OPTIMAL BOUNDS FOR THE SINE AND HYPERBOLIC TANGENT MEANS II

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Abstract. We provide the optimal bounds for the tangent and hyperbolic sine means in terms of various weighted means of the arithmetic and contraharmonic means.

Contents

1. Introduction, definitions and notations

The means of positive arguments

(sine mean)
$$\mathsf{M}_{\sin}(x,y)=\begin{cases}\dfrac{x-y}{2\sin\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}$$

and

(hyperbolic tangent mean)
$$\mathsf{M}_{\tanh}(x,y)=\begin{cases}\dfrac{x-y}{2\tanh\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}$$

have been introduced in [4], where one of the authors investigates the means of the form

$$\mathsf{M}_f(x,y)=\begin{cases}\dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x\neq y\\ x & x=y\end{cases}. \tag{1}$$

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It was shown that every symmetric and homogeneous mean defined for positive arguments can be represented in the form (1) and that every function $f:(0,1)\to\mathbb{R}$ (called Seiffert function) satisfying

$$\frac{z}{1+z}\le f(z)\le\frac{z}{1-z}$$

produces a mean. The correspondence between means and Seiffert functions is given by the formula

$$f(z)=\frac{z}{M(1-z,1+z)},\quad \text{where}\quad z=\frac{|x-y|}{x+y}.$$

The aim of this paper is to determine various optimal bounds for the $\mathsf{M}_{\tanh}$ and $\mathsf{M}_{\sin}$ by the arithmetic and contraharmonic means (denoted here by A and C). Note that similar results for the arithmetic and the root mean square have been obtained in [3].

For two means M,N, the symbol $M<N$ denotes that for all positive $x\neq y$ the inequality $M(x,y)<N(x,y)$ holds .

Our main tool will be the obvious fact that if for two Seiffert means the inequality $f<g$ holds, then their corresponding means satisfy $M_f>M_g$. Thus every inequality between means can be replaced by the inequality between their Seiffert functions.

Remark 1.1. Note that the Seiffert function of the contraharmonic mean $\mathsf{C}(x,y)=\frac{x^2+y^2}{x+y}$ is $\mathsf{c}(z)=\frac{z}{1+z^2}$ and that of the arithmetic mean $\mathsf{A}(x,y)=\frac{x+y}{2}$ is the identity function $\mathsf{a}(z)=z$. Clearly, the Seiffert functions of $\mathsf{M}_{\sin}$ and $\mathsf{M}_{\tanh}$ are the functions sin and tanh, respectively.

Remark 1.2. Throughout this paper all means are defined on the interval $(0,\infty)$.

For the reader's convenience, in the following sections we place the main results with their proofs, while all lemmas and technical details can be found in the last section of this paper.

The motivation for our research are the inequalities $\mathsf{A}<\mathsf{M}_{\sin}<\mathsf{M}_{\tanh}<\mathsf{C}$ proven in [4, Lemma 3.2].

2. Linear bounds

Given three means $K<L<M$, one may try to find the best α,β satisfying double inequality $(1-\alpha)K+\alpha M<L<(1-\beta)K+\beta M$ or equivalently $\alpha<\frac{L-K}{M-K}<\beta$. If k,l,m are respective Seiffert functions, then the latter can be written as

$$\alpha<\frac{\frac{1}{l}-\frac{1}{k}}{\frac{1}{m}-\frac{1}{k}}<\beta. \tag{2}$$

Therefore the problem reduces to finding the upper and lower bound for certain function defined on the interval $(0,1)$.

Theorem 2.1. *The inequalities*

$$(1-\alpha)\,\mathsf{A}+\alpha\mathsf{C}<\mathsf{M}_{\sin}<(1-\beta)\,\mathsf{A}+\beta\mathsf{C}$$

hold if, and only if, $\alpha\le\frac{1}{6}$ *and* $\beta\ge\frac{1}{\sin 1}-1\approx 0.1884$.

Proof. Taking into account Remark 1.1 and the formula (2), we should investigate the function

$$h(z) = \frac{\frac{1}{\sin z} - \frac{1}{z}}{\frac{1+z^2}{z} - \frac{1}{z}} = \frac{1}{z \sin z} - \frac{1}{z^2}$$

and find its bounds on the interval $(0, 1)$. We intend to show that h increases. Its derivative equals

$$h'(z) = \frac{2\sin^2 z - z^2 \cos z - z \sin z}{z^3 \sin^2 z} =: \frac{s(z)}{z^3 \sin^2 z}.$$

Since $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ and $\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ we can write

$$s(z) > 2\left(z - \frac{z^3}{3!}\right)^2 - z^2\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!}\right) - z\left(z - \frac{z^3}{3!} + \frac{z^5}{5!}\right) = \frac{z^6}{180} > 0,$$

so $h'(z) > 0$. We complete the proof by noting that $\lim_{z\to 0} h(z) = 1/6$. □

Theorem 2.2. *The inequalities*

$$(1-\alpha)\,\mathsf{A} + \alpha \mathsf{C} < \mathsf{M}_{\tanh} < (1-\beta)\,\mathsf{A} + \beta \mathsf{C}$$

hold if, and only if, $\alpha \le \frac{1}{\tanh 1} - 1 \approx 0.3130$ *and* $\beta \ge \frac{1}{3}$.

Proof. We use once more Remark 1.1 and the formula (2), and investigate the function $h : (0, 1) \to \mathbb{R}$ given by the formula

$$h(z) = \frac{\dfrac{1}{\tanh z} - \frac{1}{z}}{\frac{1+z^2}{z} - \frac{1}{z}} = \frac{1}{z}\left(\frac{1}{\tanh z} - \frac{1}{z}\right) =: \frac{s(z)}{z}.$$

The function s satisfies $\lim_{z\to 0} s(z) = 0$ and $s''(z) = \frac{2}{\sinh^3 z}\left(\cosh z - \frac{\sinh^3 z}{z^3}\right) < 0$ (by Lemma 7.1), so by Property 7.2, the function h decreases. To complete the proof note that $\lim_{z\to 0} h(z) = 1/3$. □

3. Harmonic bounds

In this section, we look for the optimal bounds for means $K < L < M$ of the form $\frac{1-\alpha}{M} + \frac{\alpha}{K} < \frac{1}{L} < \frac{1-\beta}{M} + \frac{\beta}{K}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l - m}{k - m} < \beta. \tag{3}$$

We shall use the above to prove two theorems.

Theorem 3.1. *The inequalities*

$$\frac{1-\alpha}{\mathsf{C}} + \frac{\alpha}{\mathsf{A}} < \frac{1}{\mathsf{M}_{\sin}} < \frac{1-\beta}{\mathsf{C}} + \frac{\beta}{\mathsf{A}}$$

hold if, and only if, $\alpha \le 2\sin 1 - 1 \approx 0.6829$ *and* $\beta \ge \frac{5}{6}$.

Proof. By formula (3), we shall consider the function

$$h(z) = \frac{\sin z - \frac{z}{1+z^2}}{z - \frac{z}{1+z^2}} = \frac{(1+z^2)\sin z - z}{z^3}$$

in the interval $(0, 1)$. We shall show that h decreases. A simple calculation shows that

$$h'(z) = \frac{(z^3 + z)\cos z - (z^2 + 3)\sin z + 2z}{z^4} =: \frac{s(z)}{z^4}.$$

The function s satisfies $s(0)=s'(0)=s''(0)=0$ and

$$s'''(z)=z\left((z^2-11)\sin z-8z\cos z\right)<0.$$

Therefore s is negative and so is h'. We complete the proof by noting that $\lim_{z\to 0}h(z)=5/6$. □

Theorem 3.2. *The inequalities*

$$\frac{1-\alpha}{\mathsf{C}}+\frac{\alpha}{\mathsf{A}}<\frac{1}{\mathsf{M}_{\tanh}}<\frac{1-\beta}{\mathsf{C}}+\frac{\beta}{\mathsf{A}}$$

hold if, and only if, $\alpha\le 2\tanh 1-1\approx 0.5232$ *and* $\beta\ge\frac{2}{3}$.

Proof. We use once more Remark 1.1 and the formula (3), and investigate the function

$$h(z)=\frac{\tanh z-\frac{z}{1+z^2}}{z-\frac{z}{1+z^2}}=\frac{(1+z^2)\tanh z-z}{z^3}.$$

We shall show that h is decreasing. We have

$$h'(z)=\frac{z^3+z-(z^2+3)\sinh z\cosh z+2z\cosh^2 z}{z^4\cosh^2 z}=:\frac{s(z)}{z^4\cosh^2 z}.$$

The function s satisfies $s(0)=s'(0)=s''(0)=s'''(0)=0$ and

$$s^{(4)}(z)=-16\left((z^2+2)\sinh z\cosh z+z\sinh^2 z+z\cosh^2 z\right)<0.$$

Thus, s is negative and so is h'. We complete the proof by noting that $\lim_{z\to 0}h(z)=2/3$. □

4. Quadratic bounds

Given three means $K<L<M$, one may try to find the best α,β satisfying double inequality $\sqrt{(1-\alpha)K^2+\alpha M^2}<L<\sqrt{(1-\beta)K^2+\beta M^2}$ or equivalently $\alpha<\frac{L^2-K^2}{M^2-K^2}<\beta$. If k,l,m are respective Seiffert functions, then the latter can be written as

$$\alpha<\frac{\frac{1}{l^2}-\frac{1}{k^2}}{\frac{1}{m^2}-\frac{1}{k^2}}<\beta. \tag{4}$$

Thus, the problem reduces to finding the upper and lower bound for certain function defined on the interval $(0,1)$.

Theorem 4.1. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{A}^2+\alpha\mathsf{C}^2}<\mathsf{M}_{\sin}<\sqrt{(1-\beta)\,\mathsf{A}^2+\beta\mathsf{C}^2}$$

hold if, and only if, $\alpha\le\frac{1}{3\tan^2 1}\approx 0.1374$ *and* $\beta\ge\frac{1}{6}$.

Proof. By the formula (4), we should investigate the function

$$h(z)=\frac{\dfrac{1}{\sin^2 z}-\dfrac{1}{z^2}}{\dfrac{(1+z^2)^2}{z^2}-\dfrac{1}{z^2}}=\frac{\dfrac{z^2}{\sin^2 z}-1}{(1+z^2)^2-1}.$$

We shall show that h decreases. By Lemma 7.2, it is enough to prove that function $r(z)=\left(z^2/\sin^2 z-1\right)'/((z^2+1)^2-1)'$ decreases. By a simple calculation we obtain

$$r(z)=\frac{\sin z-z\cos z}{2(z^2+1)\sin^3 z}$$

and

$$r'(z) = \frac{2(2z^3+z) - (z^2+3)\sin 2z + 2z(z^2+2)\cos 2z}{4(z^2+1)^2 \sin^4 z} =: \frac{s(z)}{4(z^2+1)^2 \sin^4 z}.$$

Since $\sin 2x > 2x - (2x)^3/3! + (2x)^5/5! - (2x)^7/7!$ and $\cos 2x < 1 - (2x)^2/2! + (2x)^4/4! - (2x)^6/6! + (2x)^8/8!$ we have

$$\begin{aligned} s(z) &< 2(2z^3+z) - (z^2+3)\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \frac{(2z)^7}{7!}\right) \\ &+ 2z(z^2+2)\left(1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \frac{(2z)^6}{6!} + \frac{(2z)^8}{8!}\right) \\ &= \frac{4}{315} z^5 (z^6 - 10z^4 + 62z^2 - 63) < 0. \end{aligned}$$

Thus r' is negative which shows that h decreases. We complete the proof by noting that $\lim\limits_{z\to 0} h(z) = 1/6$. □

And here comes the hyperbolic tangent version of the previous theorem.

Theorem 4.2. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{A}^2 + \alpha \mathsf{C}^2} < \mathsf{M}_{\tanh} < \sqrt{(1-\beta)\,\mathsf{A}^2 + \beta \mathsf{C}^2}$$

hold if, and only if, $\alpha \le \frac{4e^2}{3(e^2-1)^2} \approx 0.2414$ *and* $\beta \ge \frac{1}{3}$.

Proof. The function to be considered here is

$$h(z) = \frac{\dfrac{1}{\tanh^2 z} - \dfrac{1}{z^2}}{\dfrac{(1+z^2)^2}{z^2} - \dfrac{1}{z^2}} = \frac{\dfrac{z^2}{\tanh^2 z} - 1}{(1+z^2)^2 - 1}.$$

We have

$$h'(z) = -\frac{(4z^5 + 8z^3)\cosh z + (z^4 + 6z^2 + 6)\sinh z + (z^4 - 2z^2 - 2)\sinh 3z}{2z^3(z^2+2)^2 \cosh^3 z}$$

Denote $s(z) = (4z^5 + 8z^3)\cosh z + (z^4 + 6z^2 + 6)\sinh z + (z^4 - 2z^2 - 2)\sinh 3z$. Then, by Lemma 7.3, we have

$$\begin{aligned} s(z) &> (4z^5 + 8z^3)\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!}\right) \\ &+ (z^4 + 6z^2 + 6)\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!}\right) \\ &+ (z^4 - 2z^2 - 2)\left(3z + \frac{(3z)^3}{3!} + \frac{(3z)^5}{5!} + 2\frac{(3z)^7}{7!}\right) \\ &= z^7 \frac{4403z^4 + 2402z^2 + 6378}{7!} > 0. \end{aligned}$$

Thus, h decreases from $\lim_{z\to 0} h(z) = 1/3$ to $h(1) = \frac{4e^2}{3(e^2-1)^2}$. □

5. Bounds by weighted power mean of order −2

In this section, we look for the optimal bounds for means $K < L < M$ of the form $\sqrt{\frac{1-\alpha}{M^2}+\frac{\alpha}{K^2}} < \frac{1}{L} < \sqrt{\frac{1-\beta}{M^2}+\frac{\beta}{K^2}}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l^2-m^2}{k^2-m^2} < \beta. \tag{5}$$

Theorem 5.1. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{C}^2}+\frac{\alpha}{\mathsf{A}^2}} < \frac{1}{\mathsf{M}_{\sin}} < \sqrt{\frac{1-\beta}{\mathsf{C}^2}+\frac{\beta}{\mathsf{A}^2}}$$

hold if, and only if, $\alpha \le \frac{4\sin^2 1-1}{3} \approx 0.6108$ *and* $\beta \ge \frac{5}{6} \approx 0.8333$.

Proof. To prove our result we use the formula (5), and investigate the upper and lower bound in the unit open interval of the function

$$h(z) = \frac{\sin^2 z - \frac{z^2}{(1+z^2)^2}}{z^2 - \frac{z^2}{(1+z^2)^2}} = \frac{(1+z^2)^2\sin^2 z - z^2}{z^4(2+z^2)}.$$

We have

$$h'(z) = \frac{(z^2+1)\left(-z^4+z^2-4+z(z^4+3z^2+2)\sin 2z+(z^4+3z^2+4)\cos 2z\right)}{z^5(z^2+2)^2}$$

and

$$\begin{aligned}
-z^4+&z^2-4+z(z^4+3z^2+2)\sin 2z+(z^4+3z^2+4)\cos 2z\\
&< -z^4+z^2-4+z(z^4+3z^2+2)\left(2z-\frac{(2z)^3}{3!}+\frac{(2z)^5}{5!}\right)\\
&\quad+(z^4+3z^2+4)\left(1-\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}\right)\\
&= \frac{2}{15}z^6(2z^4+z^2-11) < 0.
\end{aligned}$$

Therefore h decreases from $\lim_{z\to 0} h(z) = 5/6$ to $h(1) = \frac{4\sin^2 1-1}{3} \approx 0.6108$. □

Theorem 5.2. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{C}^2}+\frac{\alpha}{\mathsf{A}^2}} < \frac{1}{\mathsf{M}_{\tanh}} < \sqrt{\frac{1-\beta}{\mathsf{C}^2}+\frac{\beta}{\mathsf{A}^2}}$$

hold if, and only if, $\alpha \le \frac{4\tanh^2 1-1}{3} \approx 0.4400$ *and* $\beta \ge \frac{2}{3}$.

Proof. This time we investigate the function

$$h(z) = \frac{\tanh^2 z - \frac{z^2}{(1+z^2)^2}}{z^2 - \frac{z^2}{(1+z^2)^2}} = \frac{(1+z^2)^2\tanh^2 z - z^2}{z^4(2+z^2)}.$$

This function decreases in $(0,1)$, because by Lemma 7.3

$$h'(z) = \frac{z^2+1}{2z^5(2+z^2)^2\cosh^3 z}\begin{pmatrix}(4z^5+12z^3+8z)\sinh z\\ +(z^4+9z^2+4)\cosh z\\ -(z^4+z^2+4)\cosh 3z\end{pmatrix}$$

$$< \frac{z^2+1}{2z^5(2+z^2)^2\cosh^3 z}\begin{pmatrix}(4z^5+12z^3+8z)\left(z+\frac{z^3}{3!}+2\frac{z^5}{5!}\right)\\ +(z^4+9z^2+4)\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+2\frac{z^6}{6!}\right)\\ -(z^4+z^2+4)\left(1+\frac{(3z)^2}{2!}+\frac{(3z)^4}{4!}+\frac{(3z)^6}{6!}\right)\end{pmatrix}$$

$$= -\frac{z(z^2+1)}{1440(2+z^2)^2\cosh^3 z}(679z^4+2487z^2+3532) < 0.$$

So the function h assumes values between $h(1)$ and $\lim_{z\to 0} h(z) = 2/3$. □

6. Bounds with varying arguments.

If N is a mean then the formula $N^{\{t\}}(x,y) = N\left(\frac{x+y}{2}+t\frac{x-y}{2}, \frac{x+y}{2}-t\frac{x-y}{2}\right)$ defines a homotopy between the arithmetic mean $\mathsf{A} = N^{\{0\}}$ and $N = N^{\{1\}}$. Therefore, if $\mathsf{A} < M < N$, it makes sense to ask what are the optimal numbers α, β satisfying $N^{\{\alpha\}} < M < N^{\{\beta\}}$. Theorem 6.1 from [4] gives a method for finding such numbers in terms of the Seiffert functions of the means involved. It says

Theorem 6.1. *For a Seiffert function k, denote $\widehat{k}(z) = k(z)/z$. Let M and N be two means with Seiffert functions m and n, respectively. Suppose that $\widehat{n}(z)$ is strictly monotone and let $p_0 = \inf_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$ and $q_0 = \sup_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$.*
If $\mathsf{A}(x,y) < M(x,y) < N(x,y)$ for all $x \neq y$, then the inequalities

$$N^{\{p\}}(x,y) \leqslant M(x,y) \leqslant N^{\{q\}}(x,y)$$

hold if, and only if, $p \leqslant p_0$ and $q \geqslant q_0$.
If $N(x,y) < M(x,y) < \mathsf{A}(x,y)$ for all $x \neq y$, then the inequalities

$$N^{\{q\}}(x,y) \leqslant M(x,y) \leqslant N^{\{p\}}(x,y)$$

hold if, and only if, $p \leqslant p_0$ and $q \geqslant q_0$.

In case of $N = \mathsf{C}$ we see that $\widehat{c}(z) = \frac{1}{1+z^2}$ and $\widehat{c}^{-1}(x) = \sqrt{x^{-1}-1}$.

Theorem 6.2. *The inequalities*

$$\mathsf{C}\left(\frac{x+y}{2}+\alpha\frac{x-y}{2}, \frac{x+y}{2}-\alpha\frac{x-y}{2}\right) < \mathsf{M}_{\sin} < \mathsf{C}\left(\frac{x+y}{2}+\beta\frac{x-y}{2}, \frac{x+y}{2}-\beta\frac{x-y}{2}\right)$$

hold if, and only if, $\alpha \leq \sqrt{\frac{1}{6}} \approx 0.4082$ and $\beta \geq \sqrt{\frac{1}{\sin 1}-1} \approx 0.4340$.

Proof. According to Theorem 6.1, we shall consider the function

$$h(z) = \frac{\sqrt{\frac{z}{\sin z}-1}}{z}$$

and its upper and lower bounds in the interval $(0,1)$. The range of its square was found in the proof of Theorem 2.1. □

Theorem 6.3. *The inequalities*

$$\mathsf{C}\left(\frac{x+y}{2}+\alpha\frac{x-y}{2}, \frac{x+y}{2}-\alpha\frac{x-y}{2}\right) < \mathsf{M}_{\tanh} < \mathsf{C}\left(\frac{x+y}{2}+\beta\frac{x-y}{2}, \frac{x+y}{2}-\beta\frac{x-y}{2}\right)$$

hold if, and only if, $\alpha \leq \sqrt{\coth 1 - 1} \approx 0.5595$ and $\beta \geq 0.5774$.

Proof. According to Theorem 6.1, we shall consider the function

$$h(z) = \frac{\sqrt{\frac{z}{\tanh z} - 1}}{z},$$

but we found the range of its square in the proof of Theorem 2.2. □

7. Tools and lemmas

In this section, we place all technical details needed to prove our main results.

Property 7.1. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is convex if, and only if, for every $\theta \in I$ its divided difference $\frac{f(x)-f(\theta)}{x-\theta}$ increases for $x \neq \theta$.

Its proof is an elementary exercise for the reader, and it is a starting point in the theory of higher order convexity. Simple consequence of Property 7.1 is

Property 7.2. If a function $f : (a,b) \to \mathbb{R}$ is convex and $\lim_{x\to a} f(x) = \Theta$, then the function $\frac{f(x)-\Theta}{x-a}$ increases.

Lemma 7.1 (Lazarević [2])**.** *Consider the functions* $g_u : [0,\infty) \to \mathbb{R}$

$$g_u(x) = \cosh^u x \sinh x - x, \quad -1 < u < 0.$$

For $-1/3 \le u < 0$*, the functions* g_u *are positive. For* $-1 < u < -1/3$*, there exists* $x_u > 0$*, such that* g_u *is negative in* $(0, x_u)$ *and positive in* (x_u, ∞)*.*

Proof. We have $g_u(0) = g_u'(0) = 0$ and

$$g_u''(x) = u(u-1)\sinh x \cosh^u x \left[\tanh^2 x + \frac{1+3u}{u(u-1)}\right].$$

If $-1/3 \le u < 0$, we have $\frac{1+3u}{u(u-1)} \ge 0$, so g_u is convex thus positive. For $-1 < u < -1/3$, the equation $\tanh^2 x + \frac{1+3u}{u(u-1)} = 0$ has exactly one solution ξ_u, so g_u is concave and negative on $(0, \xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u. □

The next lemma can be found in [1, Theorem 1.25].

Lemma 7.2. *Suppose* $f, g : (a,b) \to \mathbb{R}$ *are differentiable with* $g'(x) \neq 0$ *and such that* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *or* $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0$*. Then*

(1) *if* $\frac{f'}{g'}$ *is increasing on* (a,b)*, then* $\frac{f}{g}$ *is increasing on* (a,b)*,*
(2) *if* $\frac{f'}{g'}$ *is decreasing on* (a,b)*, then* $\frac{f}{g}$ *is decreasing on* (a,b)*.*

Lemma 7.3. *For* $0 < x < 1$*, the following inequalities hold*

a) $\sinh 3x < 3x + \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} + 2\frac{(3x)^7}{7!}$,

b) $\sinh x < x + \frac{x^3}{3!} + 2\frac{x^5}{5!}$,

c) $\cosh x < 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + 2\frac{x^6}{6!}$.

Proof. Proof a)

$$\sinh(3x) - 3x - \frac{(3x)^3}{3!} - \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} = \frac{(3x)^9}{9!} + \frac{(3x)^{11}}{11!} + \dots$$
$$< \frac{(3x)^7}{7!}\left(\frac{3^2}{8\cdot 9} + \frac{3^4}{8\cdot 9\cdot 10\cdot 11} + \dots\right) < \frac{(3x)^7}{7!}.$$

Other proofs are similar. □

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