OPTIMAL BOUNDS FOR THE SINE AND HYPERBOLIC TANGENT MEANS III

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Abstract. We provide the optimal bounds for the sine and hyperbolic tangent means in terms of various weighted means of the arithmetic and maximum means.

1. Introduction, definitions and notations

Contents

The means

(sine mean)
$$\mathsf{M}_{\sin}(x,y)=\begin{cases}\dfrac{x-y}{2\sin\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases},$$

and

(hyperbolic tangent mean)
$$\mathsf{M}_{\tanh}(x,y)=\begin{cases}\dfrac{x-y}{2\tanh\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}$$

defined for positive x,y, have been introduced in [4], where one of the authors investigates means of the form

$$\mathsf{M}_f(x,y)=\begin{cases}\dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x\neq y\\ x & x=y\end{cases}. \tag{1}$$

It was shown that every mean $M(x,y),\ x,y\in\mathbb{R}_+$ that is symmetric ($M(x,y)=M(y,x)$) and homogeneous ($M(\lambda x,\lambda y)=\lambda M(x,y),\ \lambda>0$) can be represented

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in the form (1) and that every function $f : (0,1) \to \mathbb{R}$ (called Seiffert function) satisfying

$$\frac{z}{1+z} \le f(z) \le \frac{z}{1-z}$$

produces a mean. Looking at the calculations below

$$\begin{aligned} M(x,y) &= \frac{x+y}{2} M\left(\frac{x+y-|y-x|}{x+y}, \frac{x+y+|y-x|}{x+y}\right) \\ &= \frac{|y-x|}{2\frac{z}{M(1-z,1+z)}} = \frac{|y-x|}{2f(z)}, \quad \text{where} \quad z = \frac{|x-y|}{x+y}, \end{aligned} \tag{2}$$

we see that the Seiffert function corresponding to M is given by the formula

$$f(z) = \frac{z}{M(1-z,1+z)}.$$

For two means M, N, the symbol $M < N$ means that the inequality $M(x,y) < N(x,y)$ holds for all $x \neq y$.

Our main tool will be the obvious fact that if for two Seiffert functions the inequality $f < g$ holds, then their corresponding means satisfy $M_f > M_g$. Consequently, every inequality between means can be replaced by the inequality between their Seiffert functions.

Optimal bounds for the sine and hyperbolic tangent means were obtained by the authors in [2, 3].

The aim of this paper is to determine various optimal (i.e. the best possible in the class of inequalities considered) bounds for $\mathsf{M}_{\tan}$ and $\mathsf{M}_{\sinh}$ by the arithmetic and maximum means.

Remark 1.1. Note that the maximum mean

$$\mathsf{MAX}(x,y) \quad \text{has Seiffert function} \quad \mathsf{max}(z) = \frac{z}{1+z},$$

and the arithmetic mean

$$\mathsf{A}(x,y) = \frac{x+y}{2} \quad \text{has Seiffert function} \quad \mathsf{a}(z) = z.$$

For the reader's convenience, in the following sections we place the main results with their proofs, while all lemmas and technical details can be found in the last section of this paper.

The motivation for our research are the inequalities

$$\mathsf{A} < \mathsf{M}_{\sin} < \mathsf{M}_{\tanh} < \mathsf{MAX},$$

proven in [4, Lemma 3.2].

2. Linear bounds

Given three means $K < L < M$, one may try to find the best α, β satisfying the double inequality $(1-\alpha)K + \alpha M < L < (1-\beta)K + \beta M$, or equivalently

$$\alpha < \frac{L-K}{M-K} < \beta.$$

If k, l, m are the respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l} - \frac{1}{k}}{\frac{1}{m} - \frac{1}{k}} < \beta. \tag{3}$$

Thus the problem reduces to finding the upper and lower bound for certain function defined on the interval $(0,1)$.

Theorem 2.1. *The inequalities*

$$(1-\alpha)\,\mathsf{A}+\alpha\mathsf{MAX}<\mathsf{M}_{\sin}<(1-\beta)\,\mathsf{A}+\beta\mathsf{MAX}$$

hold if, and only if, $\alpha\le 0$ *and* $\beta\ge\frac{1}{\sin 1}-1\approx 0.1884$.

Proof. By formula (3) and Remark 1.1, we should investigate the function

$$h(z)=\frac{\frac{1}{\sin z}-\frac{1}{z}}{\frac{1+z}{z}-\frac{1}{z}}=\frac{1}{\sin z}-\frac{1}{z},\quad z\in(0,1).$$

It follows from Lemma 7.2 that

$$h'(z)=\frac{\sin^2 z-z^2\cos z}{z^2\sin^2 z}>0,$$

so h increases from $\lim_{z\to 0}h(z)=0$ to $h(1)=\frac{1}{\sin 1}-1\approx 0.1884$. □

Theorem 2.2. *The inequalities*

$$(1-\alpha)\,\mathsf{A}+\alpha\mathsf{MAX}<\mathsf{M}_{\tanh}<(1-\beta)\,\mathsf{A}+\beta\mathsf{MAX}$$

hold if, and only if, $\alpha\le 0$ *and* $\beta\ge\frac{1}{\tanh 1}-1\approx 0.3130$.

Proof. We use once more formula (3) and investigate the function

$$h(z)=\frac{\dfrac{1}{\tanh z}-\frac{1}{z}}{\frac{1+z}{z}-\frac{1}{z}}=\frac{1}{\tanh z}-\frac{1}{z},\quad z\in(0,1).$$

Since

$$h'(z)=\frac{\sinh^2 z-z^2}{z^2\sinh^2 z}>0,$$

so h increases from $\lim_{z\to 0}h(z)=0$ to $h(1)=\frac{1}{\tanh 1}-1\approx 0.3130$. □

3. Harmonic bounds

In this section, we look for the optimal bounds for means $K<L<M$ of the form

$$\frac{1-\alpha}{M}+\frac{\alpha}{K}<\frac{1}{L}<\frac{1-\beta}{M}+\frac{\beta}{K},$$

which can be written as

$$\alpha<\frac{\frac{1}{L}-\frac{1}{M}}{\frac{1}{K}-\frac{1}{M}}<\beta,$$

or — in terms of their Seiffert functions,

$$\alpha<\frac{l-m}{k-m}<\beta. \tag{4}$$

We shall use the above to prove two theorems.

Theorem 3.1. *The inequalities*

$$\frac{1-\alpha}{\mathsf{MAX}}+\frac{\alpha}{\mathsf{A}}<\frac{1}{\mathsf{M}_{\sin}}<\frac{1-\beta}{\mathsf{MAX}}+\frac{\beta}{\mathsf{A}}$$

hold if, and only if, $\alpha\le 2\sin 1-1\approx 0.6829$ *and* $\beta\ge 1$.

Proof. According to formula (4), we investigate the function

$$h(z)=\frac{\sin z-\frac{z}{1+z}}{z-\frac{z}{1+z}}=\frac{(1+z)\sin z-z}{z^2},\quad z\in(0,1).$$

We shall show that h decreases. We have

$$h'(z)=\frac{z-(z+2)\sin z+(1+z)z\cos z}{z^3}=:\frac{g(z)}{z^3}.$$

The function g satisfies $g(0)=g'(0)=0$ and

$$g''(z)=-z\,(3\sin z+(1+z)\cos z)<0.$$

Thus g is negative and so is h'. We complete the proof by noting that $\lim_{z\to 0}h(z)=1$. ☐

Theorem 3.2. *The inequalities*

$$\frac{1-\alpha}{\mathsf{MAX}}+\frac{\alpha}{\mathsf{A}}<\frac{1}{\mathsf{M}_{\tanh}}<\frac{1-\beta}{\mathsf{MAX}}+\frac{\beta}{\mathsf{A}}$$

hold if, and only if, $\alpha\le 2\tanh 1-1\approx 0.5232$ *and* $\beta\ge 1$.

Proof. Taking into account formula (4), we should investigate the function

$$h(z)=\frac{\tanh z-\frac{z}{1+z}}{z-\frac{z}{1+z}}=\frac{(1+z)\tanh z-z}{z^2},\quad z\in(0,1).$$

We shall show that h decreases. A simple calculation shows that

$$h'(z)=\frac{z\cosh^2 z+z(1+z)-(2+z)\sinh z\cosh z}{z^3\cosh^2 z}=:\frac{g(z)}{z^3\cosh^2 z}.$$

The function g satisfies $g(0)=g'(0)=g''(0)=0$ and

$$g'''(z)=-2((2z+1)\sinh^2 z+(2z+1)\cosh^2 z+2(3-2z)\sinh z\cosh z)<0.$$

Therefore g is negative and so is h'. We complete the proof by noting that $\lim_{z\to 0}h(z)=1$. ☐

4. Quadratic bounds

Given three means $K<L<M$, one may try to find the best α,β satisfying the double inequality $\sqrt{(1-\alpha)K^2+\alpha M^2}<L<\sqrt{(1-\beta)K^2+\beta M^2}$, or equivalently

$$\alpha<\frac{L^2-K^2}{M^2-K^2}<\beta.$$

If k,l,m are the respective Seiffert functions, then the latter can be written as

$$\alpha<\frac{\frac{1}{l^2}-\frac{1}{k^2}}{\frac{1}{m^2}-\frac{1}{k^2}}<\beta. \tag{5}$$

Thus, the problem reduces to finding the upper and lower bound for a certain function defined on the interval $(0,1)$.

Theorem 4.1. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{A}^2+\alpha\mathsf{MAX}^2}<\mathsf{M}_{\sin}<\sqrt{(1-\beta)\,\mathsf{A}^2+\beta\mathsf{MAX}^2}$$

hold if, and only if, $\alpha\le 0$ *and* $\beta\ge\frac{\cot^2 1}{3}\approx .1374$.

Proof. By formula (5), we should investigate the function

$$h(z) = \frac{\dfrac{1}{\sin^2 z} - \dfrac{1}{z^2}}{\left(\dfrac{1+z}{z}\right)^2 - \dfrac{1}{z^2}} = \frac{z^2 - \sin^2 z}{(z^2+2z)\sin^2 z}, \quad z \in (0,1).$$

It derivative equals

$$h'(z) = \frac{(3+3z+4z^2)\sin z - 4z^3(z+2)\cos z - (z+1)\sin 3z}{2z^2(z+2)^2\sin^3 z} =: \frac{g(z)}{2z^2(z+2)^2\sin^3 z}.$$

Using Taylor series expansion and the Leibniz test we obtain

$$\begin{aligned} g(z) &> (3+3z+4z^2)\left(z - \frac{z^3}{3!}\right) - 4z^3(z+2)\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!}\right) \\ &- (z+1)\left(3z - \frac{(3z)^3}{3!} + \frac{(3z)^5}{5!}\right) \\ &= \frac{z^5}{120}(157 - 3z - 40z^2 - 20z^3) > 0 \quad \text{for } 0 < z < 1. \end{aligned}$$

Therefore h increases from $\lim_{z\to 0} h(z) = 0$ to $h(1) = \frac{\cot^2 1}{3} \approx .1374$.

□

And here comes the hyperbolic tangent version of the previous theorem.

Theorem 4.2. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{A}^2 + \alpha\mathsf{MAX}^2} < \mathsf{M}_{\tanh} < \sqrt{(1-\beta)\,\mathsf{A}^2 + \beta\mathsf{MAX}^2}$$

hold if, and only if, $\alpha \le 0$ *and* $\beta \ge \frac{1}{3\sinh^2 1} \approx 0.2414$.

Proof. The function to be considered here is

$$h(z) = \frac{\dfrac{1}{\tanh^2 z} - \dfrac{1}{z^2}}{\left(\dfrac{1+z}{z}\right)^2 - \dfrac{1}{z^2}} = \frac{z^2\cosh^2 z - \sinh^2 z}{(z^2+2z)\sinh^2 z}, \quad z \in (0,1).$$

Its derivative equals

$$\begin{aligned} h'(z) &= \frac{(1+z+z^2)\sinh 3z - (3+3z-z^2)\sinh z - 4z^3(z+2)\cosh z}{2z^2(z+1)^2\sinh^3 z} \\ &=: \frac{g(z)}{2z^2(z+1)^2\sinh^3 z}. \end{aligned}$$

Using Taylor expansion and Lemma 7.1 we obtain

$$\begin{aligned} g(z) &> (1+z+z^2)\left(3z + \frac{(3z)^3}{3!} + \frac{(3z)^5}{5!}\right) + z^2\left(z + \frac{z^3}{3!} + \frac{z^5}{5!}\right) \\ &- 3(1+z)\left(z + \frac{z^3}{3!} + 2\frac{z^5}{5!}\right) - 4z^3(z+2)\left(1 + \frac{z^2}{2!} + 2\frac{z^4}{4!}\right) \\ &= \frac{z^5}{120}(317 - 3z + 164z^2 - 40z^3) > 0. \end{aligned}$$

This means that h increases from $\lim_{z\to 0} h(z) = 0$ to $h(1) = \frac{1}{3\sinh^2 1} \approx 0.2414$. □

5. Bounds by weighted power mean of order −2

In this section, we look for the optimal bounds for means $K < L < M$ of the form $\sqrt{\frac{1-\alpha}{M^2} + \frac{\alpha}{K^2}} < \frac{1}{L} < \sqrt{\frac{1-\beta}{M^2} + \frac{\beta}{K^2}}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l^2 - m^2}{k^2 - m^2} < \beta. \tag{6}$$

Theorem 5.1. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{MAX}^2} + \frac{\alpha}{\mathsf{A}^2}} < \frac{1}{\mathsf{M}_{\sin}} < \sqrt{\frac{1-\beta}{\mathsf{MAX}^2} + \frac{\beta}{\mathsf{A}^2}}$$

hold if, and only if, $\alpha \le \frac{1-2\cos 2}{3} \approx 0.6108$ *and* $\beta \ge 1$.

Proof. According to formula (6), we investigate the function

$$h(z) = \frac{\sin^2 z - \left(\frac{z}{1+z}\right)^2}{z^2 - \left(\frac{z}{1+z}\right)^2} = \frac{(1+z)^2 \sin^2 z - z^2}{z^3(z+2)}, \quad z \in (0,1).$$

Its derivative satisfies

$$\begin{aligned}
h'(z) &= \frac{z+1}{z^4(z+2)^2}(z^2 - 3z - 3 + z(z^2+3z+2)\sin 2z + (z^2+3z+3)\cos 2z) \\
&< \frac{z+1}{z^4(z+2)^2}\begin{pmatrix} z^2 - 3z - 3 \\ +z(z^2+3z+2)\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!}\right) \\ +(z^2+3z+3)\left(1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!}\right) \end{pmatrix} \\
&= -\frac{2(z+1)}{15(z+2)^2}(5 + 15z + z^2 - 6z^3 - 2z^4) < 0.
\end{aligned}$$

This means that h decreases from $\lim_{z\to 0} h(z) = 1$ to $h(1) = \frac{1-2\cos 2}{3} \approx 0.6108$. □

Theorem 5.2. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{MAX}^2} + \frac{\alpha}{\mathsf{A}^2}} < \frac{1}{\mathsf{M}_{\tanh}} < \sqrt{\frac{1-\beta}{\mathsf{MAX}^2} + \frac{\beta}{\mathsf{A}^2}}$$

hold if, and only if, $\alpha \le \frac{4\tanh^2 1 - 1}{3} \approx 0.4400$ *and* $\beta \ge 1$.

Proof. We follow the same line as in the previous proof. Let

$$h(z) = \frac{\tanh^2 z - \left(\frac{z}{1+z}\right)^2}{z^2 - \left(\frac{z}{1+z}\right)^2} = \frac{(1+z)^2 \tanh^2 z - z^2}{z^3(z+2)}, \quad z \in (0,1).$$

Its derivative

$$h'(z) = \frac{z+1}{2z^4(z+2)^2\cosh^3 z} g(z),$$

where

$$g(z) = 4z(z^2+3z+2)\sinh z + (4z^2+3z+3)\cosh z - 3(z+1)\cosh 3z.$$

By Lemma 7.1 we have

$$\begin{aligned} g(z) &< 4z(z^2+3z+2)\left(z+\frac{z^3}{3!}+2\frac{z^5}{5!}\right)+(4z^2+3z+3)\left(1+\frac{z^2}{2!}+2\frac{z^4}{4!}\right) \\ &\quad - 3(z+1)\left(1+\frac{(3z)^2}{2!}+\frac{(3z)^4}{4!}\right) \\ &= -\frac{z^4}{120}(305+945z-136z^2-24z^3-8z^4) < 0. \end{aligned}$$

Thus h decreases from $\lim_{z\to 0} h(z) = 1$ to $h(1) = \frac{4\tanh^2 1 - 1}{3} \approx 0.4400$. □

6. Bounds with varying arguments

If N is a mean, then the formula $N^{\{t\}}(x,y) = N\left(\frac{x+y}{2}+t\frac{x-y}{2}, \frac{x+y}{2}-t\frac{x-y}{2}\right)$ defines a homotopy between the arithmetic mean $\mathsf{A} = N^{\{0\}}$ and $N = N^{\{1\}}$. Therefore if $\mathsf{A} < M < N$, it make sense to ask what are the optimal numbers α, β satisfying $N^{\{\alpha\}} < M < N^{\{\beta\}}$. Theorem 6.1 from [4] gives a method for finding such numbers in terms of the Seiffert functions of the means involved.

Theorem 6.1 (Witkowski [4])**.** *For a Seiffert function k denote by $\widehat{k}(z) = k(z)/z$. Let M and N be two means with Seiffert functions m and n, respectively. Suppose that $\widehat{n}(z)$ is strictly monotone and let $p_0 = \inf\limits_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$ and $q_0 = \sup\limits_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$. If $\mathsf{A}(x,y) < M(x,y) < N(x,y)$ for all $x \neq y$ then the inequalities*

$$N^{\{p\}}(x,y) \leqslant M(x,y) \leqslant N^{\{q\}}(x,y)$$

*hold if, and only if, $p \leqslant p_0$ and $q \geqslant q_0$.
If $N(x,y) < M(x,y) < \mathsf{A}(x,y)$ for all $x \neq y$ then the inequalities*

$$N^{\{q\}}(x,y) \leqslant M(x,y) \leqslant N^{\{p\}}(x,y)$$

hold if, and only if, $p \leqslant p_0$ and $q \geqslant q_0$.

In case of $N = \mathsf{MAX}$ we see that $\widehat{\mathsf{max}}(z) = \frac{1}{1+z}$ and $\widehat{\mathsf{max}}^{-1}(x) = x^{-1} - 1$.

Theorem 6.2. *The inequalities*

$$\mathsf{MAX}\left(\tfrac{x+y}{2}+\alpha\tfrac{x-y}{2}, \tfrac{x+y}{2}-\alpha\tfrac{x-y}{2}\right) < \mathsf{M}_{\sin} < \mathsf{MAX}\left(\tfrac{x+y}{2}+\beta\tfrac{x-y}{2}, \tfrac{x+y}{2}-\beta\tfrac{x-y}{2}\right)$$

hold if, and only if, $\alpha \leq 0$ and $\beta \geq \frac{1}{\sin 1} - 1 \approx 0.1884$.

Proof. Here, we investigate the function

$$h(z) = \frac{\widehat{\mathsf{max}}^{-1}\left(\frac{\sin z}{z}\right)}{z} = \frac{1}{\sin z} - \frac{1}{z},$$

but we found its range in the proof of Theorem 2.1. □

Theorem 6.3. *The inequalities*

$$\mathsf{MAX}\left(\tfrac{x+y}{2}+\alpha\tfrac{x-y}{2}, \tfrac{x+y}{2}-\alpha\tfrac{x-y}{2}\right) < \mathsf{M}_{\tanh} < \mathsf{MAX}\left(\tfrac{x+y}{2}+\beta\tfrac{x-y}{2}, \tfrac{x+y}{2}-\beta\tfrac{x-y}{2}\right)$$

hold if, and only if, $\alpha \leq 0$ and $\beta \geq \frac{1}{\tanh 1} - 1 \approx 0.3130$.

Proof. According to Theorem 6.1 we investigate the function

$$h(z) = \frac{\widehat{\mathsf{max}}^{-1}\left(\frac{\tanh z}{z}\right)}{z} = \frac{1}{\tanh z} - \frac{1}{z}.$$

The remainder of the proof follows by the same arguments as mentioned in the proof of Theorem 2.2 and hence the proof is complete. □

7. Tools and lemmas

In this section, we place all the technical details needed to prove our main results.

Lemma 7.1. *For $0 < x < 1$, the following inequalities hold*

a) $\cosh x < 1 + \frac{x^2}{2!} + 2\frac{x^4}{4!}$,

b) $\sinh x < x + \frac{x^3}{3!} + 2\frac{x^5}{5!}$,

Proof. a)

$$\cosh x - 1 - \frac{x^2}{2!} - \frac{x^4}{4!} = \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots < \frac{x^4}{4!}\left(\frac{1}{7\cdot 8} + \frac{1}{7\cdot 8\cdot 9\cdot 10} + \dots\right) < \frac{x^4}{4!}.$$

b)

$$\sinh x - x - \frac{x^3}{3!} - \frac{x^5}{5!} = \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots < \frac{x^5}{5!}\left(\frac{1}{6\cdot 7} + \frac{1}{6\cdot 7\cdot 8\cdot 9} + \dots\right) < \frac{x^5}{5!}.$$

□

Lemma 7.2 (Mitrinović & Adamović [1])**.** *Consider the functions $f_u : [0, \pi/2) \to \mathbb{R}$*

$$f_u(x) = \cos^u x \sin x - x, \quad -1 < u < 0.$$

For $-1 \le u \le -\frac{1}{3}$, the functions f_u are positive. For $-\frac{1}{3} < u < 0$, there exists $0 < x_u < \frac{\pi}{2}$ such that f_u is negative in $(0, x_u)$ and positive in (x_u, ∞).

Proof. We have $f_u(0) = f_u'(0) = 0$ and

$$f_u''(x) = u(u-1)\sin x \cos^u x\left[\tan^2 x - \frac{1+3u}{u(u-1)}\right].$$

If $-1 \le u < -1/3$, we have $\frac{3u+1}{u(u-1)} \le 0$, so f_u is convex, thus positive.

For $-1/3 < u < 0$, the equation $\tan^2 x - \frac{1+3u}{u(u-1)} = 0$ has exactly one solution ξ_u, so f_u is concave and negative on $(0, \xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u. □

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