OPTIMAL BOUNDS FOR THE SINE AND HYPERBOLIC TANGENT MEANS IV

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Abstract. We provide optimal bounds for the sine and hyperbolic tangent means in terms of various weighted means of the arithmetic and centroidal means.

Contents

1. Introduction, definitions and notations

The means

(sine mean)
$$\mathsf{M}_{\sin}(x,y)=\begin{cases}\dfrac{x-y}{2\sin\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}$$

and

(hyperbolic tangent mean)
$$\mathsf{M}_{\tanh}(x,y)=\begin{cases}\dfrac{x-y}{2\tanh\frac{x-y}{x+y}} & x\neq y\\ x & x=y\end{cases}$$

defined for positive arguments, have been introduced in [8], where one of the authors investigates means of the form

$$\mathsf{M}_f(x,y)=\begin{cases}\dfrac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x\neq y\\ x & x=y\end{cases}. \tag{1}$$

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It was shown that every symmetric and homogeneous mean of positive arguments can be represented in the form (1) and that every function $f : (0, 1) \to \mathbb{R}$ (called Seiffert function) satisfying

$$\frac{z}{1+z} \le f(z) \le \frac{z}{1-z}$$

produces a mean. The correspondence between means and Seiffert functions is given by the formula

$$f(z) = \frac{z}{M(1-z, 1+z)}, \quad \text{where} \quad z = \frac{|x-y|}{x+y}.$$

The aim of this paper is to determine various optimal bounds for the $\mathsf{M}_{\tanh}$ and $\mathsf{M}_{\sin}$ with the arithmetic and centroidal means (denoted here by A and Ce). Similar bounds by the arithmetic and contraharmonic means were obtained in [5], and by arithmetic and quadratic means in [6]. For other bounds of Seiffert-like means by the arithmetic and centroidal means, see e.q. [7, 2, 3, 9].

For two means M, N, the symbol $M < N$ denotes that for all positive $x \ne y$ the inequality $M(x, y) < N(x, y)$ holds.

Our main tool will be the obvious fact that if for two Seiffert functions the inequality $f < g$ holds, then their corresponding means satisfy $M_f > M_g$. Thus every inequality between means can be replaced by the inequality between their Seiffert functions.

Remark 1.1. Throughout this paper all means are defined on $(0, \infty)^2$.

Remark 1.2. Note that the Seiffert function of the centroidal mean $\mathsf{Ce}(x, y) = \frac{2}{3}\frac{x^2+xy+y^2}{x+y}$ is $\mathsf{ce}(z) = \frac{3z}{3+z^2}$ and that of the arithmetic mean $\mathsf{A}(x, y) = \frac{x+y}{2}$ is the identity function $\mathsf{a}(z) = z$. Clearly, the Seiffert functions of $\mathsf{M}_{\sin}$ and $\mathsf{M}_{\tanh}$ are the functions sin and tanh, respectively.

For the reader's convenience, in the following sections we place the main results with their proofs, while all lemmas and technical details can be found in the last section of this paper.

The motivation for our research are the inequalities $\mathsf{A} < \mathsf{M}_{\sin} < \mathsf{M}_{\tanh} < \mathsf{Ce}$ proven in [8, Lemma 3.1] and Lemma 7.1.

2. Linear bounds

Given three means $K < L < M$, one may try to find the best α, β satisfying the double inequality $(1-\alpha)K + \alpha M < L < (1-\beta)K + \beta M$ or equivalently $\alpha < \frac{L-K}{M-K} < \beta$. If k, l, m are respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l} - \frac{1}{k}}{\frac{1}{m} - \frac{1}{k}} < \beta. \tag{2}$$

Therefore the problem reduces to finding upper and lower bounds for certain functions defined on the interval $(0, 1)$.

Theorem 2.1. *The inequalities*

$$(1-\alpha)\,\mathsf{A} + \alpha\mathsf{Ce} < \mathsf{M}_{\sin} < (1-\beta)\,\mathsf{A} + \beta\mathsf{Ce}$$

hold if, and only if, $\alpha \le \frac{1}{2}$ *and* $\beta \ge \frac{3}{\sin 1} - 3 \approx 0.5652$.

Proof. By formula (2) and Remark 1.2, we investigate the function

$$h(z)=\frac{\frac{1}{\sin z}-\frac{1}{z}}{\frac{3+z^2}{3z}-\frac{1}{z}}=\frac{3}{z\sin z}-\frac{3}{z^2}.$$

We shall show that h increases. Observe that

$$h'(z)=3\frac{2\sin^2 z-z^2\cos z-z\sin z}{z^3\sin^2 z}=:3\frac{s(z)}{z^3\sin^2 z}.$$

Using the known inequalities $x-x^3/3!<\sin x<x-x^3/3!+x^5/5!$ and $\cos x<1-x^2/2!+x^4/4!$ we get

$$s(z)>2\left(z-\frac{z^3}{3!}\right)^2-z^2\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}\right)-z\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}\right)=\frac{z^6}{180}>0,$$

so $h'(z)>0$. We complete the proof by noting that $\lim_{z\to 0}h(z)=1/2$. □

Theorem 2.2. *The inequalities*

$$(1-\alpha)\,\mathsf{A}+\alpha\mathsf{Ce}<\mathsf{M}_{\tanh}<(1-\beta)\,\mathsf{A}+\beta\mathsf{Ce}$$

hold if, and only if, $\alpha\le\frac{3}{\tanh 1}-3\approx 0.9391$ *and* $\beta\ge 1$.

Proof. We use Remark 1.2 and formula (2) once more and investigate the function

$$h(z)=\frac{\dfrac{1}{\tanh z}-\frac{1}{z}}{\frac{1+z^2/3}{z}-\frac{1}{z}}=\frac{3}{z}\left(\frac{1}{\tanh z}-\frac{1}{z}\right)=:\frac{3s(z)}{z}.$$

The function s satisfies $\lim_{z\to 0}s(z)=0$ and $s''(z)=\frac{2}{\sinh^3 z}\left(\cosh z-\frac{\sinh^3 z}{z^3}\right)<0$ (by Lemma 7.2), so s is concave and, by Property 7.2, its divided difference (and consequently the function h) decreases. To complete the proof note that $\lim_{z\to 0}h(z)=1$. □

3. Harmonic bounds

In this section, we look for optimal bounds for means $K<L<M$ of the form $\frac{1-\alpha}{M}+\frac{\alpha}{K}<\frac{1}{L}<\frac{1-\beta}{M}+\frac{\beta}{K}$ or, in terms of their Seiffert functions,

$$\alpha<\frac{l-m}{k-m}<\beta. \tag{3}$$

We shall use the above to prove two theorems.

Theorem 3.1. *The inequalities*

$$\frac{1-\alpha}{\mathsf{Ce}}+\frac{\alpha}{\mathsf{A}}<\frac{1}{\mathsf{M}_{\sin}}<\frac{1-\beta}{\mathsf{Ce}}+\frac{\beta}{\mathsf{A}}$$

hold if, and only if, $\alpha\le 4\sin 1-3\approx 0.3659$ *and* $\beta\ge\frac{1}{2}$.

Proof. According to formula (3), we investigate the function

$$h(z)=\frac{\sin z-\frac{3z}{z^2+3}}{z-\frac{3z}{z^2+3}}=\frac{(z^2+3)\sin z-3z}{z^3}.$$

We shall show that h decreases. We have

$$h'(z)=\frac{(z^3+3z)\cos z-(z^2+9)\sin z+6z}{z^4}:=\frac{s(z)}{z^4}.$$

The function s satisfies $s(0) = s'(0) = s''(0) = 0$ and

$$s'''(z) = z\left((z^2-9)\sin z - 8z\cos z\right) < 0.$$

Thus s is negative and so is h'. We complete the proof by noting that $\lim_{z\to 0} h(z) = 1/2$. □

Theorem 3.2. *The inequalities*

$$\frac{1-\alpha}{\mathsf{C}} + \frac{\alpha}{\mathsf{A}} < \frac{1}{\mathsf{M}_{\tanh}} < \frac{1-\beta}{\mathsf{C}} + \frac{\beta}{\mathsf{A}}$$

hold if, and only if, $\alpha \le 0$ *and* $\beta \ge 4\tanh 1 - 3 \approx .0464$.

Proof. We use Remark 1.2 and formula (3) once more and investigate the function

$$h(z) = \frac{\tanh z - \frac{3z}{3+z^2}}{z - \frac{3z}{3+z^2}} = \frac{(3+z^2)\tanh z - 3z}{z^3}.$$

We shall show that h increases. We have

$$\begin{aligned} h'(z) &= \frac{6z + z(3+z^2)\frac{1}{\cosh^2 z} - (9+z^2)\frac{\sinh z}{\cosh z}}{z^4} \\ &= \frac{12z + 2z^3 + 6z\cosh 2z - (9+z^2)\sinh 2z}{2z^4\cosh^2 z} =: \frac{s(z)}{2z^4\cosh^2 z}. \end{aligned}$$

By Lemma 7.4 we get

$$\begin{aligned} s(z) &> 12z + 2z^3 + 6z\left(1 + \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!}\right) \\ &\quad - (9+z^2)\left(2z + \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} + 2\frac{(2z)^7}{7!}\right) = \frac{4z^5}{315}(21 - 15z^2 - 4z^4) > 0. \end{aligned}$$

Therefore h increases from $\lim_{z\to 0} h(z) = 0$ to $h(1)$. □

4. Quadratic bounds

Given three means $K < L < M$, one may try to find the best α, β satisfying the double inequality $\sqrt{(1-\alpha)K^2 + \alpha M^2} < L < \sqrt{(1-\beta)K^2 + \beta M^2}$ or equivalently $\alpha < \frac{L^2-K^2}{M^2-K^2} < \beta$. If k, l, m are respective Seiffert functions, then the latter can be written as

$$\alpha < \frac{\frac{1}{l^2} - \frac{1}{k^2}}{\frac{1}{m^2} - \frac{1}{k^2}} < \beta. \tag{4}$$

Thus, the problem reduces to finding upper and lower bounds for certain functions defined on the interval $(0, 1)$.

Theorem 4.1. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{A}^2 + \alpha\mathsf{Ce}^2} < \mathsf{M}_{\sin} < \sqrt{(1-\beta)\,\mathsf{A}^2 + \beta\mathsf{Ce}^2}$$

hold if, and only if, $\alpha \le \frac{1}{2}$ *and* $\beta \ge \frac{9}{7}\cot^2 1 \approx 0.5301$.

Proof. Using formula (4) we investigate the function

$$h(z) = \frac{\dfrac{1}{\sin^2 z} - \dfrac{1}{z^2}}{\dfrac{(z^2+3)^2}{9z^2} - \dfrac{1}{z^2}} = \frac{\dfrac{z^2}{\sin^2 z} - 1}{\dfrac{(z^2+3)^2}{9} - 1}.$$

To show that h increases we use Lemma 7.3. A simple calculation shows that

$$r(z)=\frac{\left(\frac{z^2}{\sin^2 z}-1\right)'}{\left(\dfrac{(z^2+3)^2}{9}-1\right)'}=\frac{9(\sin z-z\cos z)}{2(z^2+3)\sin^3 z}$$

and

$$r'(z)=9\frac{2(2z^3+5z)-(z^2+9)\sin 2z+2z(z^2+4)\cos 2z}{4(z^2+3)^2\sin^4 z}:=9\frac{s(z)}{4(z^2+3)^2\sin^4 z}.$$

From $\sin 2x<2x-(2x)^3/3!+(2x)^5/5!$ and $\cos 2x>1-(2x)^2/2!+(2x)^4/4!-(2x)^6/6!$ we get

$$\begin{aligned}s(z)&>2(2z^3+5z)-(z^2+9)\left(2z-\frac{(2z)^3}{3!}+\frac{(2z)^5}{5!}\right)\\&+2z(z^2+4)\left(1-\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}-\frac{(2z)^6}{6!}\right)\\&=\frac{4}{45}z^5(-2z^4+4z^2+3)>0.\end{aligned}$$

Thus r' is positive and both r and h increase. We complete the proof by noting that $\lim_{z\to 0}h(z)=1/2$. □

And here comes the hyperbolic tangent version of the previous theorem.

Theorem 4.2. *The inequalities*

$$\sqrt{(1-\alpha)\,\mathsf{A}^2+\alpha\mathsf{Ce}^2}<\mathsf{M}_{\tanh}<\sqrt{(1-\beta)\,\mathsf{A}^2+\beta\mathsf{Ce}^2}$$

hold if, and only if, $\alpha\le\frac{9}{7}(\coth^2 1-1)\approx .9309$ *and* $\beta\ge 1$.

Proof. We shall use the identity $\tanh^2 z=\frac{\cosh 2z-1}{\cosh 2z+1}$.
The function to be considered here is

$$h(z)=\frac{\dfrac{1}{\tanh^2 z}-\dfrac{1}{z^2}}{\dfrac{(z^2+3)^2}{9z^2}-\dfrac{1}{z^2}}=9\frac{1+z^2+(z^2-1)\cosh 2z}{(z^4+6z^2)(\cosh 2z-1)},$$

and its derivative equals

$$h'(z)=-\frac{18z\sinh z}{(z^4+6z^2)^2(\cosh 2z-1)^2}s(z),$$

where

$$s(z)=4z^3(z^2+6)\cosh z+(z^4+6z^2+18)\sinh z-(6+2z^2-z^4)\sinh 3z$$

and by Lemma 7.4

$$\begin{aligned}s(z)&>\begin{pmatrix}4z^3(z^2+6)\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\frac{z^8}{8!}\right)\\+(z^4+6z^2+18)\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}\right)\\-(6+2z^2-z^4)\left(3z+\frac{(3z)^3}{3!}+\frac{(3z)^5}{5!}+\frac{(3z)^7}{7!}+2\frac{(3z)^9}{9!}\right)\end{pmatrix}\\&=z^7\frac{2189z^6+4502z^4+14430z^2+21504}{20160}>0.\end{aligned}$$

This shows that $h' < 0$ so h decreases from $\lim_{z\to 0} h(z) = 1$ to $h(1) = \frac{9}{7}(\coth^2 1 - 1) \approx .9309$. $\square$

5. Bounds with the weighted power mean of order -2

In this section, we look for optimal bounds for means $K < L < M$ of the form $\sqrt{\frac{1-\alpha}{M^2} + \frac{\alpha}{K^2}} < \frac{1}{L} < \sqrt{\frac{1-\beta}{M^2} + \frac{\beta}{K^2}}$ or, in terms of their Seiffert functions,

$$\alpha < \frac{l^2 - m^2}{k^2 - m^2} < \beta. \tag{5}$$

Theorem 5.1. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{Ce}^2} + \frac{\alpha}{\mathsf{A}^2}} < \frac{1}{\mathsf{M}_{\sin}} < \sqrt{\frac{1-\beta}{\mathsf{Ce}^2} + \frac{\beta}{\mathsf{A}^2}}$$

hold if, and only if, $\alpha \le \frac{16\sin^2 1 - 9}{7} \approx 0.3327$ *and* $\beta \ge \frac{1}{2}$.

Proof. Taking formula (5) into account we should investigate the function

$$h(z) = \frac{\sin^2 z - \frac{9z^2}{(z^2+3)^2}}{z^2 - \frac{9z^2}{(z^2+3)^2}} = \frac{(z^2+3)^2 \sin^2 z - 9z^2}{z^4(z^2+6)}.$$

We shall show that h decreases. We have

$$\begin{aligned} h'(z) &= \frac{(z^2+3)\left(-z^4 + 27z^2 - 36 + (z^2+3)(z^2+6)z\sin 2z + (z^4+9z^2+36)\cos 2z\right)}{z^5(z^2+6)^2} \\ &:= \frac{(z^2+3)s(z)}{z^5(z^2+6)^2}. \end{aligned}$$

From $\sin 2x < 2x - (2x)^3/3! + (2x)^5/5! - (2x)^7/7! + (2x)^9/9!$ and $\cos 2x < 1 - (2x)^2/2! + (2x)^4/4! - (2x)^6/6! + (2x)^8/8!$ we obtain

$$\begin{aligned} s(z) &< -z^4 + 27z^2 - 36 + (z^2+3)(z^2+6)z\left(2z - \frac{(2z)^3}{3!} + \frac{(2z)^5}{5!} - \frac{(2z)^7}{7!} + \frac{(2z)^9}{9!}\right) \\ &\quad + (z^4+9z^2+36)\left(1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \frac{(2z)^6}{6!} + \frac{(2z)^8}{8!}\right) \\ &= \frac{2}{2835}z^6(2z^8 - 9z^6 + 45z^4 + 999z^2 - 6237) < 0. \end{aligned}$$

Thus $h'(z) < 0$. We complete the proof by noting that $\lim_{z\to 0} h(z) = 1/2$. $\square$

Theorem 5.2. *The inequalities*

$$\sqrt{\frac{1-\alpha}{\mathsf{Ce}^2} + \frac{\alpha}{\mathsf{A}^2}} < \frac{1}{\mathsf{M}_{\tanh}} < \sqrt{\frac{1-\beta}{\mathsf{Ce}^2} + \frac{\beta}{\mathsf{A}^2}}$$

hold if, and only if, $\alpha \le 0$ *and* $\beta \ge \frac{16\tanh^2 1 - 9}{7} \approx .0401$.

Proof. This time we investigate the function

$$h(z) = \frac{\tanh^2 z - \frac{9z^2}{(3+z^2)^2}}{z^2 - \frac{9z^2}{(3+z^2)^2}} = \frac{(z^2+3)^2\tanh^2 z - 9z^2}{z^6 + 6z^4}.$$

This function increases, because by Lemma 7.4

$$h'(z) = \frac{z^2+3}{2z^5(z^2+6)^2\cosh^3 z}\begin{pmatrix} 4z(z^2+3)(z^2+6)\sinh z \\ +(z^4+63z^2+36)\cosh z \\ -(z^4-9z^2+36)\cosh 3z \end{pmatrix}$$

$$> \frac{z^2+3}{2z^5(z^2+6)^2\cosh^3 z}\begin{pmatrix} 4z(z^2+3)(z^2+6)\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}\right) \\ +(z^4+63z^2+36)\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\frac{z^8}{8!}\right) \\ -(z^4-9z^2+36)\left(1+\frac{(3z)^2}{2!}+\frac{(3z)^4}{4!}+\frac{(3z)^6}{6!}+\frac{3}{2}\times\frac{(3z)^8}{8!}\right) \end{pmatrix}$$

$$= \frac{z(z^2+3)}{2(z^2+6)^2\cosh^3 z}\times\frac{-19617z^6+99001z^4-156324z^2+258048}{80640} > 0.$$

So the function h assumes values between $\lim_{z\to 0} h(z) = 0$ and $h(1)$. □

6. Bounds with varying arguments

If N is a mean, then the formula $N^{\{t\}}(x,y) = N\left(\frac{x+y}{2}+t\frac{x-y}{2}, \frac{x+y}{2}-t\frac{x-y}{2}\right)$ defines a homotopy between the arithmetic mean $\mathsf{A} = N^{\{0\}}$ and $N = N^{\{1\}}$. Therefore, if $\mathsf{A} < M < N$, it makes sense to ask what the optimal numbers α, β are satisfying $N^{\{\alpha\}} < M < N^{\{\beta\}}$. Theorem 6.1 from [8] gives a method for finding such numbers in terms of the Seiffert functions of the means involved. It says

Theorem 6.1. *For a Seiffert function k, denote $\widehat{k}(z) = k(z)/z$. Let M and N be two means with Seiffert functions m and n, respectively. Suppose that $\widehat{n}(z)$ is strictly monotone and let $p_0 = \inf_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$ and $q_0 = \sup_z \frac{\widehat{n}^{-1}(\widehat{m}(z))}{z}$.*
If $\mathsf{A}(x,y) < M(x,y) < N(x,y)$ for all $x \neq y$, then the inequalities

$$N^{\{p\}}(x,y) \leqslant M(x,y) \leqslant N^{\{q\}}(x,y)$$

hold if, and only if, $p \leqslant p_0$ and $q \geqslant q_0$.
If $N(x,y) < M(x,y) < \mathsf{A}(x,y)$ for all $x \neq y$, then the inequalities

$$N^{\{q\}}(x,y) \leqslant M(x,y) \leqslant N^{\{p\}}(x,y)$$

hold if, and only if, $p \leqslant p_0$ and $q \geqslant q_0$.

In the case of $N = \mathsf{Ce}$ we see that $\widehat{ce}(z) = \frac{3}{z^2+3}$ and $\widehat{ce}^{-1}(x) = \sqrt{3x^{-1}-3}$.

Theorem 6.2. *The inequalities*

$$\mathsf{Ce}\left(\tfrac{x+y}{2}+\alpha\tfrac{x-y}{2}, \tfrac{x+y}{2}-\alpha\tfrac{x-y}{2}\right) < \mathsf{M}_{\sin} < \mathsf{Ce}\left(\tfrac{x+y}{2}+\beta\tfrac{x-y}{2}, \tfrac{x+y}{2}-\beta\tfrac{x-y}{2}\right)$$

hold if, and only if, $\alpha \le \sqrt{\frac{1}{2}} \approx 0.7071$ and $\beta \ge \sqrt{\frac{3}{\sin 1}-3} \approx 0.7518$.

Proof. Using Theorem 6.1 we should find the range of the function

$$h(z) = \frac{\sqrt{3\frac{z}{\sin z}-3}}{z}.$$

The monotonicity of the function h^2 follows from the proof of Theorem 2.1, so evaluation of the values of h at the endpoints completes the proof. □

Theorem 6.3. *The inequalities*

$$\mathsf{Ce}\left(\tfrac{x+y}{2}+\alpha\tfrac{x-y}{2}, \tfrac{x+y}{2}-\alpha\tfrac{x-y}{2}\right) < \mathsf{M}_{\tanh} < \mathsf{Ce}\left(\tfrac{x+y}{2}+\beta\tfrac{x-y}{2}, \tfrac{x+y}{2}-\beta\tfrac{x-y}{2}\right)$$

hold if, and only if, $\alpha \le \sqrt{3\coth 1 - 3} \approx 0.9691$ and $\beta \ge 1$.

Proof. According to Theorem 6.1, we shall consider the function

$$h(z) = \frac{\sqrt{3\frac{z}{\tanh z} - 3}}{z},$$

but we found the range of its square in the proof of Theorem 2.2. □

7. Tools and lemmas

In this section, we place all the technical details needed to prove our main results.

Property 7.1. A function $f : (a, b) \to \mathbb{R}$ is convex if, and only if, for every $a < \theta < b$ its divided difference $\frac{f(x)-f(\theta)}{x-\theta}$ increases for $x \neq \theta$.

A simple consequence of Property 7.1 is

Property 7.2. If a function $f : (a, b) \to \mathbb{R}$ is convex and $\lim_{x\to a} f(x) = \Theta$, then the function $\frac{f(x)-\Theta}{x-a}$ increases.

Lemma 7.1. *For all positive $x \neq y$ the inequality $\mathsf{M}_{\tanh}(x, y) < \mathsf{Ce}(x, y)$ holds.*

Proof. Using Seiffert's functions we have to proof that $h(z) = \tanh z - \frac{3z}{3+z^2} > 0$ for $0 < z < 1$. Note that

$$\begin{aligned}\cosh z &= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \frac{z^8}{8!} \dots \\ &< 1 + \frac{z^2}{2!} + \frac{z^4}{4!}\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) = 1 + \frac{z^2}{2} + \frac{z^4}{12}.\end{aligned} \tag{6}$$

This yields

$$\begin{aligned}h'(z) &= \frac{1}{\cosh^2 z} - \frac{9-3z^2}{(3+z^2)^2} > \frac{1}{(1+\frac{z^2}{2}+\frac{z^4}{12})^2} - \frac{9-3z^2}{(3+z^2)^2} \\ &= \frac{3z^4(z^6+9z^4+24z^2+12)}{(3+z^2)^2(12+6z^2+z^4)^2} > 0,\end{aligned}$$

which, combined with $h(0) = 0$ completes the proof. □

Lemma 7.2 (Lazarević [4])**.** *Consider the functions $g_u : [0, \infty) \to \mathbb{R}$*

$$g_u(x) = \cosh^u x \sinh x - x, \quad -1 < u < 0.$$

For $-1/3 \le u < 0$, the functions g_u are positive. For $-1 < u < -1/3$, there exists $x_u > 0$, such that g_u is negative in $(0, x_u)$ and positive in (x_u, ∞).

Proof. We have $g_u(0) = g_u'(0) = 0$ and

$$g_u''(x) = u(u-1)\sinh x \cosh^u x\left[\tanh^2 x + \frac{1+3u}{u(u-1)}\right].$$

If $-1/3 \le u < 0$, we have $\frac{1+3u}{u(u-1)} \ge 0$, so g_u is convex thus positive. For $-1 < u < -1/3$, the equation $\tanh^2 x + \frac{1+3u}{u(u-1)} = 0$ has exactly one solution ξ_u, so g_u is concave and negative on $(0, \xi_u)$. Then it becomes convex and tends to infinity, thus assumes zero at exactly one point x_u. □

The next lemma can be found in [1, Theorem 1.25].

Lemma 7.3. *Suppose $f, g : (a, b) \to \mathbb{R}$ are differentiable with $g'(x) \neq 0$ and such that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to b} f(x) = \lim_{x\to b} g(x) = 0$. Then*

(1) if $\frac{f'}{g'}$ *is increasing on* (a,b)*, then* $\frac{f}{g}$ *is increasing on* (a,b)*,*

(2) if $\frac{f'}{g'}$ *is decreasing on* (a,b)*, then* $\frac{f}{g}$ *is decreasing on* (a,b)*.*

Lemma 7.4. *For* $0<x<1$*, the following inequalities hold*

a) $\sinh 3x < 3x + \dfrac{(3x)^3}{3!} + \dfrac{(3x)^5}{5!} + \dfrac{(3x)^7}{7!} + 2\dfrac{(3x)^9}{9!}$,

b) $\sinh 2x < 2x + \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + 2\frac{(2x)^7}{7!}$,

c) $\cosh 3x < 1 + \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \frac{(3x)^6}{6!} + \frac{3}{2} \times \frac{(3x)^8}{8!}$.

Proof. Proof a)

$$\begin{aligned}\sinh(3x) - 3x - \frac{(3x)^3}{3!} - \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} - \frac{(3x)^9}{9!}\\ = \frac{(3x)^{11}}{11!} + \frac{(3x)^{13}}{13!} + \cdots < \frac{(3x)^9}{9!}\left(\frac{3^2}{10\cdot 11} + \frac{3^4}{10\cdot 11\cdot 12\cdot 13} + \dots\right) < \frac{(3x)^9}{9!}.\end{aligned}$$

Other proofs are similar. □

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