \frac{\left(1 + \frac{z^2}{2} + O(z^4)\right)\left(z - \frac{z^3}{3!} + O(z^5)\right) - z}{z\left(1 + \frac{z^2}{2} + O(z^5)\right) - z} \\ &= \frac{\frac{z^3}{6} + O(z^5)}{\frac{z^3}{2} + O(z^5)} \xrightarrow[z\to 0]{} \frac{2}{3}, \end{aligned}$$

so h decreases from $\frac{2}{3}$ to $h(1)$. ∎

Theorem 2.2. *The inequalities*

$$\frac{1-\alpha}{\mathsf{RMS}} + \frac{\alpha}{\mathsf{A}} < \frac{1}{\mathsf{M}_{\tanh}} < \frac{1-\beta}{\mathsf{RMS}} + \frac{\beta}{\mathsf{A}}$$

hold if, and only if, $\alpha \le \frac{2\tanh 1 - \sqrt{2}}{2-\sqrt{2}} \approx 0.1860$ *and* $\beta \ge \frac{1}{3}$.

Proof. Taking into account formula (4), we should investigate the function

$$h(z)=\frac{\tanh z-\frac{z}{\sqrt{1+z^2}}}{z-\frac{z}{\sqrt{1+z^2}}}=\frac{\frac{\sqrt{1+z^2}\tanh z}{z}-1}{\sqrt{1+z^2}-1},\quad z\in(0,1).$$

We shall show that h decreases. By Lemma 6.5, it is enough to prove that the function $r(z)=\left(\sqrt{1+z^2}\tanh z/z-1\right)'/(\sqrt{z^2+1}-1)'$ decreases. We have

$$r(z)=\frac{z^3+z-\sinh z\cosh z}{z^3\cosh^2 z}$$

and

$$r'(z)=\frac{-2z^2(z^2+1)\sinh z+3\cosh^2 z\sinh z-3z\cosh z}{z^4\cosh^3 z}.$$

From the inequalities $\cosh x>1+x^2/2!$ and $\sinh x>x+x^3/3!$ and Lemma 6.1, we obtain

$$\begin{aligned}&-2z^2(z^2+1)\sinh z+3\cosh^2 z\sinh z-3z\cosh z\\&\quad<-2z^2(z^2+1)\left(z+\frac{z^3}{3!}\right)+3\left(1+\frac{z^2}{2!}+2\frac{z^4}{4!}\right)^2\left(z+\frac{z^3}{3!}+2\frac{z^5}{5!}\right)\\&\quad\quad-3z\left(1+\frac{z^2}{2!}\right)\\&\quad=\frac{z^5(z^8+22z^6+240z^4+504z^2-1536)}{2880}<0.\end{aligned}$$

Hence r' is negative which shows that h decreases. We use the same technique as above to show that $\lim_{z\to 0}h(z)=1/3$. □

3. Quadratic bounds

Given three means $K<L<M$, one may try to find the best α,β satisfying the double inequality $\sqrt{(1-\alpha)K^2+\alpha M^2}<L<\sqrt{(1-\beta)K^2+\beta M^2}$, or equivalently

$$\alpha<\frac{L^2-K^2}{M^2-K^2}<\beta.$$

If k,l,m are the respective Seiffert functions, then the latter can be written as

$$\alpha<\frac{\frac{1}{l^2}-\frac{1}{k^2}}{\frac{1}{m^2}-\frac{1}{k^2}}<\beta. \tag{5}$$

Thus, the problem reduces to finding the upper and lower bound for a certain function defined on the interval $(0,1)$.

Theorem 3.1. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{A}^2+\alpha\mathsf{RMS}^2}<\mathsf{M}_{\sin}<\sqrt{(1-\beta)\,\mathsf{A}^2+\beta\mathsf{RMS}^2}$$

hold if, and only if, $\alpha\le\frac{1}{3}$ *and* $\beta\ge\frac{1}{\sin^2 1}-1\approx 0.4123$.

Proof. By formula (5), we should investigate the function

$$h(z)=\frac{\dfrac{1}{\sin^2 z}-\dfrac{1}{z^2}}{\left(\dfrac{\sqrt{1+z^2}}{z}\right)^2-\dfrac{1}{z^2}}=\frac{1}{\sin^2 z}-\frac{1}{z^2},\quad z\in(0,1).$$

Its first derivative equals $h'(z) = \frac{2}{\sin^3 z}\left(\frac{\sin^3 z}{z^3} - \cos z\right)$. Lemma 6.3 implies that $h'(z) > 0$. Hence the function h increases. With help of Maclaurin we get

$$h(z) = \frac{z^2 - \left(z - \frac{z^3}{3!} + O(z^5)\right)^2}{z^2\left(z + O(z^3)\right)^2} = \frac{\frac{z^4}{3} + O(z^5)}{z^4 + O(z^5)} \xrightarrow[z\to 0]{} \frac{1}{3},$$

which completes the proof. □

And here comes the hyperbolic tangent version of the previous theorem.

Theorem 3.2. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{A}^2 + \alpha \mathsf{RMS}^2} < \mathsf{M}_{\tanh} < \sqrt{(1-\beta)\,\mathsf{A}^2 + \beta \mathsf{RMS}^2}$$

hold if, and only if, $\alpha \le \frac{2}{3}$ *and* $\beta \ge \frac{1}{\tanh^2 1} - 1 \approx 0.7241$.

Proof. The function to be considered here is

$$h(z) = \frac{\dfrac{1}{\tanh^2 z} - \dfrac{1}{z^2}}{\left(\dfrac{\sqrt{1+z^2}}{z}\right)^2 - \dfrac{1}{z^2}} = \frac{1}{\tanh^2 z} - \frac{1}{z^2}, \quad z \in (0,1).$$

Since $h'(z) = \frac{2}{\sinh^3 z}\left(\frac{\sinh^3 z}{z^3} - \cosh z\right) > 0$ (by Lemma 6.4), we see that the function h increases. The reader will check, using Maclaurin's series expansion that $\lim\limits_{z\to 0} h(z) = 2/3$, so the range of h is the interval $(\frac{2}{3}, h(1))$. □

4. Bounds by weighted power mean of order −2

In this section, we look for the optimal bounds for means $K < L < M$ of the form $\sqrt{\frac{1-\alpha}{M^2} + \frac{\alpha}{K^2}} < \frac{1}{L} < \sqrt{\frac{1-\beta}{M^2} + \frac{\beta}{K^2}}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l^2 - m^2}{k^2 - m^2} < \beta. \tag{6}$$

Theorem 4.1. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{RMS}^2} + \frac{\alpha}{\mathsf{A}^2}} < \frac{1}{\mathsf{M}_{\sin}} < \sqrt{\frac{1-\beta}{\mathsf{RMS}^2} + \frac{\beta}{\mathsf{A}^2}}$$

hold if, and only if, $\alpha \le -\cos 2 \approx 0.4161$ *and* $\beta \ge \frac{2}{3}$.

Proof. According to formula (6), we investigate the function

$$h(z) = \frac{\sin^2 z - \frac{z^2}{1+z^2}}{z^2 - \frac{z^2}{1+z^2}} = \frac{\sin^2 z - z^2\cos^2 z}{z^4}, \quad z \in (0,1).$$

We shall show that h decreases in the interval $(0,1)$. To this end we use known inequalities $x-\frac{x^3}{3!}<\sin x<x-\frac{x^3}{3!}+\frac{x^5}{5!}$ and $\cos x<1-\frac{x^2}{2!}+\frac{x^4}{4!}$. We have

$$\begin{aligned}h'(z)&=\frac{2}{z^5}\left(z(z^2+1)\sin z\cos z+z^2\cos^2 z-2\sin^2 z\right)\\&<\frac{2}{z^5}\left(z(z^2+1)\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}\right)\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}\right)+z^2\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}\right)^2\right.\\&\quad\left.-2\left(z-\frac{z^3}{3!}\right)^2\right)=\frac{z}{1440}\left(z^6-26z^4+232z^2-736\right)<0.\end{aligned}$$

To complete the proof note that $\lim_{z\to 0}h(z)=2/3$, - proof is left as an exercise to the reader.

□

Theorem 4.2. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{RMS}^2}+\frac{\alpha}{\mathsf{A}^2}}<\frac{1}{\mathsf{M}_{\tanh}}<\sqrt{\frac{1-\beta}{\mathsf{RMS}^2}+\frac{\beta}{\mathsf{A}^2}}$$

hold if, and only if, $\alpha\le\frac{\sinh^2 1-1}{\cosh^2 1}\approx 0.1601$ *and* $\beta\ge\frac{1}{3}$.

Proof. We follow the same line as in the previous proof. Let

$$h(z)=\frac{\tanh^2 z-\frac{z^2}{1+z^2}}{z^2-\frac{z^2}{1+z^2}}=\frac{\sinh^2 z-z^2}{z^4\cosh^2 z},\quad z\in(0,1).$$

We shall show that the function h decreases in $(0,1)$. We have

$$h'(z)=\frac{2z(z^2+1)\sinh z+(2z^2+1)\cosh z-\cosh 3z}{z^5\cosh^3 z}:=\frac{p(z)}{z^5\cosh^3 z}$$

and the function p satisfies $p(0)=p'(0)=\dots=p^{(5)}(0)=0$ and

$$\begin{aligned}p^{(6)}(z)&=2z(z^2+103)\sinh z+(38z^2+313)\cosh z-729\cosh 3z\\&<208\sinh z+351\cosh z-729\cosh z<0.\end{aligned}$$

Thus p is negative and so is h'. Since $\lim_{z\to 0}h(z)=1/3$ the assertion follows. □

5. Bounds with varying arguments

If N is a mean, then the formula $N^{\{t\}}(x,y)=N\left(\frac{x+y}{2}+t\frac{x-y}{2},\frac{x+y}{2}-t\frac{x-y}{2}\right)$ defines a homotopy between the arithmetic mean $\mathsf{A}=N^{\{0\}}$ and $N=N^{\{1\}}$. Therefore if $\mathsf{A}<M<N$, it make sense to ask what are the optimal numbers α,β satisfying $N^{\{\alpha\}}<M<N^{\{\beta\}}$. Theorem 6.1 from [6] gives a method for finding such numbers in terms of the Seiffert functions of the means involved.

Theorem 5.1 (Witkowski [6])**.** *For a Seiffert function k denote by $\widehat{k}(z)=k(z)/z$. Let M and N be two means with Seiffert functions m and n, respectively. Suppose that $\widehat{n}(z)$ is strictly monotone and let $p_0=\inf_z\frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$ and $q_0=\sup_z\frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$. If $\mathsf{A}(x,y)<M(x,y)<N(x,y)$ for all $x\neq y$ then the inequalities*

$$N^{\{p\}}(x,y)\leqslant M(x,y)\leqslant N^{\{q\}}(x,y)$$

hold if, and only if, $p \leqslant p_0$ *and* $q \geqslant q_0$.
If $N(x,y) < M(x,y) < \mathsf{A}(x,y)$ *for all* $x \neq y$ *then the inequalities*

$$N^{\{q\}}(x,y) \leqslant M(x,y) \leqslant N^{\{p\}}(x,y)$$

hold if, and only if, $p \leqslant p_0$ *and* $q \geqslant q_0$.

In case of $N = \mathsf{RMS}$ we see that $\widehat{\mathsf{rms}}(z) = \frac{1}{\sqrt{1+z^2}}$ and $\widehat{\mathsf{rms}}^{-1}(z) = \sqrt{z^{-2}-1}$.

Theorem 5.2. *The inequalities*

$$\mathsf{RMS}\left(\tfrac{x+y}{2} + \alpha\tfrac{x-y}{2}, \tfrac{x+y}{2} - \alpha\tfrac{x-y}{2}\right) < \mathsf{M}_{\sin} < \mathsf{RMS}\left(\tfrac{x+y}{2} + \beta\tfrac{x-y}{2}, \tfrac{x+y}{2} - \beta\tfrac{x-y}{2}\right)$$

hold if, and only if, $\alpha \leq \sqrt{\frac{1}{3}} \approx 0.5774$ *and* $\beta \geq \cot 1 \approx 0.6421$.

Proof. Here, we investigate the function

$$h(z) = \frac{\widehat{\mathsf{rms}}^{-1}\left(\frac{\sin z}{z}\right)}{z} = \sqrt{\frac{1}{\sin^2 z} - \frac{1}{z^2}}.$$

The monotonicity of the function h^2 follows from the proof of Theorem 3.1, so evaluation of the values of h at the endpoints completes the proof. □

Theorem 5.3. *The inequalities*

$$\mathsf{RMS}\left(\tfrac{x+y}{2} + \alpha\tfrac{x-y}{2}, \tfrac{x+y}{2} - \alpha\tfrac{x-y}{2}\right) < \mathsf{M}_{\tanh} < \mathsf{RMS}\left(\tfrac{x+y}{2} + \beta\tfrac{x-y}{2}, \tfrac{x+y}{2} - \beta\tfrac{x-y}{2}\right)$$

hold if, and only if, $\alpha \leq \sqrt{\frac{2}{3}} \approx 0.8165$ *and* $\beta \geq \frac{2e}{e^2-1} \approx 0.8509$.

Proof. According to Theorem 5.1 we investigate the function

$$h(z) = \frac{\widehat{\mathsf{rms}}^{-1}\left(\frac{\tanh z}{z}\right)}{z} = \sqrt{\frac{1}{\tanh^2 z} - \frac{1}{z^2}}.$$

Slight modification of the proof of Theorem 3.2 shows that h increases from $\sqrt{2/3}$ to $\sqrt{1/\tanh^2 1 - 1}$, which completes the proof. □

6. Tools and lemmas

In this section, we place all the technical details needed to prove our main results.

Lemma 6.1. *For* $0 < x < 1$, *the following inequalities hold*

a) $\cosh x > 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$,

b) $\sinh x < x + \dfrac{x^3}{3!} + 2\dfrac{x^5}{5!}$,

c) $\cosh 3x < 1 + \dfrac{(3x)^2}{2!} + \dfrac{(3x)^4}{4!} + 2\dfrac{(3x)^6}{6!}$,

d) $\cosh 2x < 1 + \dfrac{(2x)^2}{2!} + \dfrac{(2x)^4}{4!} + 2\dfrac{(2x)^6}{6!}$.

Proof. a) Just truncate the Taylor series of the hyperbolic cosine.
b)

$$\sinh x - x - \frac{x^3}{3!} - \frac{x^5}{5!} = \frac{x^7}{7!} + \frac{x^9}{9!} + \dots < \frac{x^5}{5!}\left(\frac{1}{6\cdot 7} + \frac{1}{6\cdot 7\cdot 8\cdot 9} + \dots\right) < \frac{x^5}{5!}.$$

c)

$$\cosh 3x - 1 - \frac{(3x)^2}{2!} - \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} = \frac{(3x)^8}{8!} + \frac{(3x)^{10}}{10!} + \dots$$
$$< \frac{(3x)^6}{6!}\left(\frac{3^2}{7\cdot 8} + \frac{3^4}{7\cdot 8\cdot 9\cdot 10}\dots\right) < \frac{(3x)^6}{6!}.$$

d) follows from c).

□

Lemma 6.2. *For all $0 < x \neq y$, the inequality*

$$\mathsf{M}_{\tanh}(x,y) < \mathsf{RMS}(x,y) \tag{7}$$

holds.

Proof. In terms of Seiffert functions the inequality (7) reads

$$\tanh z > \frac{z}{\sqrt{1+z^2}}$$

which for positive z is equivalent to $\cosh^2 z > 1 + z^2$. The latter holds, because $\cosh^2 z > (1 + z^2/2)^2 > 1 + z^2$. □

Lemma 6.3 (Mitrinović & Adamović [4])**.** *Consider the functions $f_u : [0, \pi/2) \to \mathbb{R}$*

$$f_u(x) = \cos^u x \sin x - x, \quad -1 < u < 0.$$

For $-1 \le u \le -\frac{1}{3}$, the functions f_u are positive. For $-\frac{1}{3} < u < 0$, there exists $0 < x_u < \frac{\pi}{2}$ such that f_u is negative in $(0, x_u)$ and positive in (x_u, ∞).

Proof. We have $f_u(0) = f_u'(0) = 0$ and

$$f_u''(x) = u(u-1)\sin x \cos^u x \left[\tan^2 x - \frac{1+3u}{u(u-1)}\right].$$

If $-1 \le u < -1/3$, we have $\frac{3u+1}{u(u-1)} \le 0$, so f_u is convex, thus positive.

For $-1/3 < u < 0$, the equation $\tan^2 x - \frac{1+3u}{u(u-1)} = 0$ has exactly one solution ξ_u, so f_u is concave and negative on $(0, \xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u. □

Lemma 6.4 (Lazarević [3])**.** *Consider the functions $g_u : [0, \infty) \to \mathbb{R}$*

$$g_u(x) = \cosh^u x \sinh x - x, \quad -1 < u < 0.$$

For $-1/3 \le u < 0$, the functions g_u are positive. For $-1 < u < -1/3$, there exists $x_u > 0$ such that g_u is negative in $(0, x_u)$ and positive in (x_u, ∞).

Proof. We have $g_u(0) = g_u'(0) = 0$ and

$$g_u''(x) = u(u-1)\sinh x \cosh^u x \left[\tanh^2 x + \frac{1+3u}{u(u-1)}\right].$$

If $-1/3 \le u < 0$, we have $\frac{1+3u}{u(u-1)} \ge 0$, so g_u is convex thus positive. For $-1 < u < -1/3$, the equation $\tanh^2 x + \frac{1+3u}{u(u-1)} = 0$ has exactly one solution ξ_u, so g_u is concave and negative on $(0, \xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u. □

Lemma 6.5 (Anderson et al. [1]). *Suppose $f, g : (a, b) \to \mathbb{R}$ are differentiable functions with $g'(x) \neq 0$ and such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0$. Then*

(1) if $\frac{f'}{g'}$ is increasing on (a, b), then $\frac{f}{g}$ is increasing on (a, b),

(2) if $\frac{f'}{g'}$ is decreasing on (a, b), then $\frac{f}{g}$ is decreasing on (a, b).

Proof. We shall consider the case $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0$, $g' > 0$ and $\frac{f'}{g'}$ is increasing. Other cases are analogous.

Consider the function $h = f \circ g^{-1} : (g(a), 0) \to \mathbb{R}$. Its derivative $(f \circ g^{-1})'(t) = \frac{f'}{g'}(g^{-1}(t))$ is increasing as a composition of two increasing functions, so the function h is convex. If we set $h(0) = 0$, then h remains convex on $(g(a), 0]$, which means that its divided difference $\frac{h(t)-h(0)}{t-0} = \frac{f(g^{-1}(t))}{t}$ is increasing, thus so is $\frac{f(g^{-1}(g(x))}{g(x)} = \frac{f(x)}{g(x)}$. □

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