OPERATOR MONOTONICITY OF THE CONVEX INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR

Abstract. For a continuous and positive function \( w(\lambda), \lambda > 0 \) and a positive measure on \((0, \infty)\) we consider the following convex integral transform

\[
\mathcal{C}(w, \mu)(T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu(\lambda),
\]

where the integral is assumed to exist for \( T \) a positive operator on a complex Hilbert space \( H \).

We show among others that, if \( \int_0^\infty w(\lambda) d\mu(\lambda) < \infty \) and \( B > A > 0 \), then

\[
\mathcal{C}(w, \mu)(B) - \mathcal{C}(w, \mu)(A) \leq \left( \int_0^\infty w(\lambda) d\mu(\lambda) \right) (B - A),
\]

which shows that the function \( \int_0^\infty w(\lambda) d\mu(\lambda) t \) is operator monotone on \((0, \infty)\). From this we derive that, if \( f: [0, \infty) \to \mathbb{R} \) is an operator convex function on \([0, \infty)\) that has the representation

\[
f(t) = f(0) + f'(0) t + c t^2 + \int_0^\infty \frac{t^2 \lambda}{\lambda + t} d\mu(\lambda),
\]

where \( c \geq 0 \) and \( \mu \) a positive measure on \((0, \infty)\) with finite expectation \( E(\mu) := \int_0^\infty \lambda d\mu(\lambda) \), then \( E(\mu) + f'(0) t + c t^2 - f(t) \) is operator monotone on \((0, \infty)\). Several examples involving the exponential and logarithmic functions are also given.

1. Introduction

Consider a complex Hilbert space \((H, \langle \cdot, \cdot \rangle)\). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( \langle Tx, x \rangle \geq 0 \) for all \( x \in H \) and also an operator \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible.

We have the following integral representation for the power function when \( t > 0, r \in (0, 1] \), see for instance [1, p. 145]

\[
t^{r-1} = \sin(r\pi) \pi \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda.
\]

Observe that for \( t > 0, t \neq 1 \), we have

\[
\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln \left( \frac{u + t}{u + 1} \right)
\]

for all \( u > 0 \).

By taking the limit over \( u \to \infty \) in this equality, we derive

\[
\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},
\]

1991 Mathematics Subject Classification. 47A63, 47A60.


which gives the representation for the logarithm
\begin{equation}
\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)}
\end{equation}
for all \( t > 0 \).

Motivated by these representations, we introduce, for a continuous and positive function \( w(\lambda), \lambda > 0 \), the following integral transform
\begin{equation}
\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \quad t > 0,
\end{equation}
where \( \mu \) is a positive measure on \((0, \infty)\) and the integral (1.3) exists for all \( t > 0 \).

For \( \mu \) the Lebesgue usual measure, we put
\begin{equation}
\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \quad t > 0.
\end{equation}

If we take \( \mu \) to be the usual Lebesgue measure and the kernel \( w_r(\lambda) = \lambda^{r-1} \), \( r \in (0, 1] \), then
\begin{equation}
t^{r-1} = \sin \left( \frac{r\pi}{\pi} \right) \mathcal{D}(w_r)(t), \quad t > 0.
\end{equation}

For the same measure, if we take the kernel \( w_{1n}(\lambda) = (\lambda + 1)^{-1} \), \( t > 0 \), we have the representation
\begin{equation}
\ln t = (t - 1) \mathcal{D}(w_{1n})(t), \quad t > 0.
\end{equation}

Assume that \( T > 0 \), then by the continuous functional calculus for selfadjoint operators, we can define the positive operator
\begin{equation}
\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\mu(\lambda),
\end{equation}
where \( w \) and \( \mu \) are as above. Also, when \( \mu \) is the usual Lebesgue measure, then
\begin{equation}
\mathcal{D}(w)(T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\lambda,
\end{equation}
for \( T > 0 \).

A real valued continuous function \( f \) on \((0, \infty)\) is said to be operator monotone if \( f(A) \geq f(B) \) holds for any \( A \geq B > 0 \).

We have the following representation of operator monotone functions \([7]\), see for instance \([1, \text{p. 144-145}]\):

**Theorem 1.** A function \( f : (0, \infty) \to \mathbb{R} \) is operator monotone in \((0, \infty)\) if and only if it has the representation
\begin{equation}
f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda),
\end{equation}
where \( a \in \mathbb{R}, \ b \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that
\begin{equation}
\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty.
\end{equation}

If \( f \) is operator monotone in \([0, \infty)\), then \( a = f(0) \) in (1.9).

In the recent paper \([2]\) we obtained the following result:
Theorem 2. For all $A, B > 0$ we have the representation

\begin{equation}
D (w, \mu) (B) - D (w, \mu) (A)
= - \int_0^\infty \left( \int_0^1 (\lambda + tB + (1 - t) A)^{-1} (B - A) (\lambda + tB + (1 - t) A)^{-1} dt \right)
\times w (\lambda) d\mu (\lambda).
\end{equation}

If $B \geq A > 0$, then

\begin{equation}
D (w, \mu) (B) \leq D (w, \mu) (A),
\end{equation}

namely, the function $D (w, \mu) (\cdot)$ is operator monotone decreasing on $(0, \infty)$.

As a consequence we also obtained the following result [2]:

Corollary 1. Assume that $f : [0, \infty) \to \mathbb{R}$ is an operator monotone function on $[0, \infty)$. Then $[f (0) - f (t)] t^{-1}$ is operator monotone on $(0, \infty)$.

A real valued continuous function $f$ on an interval $I$ is said to be operator convex (operator concave) on $I$ if

\begin{equation}
(OC) \quad f ((1 - \lambda) A + \lambda B) \leq (\geq) (1 - \lambda) f (A) + \lambda f (B)
\end{equation}

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator $A$ and $B$ on a Hilbert space $H$ whose spectra are contained in $I$. Notice that a function $f$ is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 3. A function $f : (0, \infty) \to \mathbb{R}$ is operator convex in $(0, \infty)$ if and only if it has the representation

\begin{equation}
f (t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t + \lambda} d\mu (\lambda),
\end{equation}

where $a, b \in \mathbb{R}$, $c \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ such that (1.2) holds. If $f$ is operator convex in $[0, \infty)$, then $a = f (0)$ and $b = f_+^0 (0)$, the right derivative, in (1.13).

In [2] we also obtained the following result for operator convex functions:

Corollary 2. Assume that $f : [0, \infty) \to \mathbb{R}$ is an operator convex function on $[0, \infty)$. Then $[f (0) + f_+^0 (0) t - f (t)] t^{-2}$ is operator monotone on $(0, \infty)$.

For a continuous and positive function $w (\lambda), \lambda > 0$ and a positive measure $\mu$ on $(0, \infty)$, we can define the following mapping, which we call the convex integral transform,

\begin{equation}
C (w, \mu) (t) := t^2 D (w, \mu) (t), \quad t > 0.
\end{equation}
For $t > 0$ we have

$$(1.15) \quad C(w, \mu) (t) := \int_0^\infty w(\lambda) t^2 (t + \lambda)^{-1} d\mu(\lambda)$$
$$= \int_0^\infty w(\lambda) (t + \lambda - \lambda^2 (t + \lambda)^{-1} d\mu(\lambda)$$
$$= \int_0^\infty w(\lambda) \left[ (t + \lambda)^2 - 2\lambda(t + \lambda) + \lambda^2 \right] (t + \lambda)^{-1} d\mu(\lambda)$$
$$= \int_0^\infty w(\lambda) \left[ (t + \lambda) - 2\lambda + \lambda^2 (t + \lambda)^{-1} \right] d\mu(\lambda)$$
$$= \int_0^\infty w(\lambda) \left[ t - \lambda + \lambda^2 (t + \lambda)^{-1} \right] d\mu(\lambda).$$

If $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$ and $\int_0^\infty w(\lambda) \lambda d\mu(\lambda) < \infty$, then we conclude that

$$(1.16) \quad C(w, \mu) (t) = - \int_0^\infty w(\lambda) \lambda d\mu(\lambda) + t \int_0^\infty w(\lambda) d\mu(\lambda) + D(\ell^2 w, \mu) (t),$$
where $\ell(t) = t, t > 0$.

Consider, for instance, the kernel $e_{-a}(\lambda) := \exp(-a\lambda), \lambda \geq 0$ and $a > 0$. After some calculations, we obtain

$D(e_{-a}) (t) = \int_0^\infty \exp \left( \frac{-a\lambda}{t + \lambda} \right) d\lambda = E_1(at) \exp(at), \ t \geq 0$

where the exponential integral is given by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

We also have

$$\int_0^\infty w(\lambda) \lambda d\lambda = \int_0^\infty \exp(-a\lambda) \lambda d\lambda = \frac{1}{a^2}$$

and

$$\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a}.$$

Therefore

$$(1.17) \quad C(e_{-a}) (t) := t^2 D(e_{-a}, \mu) (t) = t^2 E_1(at) \exp(at), \ t > 0.$$

Since

$$D(\ell^2 e_{-a}, \mu) (t) = \int_0^\infty \frac{\lambda^2 \exp(-a\lambda)}{t + \lambda} d\lambda$$

then by (1.16) we get

$$t^2 E_1(at) \exp(at) = \frac{1}{a^2} + \frac{t}{a} + D(\ell^2 w, \mu) (t),$$

which gives

$$D(\ell^2 w, \mu) (t) = t^2 E_1(at) \exp(at) - \frac{t}{a} + \frac{1}{a^2}, \ t > 0, a > 0.$$

If we take $w_r(\lambda) = \lambda^{-r-1}, r \in (0, 1]$, then $\int_0^\infty w_r(\lambda) d\lambda = \infty$ and the equality (1.16) does not hold in this case.
Using the continuous functional calculus for selfadjoint operators in Hilbert spaces we can introduce the following convex integral transform of the positive operator $T$

$$\mathcal{C}(w, \mu) (T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} \, d\mu(\lambda),$$

provided the integral exist.

In this paper, we show among others that, if $\int_0^\infty w(\lambda) \, d\mu(\lambda) < \infty$ and $B \geq A > 0$, then

$$\mathcal{C}(w, \mu) (B) - \mathcal{C}(w, \mu) (A) \leq \left( \int_0^\infty w(\lambda) \, d\lambda \right) (B - A),$$

which shows that the function $(\int_0^\infty w(\lambda) \, d\mu(\lambda)) \, t - \mathcal{C}(w, \mu) (t)$ is operator monotone on $(0, \infty)$. From this we derive that, if $f : [0, \infty) \to \mathbb{R}$ is an operator convex function on $[0, \infty)$ that has the representation

$$f(t) = f(0) + f'_+(0) t + c t^2 + \int_0^\infty \frac{t^2 \lambda}{t + \lambda} \, d\mu(\lambda),$$

where $c \geq 0$ and $\mu$ a positive measure on $(0, \infty)$ with finite expectation $E(\mu) := \int_0^\infty \lambda d\mu(\lambda)$, then $[E(\mu) + f'_+(0)] t + c t^2 - f(t)$ is operator monotone on $(0, \infty)$.

Several examples involving the exponential and logarithmic functions are also given.

2. Main Results

In what follows we assume that the integral transform

$$D(w, \mu) (t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} \, d\mu(\lambda), \quad t > 0,$$

defined for a continuous and positive function $w(\lambda), \lambda > 0$ and $\mu$ a positive measure on $(0, \infty)$, exists for all $t > 0$.

**Theorem 4.** For all $A, B > 0$ we have the representation

$$\mathcal{C}(w, \mu) (B) - \mathcal{C}(w, \mu) (A) = \int_0^\infty w(\lambda) \left[ B - A - \lambda^2 \left( \int_0^1 (\lambda + (1 - t) A + t B)^{-1} (B - A) (\lambda + (1 - t) A + t B)^{-1} \, dt \right) \right] \, d\mu(\lambda).$$

**Proof.** For $A, B > 0$ we have, by using continuous functional calculus for selfadjoint operators and (1.15), that

$$\mathcal{C}(w, \mu) (A) = \int_0^\infty w(\lambda) \left[ A - \lambda + \lambda^2 (A + \lambda)^{-1} \right] \, d\mu(\lambda)$$

and

$$\mathcal{C}(w, \mu) (B) = \int_0^\infty w(\lambda) \left[ B - \lambda + \lambda^2 (B + \lambda)^{-1} \right] \, d\mu(\lambda).$$

This gives that

$$\mathcal{C}(w, \mu) (B) - \mathcal{C}(w, \mu) (A) = \int_0^\infty w(\lambda) \left[ B - A + \lambda^2 \left( (B + \lambda)^{-1} - (A + \lambda)^{-1} \right) \right] \, d\mu(\lambda).$$
Let $T, S > 0$. The function $f(s) = -s^{-1}$ is operator monotone on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

\begin{equation}
\nabla f_T (S) := \lim_{s \to 0} \left[ f(T + sS) - f(T) \right] \frac{1}{s} = T^{-1}ST^{-1}
\end{equation}

for $T, S > 0$.

Consider the continuous function $f$ defined on an interval $I$ for which the corresponding operator function is Gâteaux differentiable on the segment $[C, D] := \{(1-s) C + sD, s \in [0,1]\}$ for $C, D$ selfadjoint operators with spectra in $I$. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(s) := f((1-s) C + sD), \ s \in [0,1].$$

Then we have, by the properties of the Bochner integral, that

\begin{equation}
f(D) - f(C) = \int_0^1 \frac{d}{ds} (f_{C,D}(s)) \, ds = \int_0^1 \nabla f_{(1-s)C+sD}(D-C) \, ds.
\end{equation}

If we write this equality for the function $f(s) = -s^{-1}$ and $C, D > 0$, then we get the representation

\begin{equation}
C^{-1} - D^{-1} = \int_0^1 ((1-s) C + sD)^{-1} (D - C) ((1-s) C + sD)^{-1} \, ds.
\end{equation}

Now, if we take in (2.5) $C = \lambda + B$, $D = \lambda + A$, then

\begin{align*}
& (\lambda + B)^{-1} - (\lambda + A)^{-1} \\
& = \int_0^1 ((1-s) (\lambda + B) + s (\lambda + A))^{-1} (A - B) \\
& \quad \times ((1-s) (\lambda + B) + s (\lambda + A))^{-1} ds \\
& = \int_0^1 (\lambda + (1-s) B + sA)^{-1} (A - B) (\lambda + (1-s) B + sA)^{-1} ds \\
& = \int_0^1 (\lambda + (1-t) A + tB)^{-1} (A - B) (\lambda + (1-t) A + tB)^{-1} dt,
\end{align*}

where for the last equality we used the change of variable $s = 1 - t$.

By utilising (2.2) we derive (2.1). \qedhere

**Corollary 3.** Assume that the kernel $w \in L_1(\mu, [0, \infty))$, namely $\int_0^\infty w(\lambda) \, d\mu(\lambda) < \infty$. Then we have the equality

\begin{equation}
\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) = \left( \int_0^\infty w(\lambda) \, d\mu(\lambda) \right) (B - A) + \mathcal{D}(\ell^2,\mu)(B) - \mathcal{D}(\ell^2,\mu)(A),
\end{equation}

where $\ell(\lambda) = \lambda, \ \lambda \geq 0$.

Moreover, if $B \geq A > 0$, then

\begin{equation}
\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) \leq \left( \int_0^\infty w(\lambda) \, d\lambda \right) (B - A),
\end{equation}

which shows that the function $\left( \int_0^\infty w(\lambda) \, d\mu(\lambda) \right) t - \mathcal{C}(w,\mu)(t)$ is operator monotone on $(0, \infty)$. \pagebreak
**Remark 1.** If we consider the transform from the introduction (1.17),
\begin{equation}
C(c_{-a})(t) = t^2 E_1(at) \exp(at), \ t > 0, \ a > 0,
\end{equation}
then we can conclude that the function \( \frac{1}{a} t - t^2 E_1(at) \exp(at) \) is operator monotone on \( (0, \infty) \) for \( a > 0 \).

If \( B \geq A > 0 \), then we also have the operator inequality
\begin{equation}
B^2 E_1(aB) \exp(aB) - A^2 E_1(aA) \exp(aA) \leq \frac{1}{a} (B - A).
\end{equation}

**Proposition 1.** Assume that the function \( f : [0, \infty) \rightarrow \mathbb{R} \) is operator monotone in \([0, \infty)\) and has the representation (1.9) with \( b \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that the expectation \( E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty \), then for \( B \geq A > 0 \)
\begin{equation}
B f(B) - A f(A) \leq [E(\mu) + f(0)] (B - A) + b (B^2 - A^2).
\end{equation}
This is equivalent to the fact that \([E(\mu) + f(0)] t + bt^2 - tf(t)\) is operator monotone on \((0, \infty)\).

**Proof.** From (1.9) we get by multiplying with \( t > 0 \) that
\[
t f(t) = at + bt^2 + t^2 \int_0^\infty \frac{\lambda}{\lambda + t} d\mu(\lambda) = at + bt^2 + C(\ell, \mu)(t),
\]
namely
\[
C(\ell, \mu)(t) = t f(t) - f(0) t - bt^2.
\]
Since
\[
C(\ell, \mu)(B) - C(\ell, \mu)(A) = B f(B) - A f(A) - f(0) (B - A) - b (B^2 - A^2)
\]
hence by (2.7) we get
\[
B f(B) - A f(A) - f(0) (B - A) - b (B^2 - A^2) \leq E(\mu) (B - A),
\]
which is equivalent to (2.10). \hfill \square

**Proposition 2.** Assume that the function \( f : [0, \infty) \rightarrow \mathbb{R} \) is operator convex in \([0, \infty)\) and has the representation (1.13) with \( c \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that the expectation \( E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty \), then for \( B \geq A > 0 \)
\begin{equation}
f(B) - f(A) \leq [E(\mu) + f_+(0)] (B - A) + c (B^2 - A^2).
\end{equation}
This is equivalent to the fact that \([E(\mu) + f_+(0)] t + ct^2 - f(t)\) is operator monotone on \((0, \infty)\).

**Proof.** From (1.13) we get
\[
C(\ell, \mu)(t) = f(t) - f(0) - f_+(0) t - ct^2.
\]
Since
\[
C(\ell, \mu)(B) - C(\ell, \mu)(A) = f(B) - f(A) - f_+(0) (B - A) - c (B^2 - A^2)
\]
hence by (2.7) we obtain
\[
f(B) - f(A) - f_+(0) (B - A) - c (B^2 - A^2) \leq E(\mu) (B - A),
\]
which is equivalent to (2.11). \hfill \square
Theorem 5. For all \( B \geq A > 0 \), we have

\[
\int_0^1 C(w, \mu) ((1 - t) A + tB) \, dt - \frac{1}{2} \int_0^1 C(w, \mu) ((1 - t) A + tB) \, dt \leq \frac{1}{12} \left( \int_0^\infty w(\lambda) \, d\lambda \right) (B - A).
\]

Proof. Let \( t \in [0, 1] \). Then

\[
(1 - t) A + tB - \frac{A + B}{2} = \left( t - \frac{1}{2} \right) (B - A),
\]

which is positive for \( t \in \left( \frac{1}{2}, 1 \right) \) and negative for \( t \in [0, \frac{1}{2}] \).

Let \( t \in \left( \frac{1}{2}, 1 \right) \), then by (2.7) we get

\[
C(w, \mu) ((1 - t) A + tB) - C(w, \mu) \left( \frac{A + B}{2} \right) \leq \left( \int_0^\infty w(\lambda) \, d\lambda \right) \left( (1 - t) A + tB - \frac{A + B}{2} \right) = \left( \int_0^\infty w(\lambda) \, d\lambda \right) \left( t - \frac{1}{2} \right) (B - A).
\]

If we multiply (2.13) by \( t - \frac{1}{2} > 0 \), we get

\[
\left( C(w, \mu) ((1 - t) A + tB) - C(w, \mu) \left( \frac{A + B}{2} \right) \right) \left( t - \frac{1}{2} \right) \leq \left( \int_0^\infty w(\lambda) \, d\lambda \right) \left( t - \frac{1}{2} \right)^2 (B - A).
\]

We observe that the inequality (2.14) also holds for \( t \in [0, \frac{1}{2}] \). So, if we integrate (2.14) over \( t \in [0, 1] \), then we get

\[
\int_0^1 \left( C(w, \mu) ((1 - t) A + tB) - C(w, \mu) \left( \frac{A + B}{2} \right) \right) \left( t - \frac{1}{2} \right) \, dt \leq \left( \int_0^\infty w(\lambda) \, d\lambda \right) \left( \int_0^1 \left( t - \frac{1}{2} \right)^2 \, dt \right) (B - A).
\]

Observe that

\[
\int_0^1 \left( C(w, \mu) ((1 - t) A + tB) - C(w, \mu) \left( \frac{A + B}{2} \right) \right) \left( t - \frac{1}{2} \right) \, dt = \int_0^1 C(w, \mu) ((1 - t) A + tB) \left( t - \frac{1}{2} \right) \, dt - C(w, \mu) \left( \frac{A + B}{2} \right) \int_0^1 \left( t - \frac{1}{2} \right) \, dt
\]

\[
= \int_0^1 C(w, \mu) ((1 - t) A + tB) \, dt - \frac{1}{2} \int_0^1 C(w, \mu) ((1 - t) A + tB) \, dt
\]

and

\[
\int_0^1 \left( t - \frac{1}{2} \right)^2 \, dt = \frac{1}{12}
\]

and by (2.15) we get (2.12). \qed
**Remark 2.** For all $B \geq A > 0$, we have by (2.12) and (2.8) for $a = 1$ that
\begin{equation}
\int_0^1 ((1-t)A + tB)^2 E_1 ((1-t)A + tB) \exp ((1-t)A + tB) \, dt - \frac{1}{2} \int_0^1 ((1-t)A + tB)^2 E_1 ((1-t)A + tB) \exp ((1-t)A + tB) \, dt \\
\leq \frac{1}{12} (B - A).
\end{equation}

In the case of operator monotone functions we have:

**Proposition 3.** Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ and has the representation (1.9) with $b \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ such that the expectation $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$, then for $B \geq A > 0$
\begin{equation}
\int_0^1 ((1-t)A + tB) f ((1-t)A + tB) \, dt - \frac{1}{2} \int_0^1 ((1-t)A + tB) f ((1-t)A + tB) \, dt \\
\leq \frac{1}{12} [E(\mu) + f(0)] (B - A) + \frac{1}{12} b (B^2 - A^2).
\end{equation}

**Proof.** Using a similar argument as in Theorem 5, we have
\begin{equation}
\left[ (1-t)A + tB \right] f ((1-t)A + tB) - \frac{A + B}{2} f \left( \frac{A + B}{2} \right) \left( t - \frac{1}{2} \right) \\
\leq [E(\mu) + f(0)] \left( (1-t)A + tB - \frac{A + B}{2} \right) \left( t - \frac{1}{2} \right) \\
+ b \left( ((1-t)A + tB)^2 - \left( \frac{A + B}{2} \right)^2 \right) \left( t - \frac{1}{2} \right),
\end{equation}
for all $t \in [0, 1]$.

Integrating (2.18) over $t \in [0, 1]$ and taking into account that $\int_0^1 \left( t - \frac{1}{2} \right) \, dt = 0$, then we obtain
\begin{align*}
\int_0^1 ((1-t)A + tB) f ((1-t)A + tB) \left( t - \frac{1}{2} \right) \, dt &
\leq [E(\mu) + f(0)] \int_0^1 ((1-t)A + tB) \left( t - \frac{1}{2} \right) \, dt \\
&+ b \int_0^1 ((1-t)A + tB)^2 \left( t - \frac{1}{2} \right) \, dt.
\end{align*}

Observe that
\begin{align*}
\int_0^1 ((1-t)A + tB) \left( t - \frac{1}{2} \right) \, dt &= \left( \int_0^1 (1-t) \left( t - \frac{1}{2} \right) \, dt \right) A + \left( \int_0^1 t \left( t - \frac{1}{2} \right) \, dt \right) B \\
&= -\frac{1}{12} A + \frac{1}{12} B = \frac{1}{12} (B - A).
\end{align*}
and
\[
\int_0^1 ((1 - t) A + tB)^2 \left( t - \frac{1}{2} \right) dt
= \int_0^1 \left( (1 - t)^2 A^2 + (1 - t) t (AB + BA) + t^2 B^2 \right) \left( t - \frac{1}{2} \right) dt
= \left( \int_0^1 (1 - t)^2 \left( t - \frac{1}{2} \right) dt \right) A^2 + \left( \int_0^1 (1 - t) t \left( t - \frac{1}{2} \right) dt \right) (AB + BA)
+ \left( \int_0^1 t^2 \left( t - \frac{1}{2} \right) dt \right) B^2
= -\frac{1}{12} A^2 + \frac{1}{12} B^2 = \frac{1}{12} (B^2 - A^2).
\]

By employing (2.18) we derive (2.17).

By making use of a similar argument, we can also obtain the following result for operator convex functions:

**Proposition 4.** Assume that the function \( f : [0, \infty) \to \mathbb{R} \) is operator convex in \([0, \infty)\) and has the representation (1.13) with \( c \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that the expectation \( E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty \), then for \( B \geq A > 0 \)

\[
\int_0^1 f((1 - t) A + tB) dt - \frac{1}{2} \int_0^1 f((1 - t) A + tB) dt
\leq \frac{1}{12} [E(\mu) + f'(0)] (B - A) + \frac{1}{12} c (B^2 - A^2).
\]

3. **More Examples of Transforms with Finite \( \int_0^\infty w(\lambda) d\lambda \)**

We define the *upper incomplete Gamma function* as [10]

\[
\Gamma(a, z) := \int_z^\infty t^{a-1} e^{-t} dt,
\]

which for \( z = 0 \) gives *Gamma function*

\[
\Gamma(a) := \int_0^\infty t^{a-1} e^{-t} dt \text{ for } \text{Re} a > 0.
\]

We have the integral representation [11]

\[
\Gamma(a, z) = \frac{z^a e^{-z}}{\Gamma(1 - a)} \int_0^\infty \frac{t^{-a} e^{-t}}{z + t} dt
\]

for \( \text{Re} a < 1 \) and \(|\text{ph} z| < \pi\).

Now, we consider the weight \( w_{-a e^{-\cdot}}(\lambda) := \lambda^{-a} e^{-\lambda} \) for \( \lambda > 0 \). Then by (3.1) we obtain

\[
\mathcal{D}(w_{-a e^{-\cdot}})(t) = \int_0^\infty \frac{\lambda^{-a} e^{-\lambda}}{t + \lambda} d\lambda = \Gamma(1 - a) t^{-a} e^t \Gamma(a, t)
\]

for \( a < 1 \) and \( t > 0 \).

Define

\[
\mathcal{C}(w_{-a e^{-\cdot}}) := \int_0^\infty \lambda^{-a} e^{-\lambda} d\lambda = \int_0^\infty \lambda^{1-\alpha-1} e^{-\lambda} d\lambda = \Gamma(1 - a)
\]

for \( a < 1 \).
For $a = 0$ in (3.2) we get

\begin{equation}
(3.4) \quad D (w_{e^{-}}) (t) = \int_{0}^{\infty} \frac{e^{-\lambda}}{t + \lambda} d\lambda = \Gamma(1) e^{t} \Gamma(0, t) = e^{t} E_{1} (t)
\end{equation}

for $t > 0$, where the exponential integral $E_{1}$ is defined by

$$E_{1} (t) := \int_{t}^{\infty} \frac{e^{-u}}{u} du.$$

For $a = 0$ in (3.3) we get

\begin{equation}
(3.5) \quad C (w_{e^{-}}) := \int_{0}^{\infty} e^{-\lambda} d\lambda = \Gamma(1) = 1.
\end{equation}

Let $a = 1 - n$, with $n$ a natural number with $n \geq 0$, then by (3.2) we have

\begin{equation}
(3.6) \quad D (w_{n-1e^{-}}) (t) = \int_{0}^{\infty} \frac{\lambda^{n-1} e^{-\lambda}}{t + \lambda} d\lambda = \Gamma(n) t^{n-1} e^{t} \Gamma(1 - n, t) = (n-1)! t^{n-1} e^{t} \Gamma(1 - n, t).
\end{equation}

If we define the generalized exponential integral [12] by

$$E_{p} (z) := z^{p-1} \Gamma (1 - p, z) = z^{p-1} \int_{z}^{\infty} \frac{e^{-t}}{t^{p}} dt$$

then

$$t^{n-1} \Gamma(1 - n, t) = E_{n} (t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [12, Eq. 8.19.7], for $n \geq 2$

$$E_{n} (z) = \frac{(-z)^{n-1}}{(n-1)!} E_{1} (z) + \frac{e^{-z}}{(n-1)!} \sum_{k=0}^{n-2} (n-k-2)! (-z)^{k},$$

we then obtain

\begin{equation}
(3.7) \quad D (w_{n-1e^{-}}) (t) = (n-1)! e^{t} E_{n} (t)
= (n-1)! e^{t}
\times \left[ \frac{(-t)^{n-1}}{(n-1)!} E_{1} (t) + \frac{e^{-t}}{(n-1)!} \sum_{k=0}^{n-2} (n-k-2)! (-t)^{k} \right]
= \sum_{k=0}^{n-2} (-1)^{k} (n-k-2) t^{k} + (-1)^{n-1} t^{n-1} e^{t} E_{1} (t)
\end{equation}

for $n \geq 2$ and $t > 0$.

For $n \geq 2$ we get

$$C (w_{n-1e^{-}}) (t) = \int_{0}^{\infty} \lambda^{n-1} e^{-\lambda} d\lambda = \Gamma(n) = (n-1)!. $$

For $n = 2$, we derive by (3.7) that

\begin{equation}
(3.8) \quad D (w_{e^{-}}) (t) = \int_{0}^{\infty} \lambda e^{-\lambda} (t + \lambda)^{-1} d\lambda = 1 - t \exp (t) E_{1} (t)
\end{equation}

for $t > 0$. We also have

\begin{equation}
(3.9) \quad C (w_{e^{-}}) := \int_{0}^{\infty} \lambda e^{-\lambda} d\lambda = \Gamma(0) = 1.
\end{equation}
By making use of Corollary 3 we can then state:

**Proposition 5.** For $a < 1$, the function $t - t^{2-a} \exp(t) \Gamma(a, t)$ is operator monotone on $(0, \infty)$. In particular, $t - t^{2} \exp(t) E_{1}(t)$ and $t - t^{2} + t^{3} \exp(t) E_{1}(t)$ are operator monotone on $(0, \infty)$.

**Proof.** Consider the convex transform

$$
C(w, -ae^{-}) (t) = t^{2} D(w, -ae^{-}) (t) = \Gamma(1 - a)t^{2-a}e^{t}\Gamma(a, t).
$$

By Corollary 3 we have that $\Gamma(1 - a)t - \Gamma(1 - a)t^{2-a}e^{t}\Gamma(a, t)$ is operator monotone, which implies the claim, since $\Gamma(1 - a) > 0$.

We can also consider the weight $w_{(2+a^{2})^{-1}} (\lambda) := \frac{1}{\lambda^{2} + a^{2}}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$
D\left(w_{(2+a^{2})^{-1}}\right) (t) := \int_{0}^{\infty} \frac{1}{(\lambda + t)(\lambda^{2} + a^{2})} d\lambda = \frac{1}{t^{2} + a^{2}} \left[ \pi t \right]
$$

for $t > 0$ and $a > 0$. We have

$$
D\left(w_{(2+a^{2})^{-1}}\right) := \int_{0}^{\infty} \frac{1}{\lambda^{2} + a^{2}} d\lambda = \frac{\pi}{2a}.
$$

For $a = 1$ we also have

$$
D\left(w_{(2+1)^{-1}}\right) (t) := \int_{0}^{\infty} \frac{1}{(\lambda + t)(\lambda^{2} + 1)} d\lambda = \frac{1}{t^{2} + 1} \left( \frac{\pi t}{2} - \ln t \right)
$$

for $t > 0$. In this case

$$
D\left(w_{(2+1)^{-1}}\right) := \int_{0}^{\infty} \frac{1}{\lambda^{2} + 1} d\lambda = \frac{\pi}{2}.
$$

By making use of Corollary 3 we can then state:

**Proposition 6.** For $a > 0$, the function

$$
\frac{1}{t^{2} + a^{2}} \left[ t^{2} \ln \left( \frac{t}{a} \right) + \frac{\pi t}{2a} \right]
$$

is operator monotone on $(0, \infty)$. In particular

$$
\frac{1}{t^{2} + 1} \left( t^{2} \ln t + \frac{\pi t}{2} \right)
$$

is operator monotone on $(0, \infty)$.

**Proof.** We have

$$
C\left(w_{(2+a^{2})^{-1}}\right) (t) = t^{2}D\left(w_{(2+a^{2})^{-1}}\right) (t) = \frac{t^{2}}{t^{2} + a^{2}} \left[ \frac{\pi t}{2a} - \ln \left( \frac{t}{a} \right) \right].
$$
By Corollary 3 we derive that
\[
\frac{\pi}{2a} t - \frac{t^2}{t^2 + a^2} \left[ \frac{\pi t}{2a} - \ln \left( \frac{t}{a} \right) \right] = \frac{t^2}{t^2 + a^2} \ln \left( \frac{t}{a} \right) + \frac{\pi t}{2a} \left( \frac{a^2}{t^2 + a^2} \right)
\]
\[= \frac{1}{t^2 + a^2} \left[ t^2 \ln \left( \frac{t}{a} \right) + \frac{\pi t}{2a} \right]
\]
is operator monotone on \((0, \infty)\).

The interested reader can obtain other similar results by employing the examples of operator monotone/convex functions from [2]-[5], [8]-[9] and the references therein.

References


[5] T. Furuta, Precise lower bound of \(f(A) - f(B)\) for \(A > B > 0\) and non-constant operator monotone function \(f\) on \([0, \infty)\). *J. Math. Inequal.* 9 (2015), no. 1, 47–52.


1Department of Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

2DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand, Johannesburg, South Africa.