OPERATOR MONOTONICITY OF THE CONVEX INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *convex integral transform*

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, if $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$ and $B \geq A > 0$, then

$$\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) \leq \left(\int_0^\infty w(\lambda) d\lambda\right)(B - A),$$

which shows that the function $\left(\int_0^\infty w(\lambda) d\mu(\lambda)\right) t - \mathcal{C}(w,\mu)(t)$ is operator monotone on $(0,\infty)$. From this we derive that, if $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$ that has the representation

$$f(t) = f(0) + f'_+(0) t + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t + \lambda} d\mu(\lambda),$$

where $c \geq 0$ and μ a positive measure on $(0,\infty)$ with finite expectation $E(\mu) := \int_0^\infty \lambda d\mu(\lambda)$, then $\left[E(\mu) + f'_+(0)\right] t + ct^2 - f(t)$ is operator monotone on $(0,\infty)$. Several examples involving the exponential and logarithmic functions are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

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which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [7], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.9}$$

where $a\in R$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.9).

In the recent paper [2] we obtained the following result:

Theorem 2. *For all A, $B>0$ we have the representation*

$$\begin{aligned}(1.11)\quad &\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)\\ &=-\int_0^\infty\left(\int_0^1(\lambda+tB+(1-t)A)^{-1}(B-A)(\lambda+tB+(1-t)A)^{-1}dt\right)\\ &\times w(\lambda)\,d\mu(\lambda).\end{aligned}$$

If $B\geq A>0$, then

$$\mathcal{D}(w,\mu)(B)\leq\mathcal{D}(w,\mu)(A), \tag{1.12}$$

namely, the function $\mathcal{D}(w,\mu)(\cdot)$ is operator monotone decreasing on $(0,\infty)$.

As a consequence we also obtained the following result [2]:

Corollary 1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function on $[0,\infty)$. Then $[f(0)-f(t)]\,t^{-1}$ is operator monotone on $(0,\infty)$.*

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 3. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty\frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.13}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.13).

In [2] we also obtained the following result for operator convex functions:

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function on $[0,\infty)$. Then $\left[f(0)+f'_+(0)\,t-f(t)\right]t^{-2}$ is operator monotone on $(0,\infty)$*

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call the *convex integral transform,*

$$\mathcal{C}(w,\mu)(t):=t^2\mathcal{D}(w,\mu)(t),\ t>0. \tag{1.14}$$

For $t>0$ we have

$$
\begin{aligned}
(1.15)\qquad \mathcal{C}(w,\mu)(t) &:= \int_0^\infty w(\lambda)\, t^2 (t+\lambda)^{-1}\, d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)(t+\lambda-\lambda)^2 (t+\lambda)^{-1}\, d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left[(t+\lambda)^2 - 2\lambda(t+\lambda)+\lambda^2\right](t+\lambda)^{-1}\, d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left[(t+\lambda) - 2\lambda + \lambda^2 (t+\lambda)^{-1}\right] d\mu(\lambda) \\
&= \int_0^\infty w(\lambda)\left[t - \lambda + \lambda^2 (t+\lambda)^{-1}\right] d\mu(\lambda).
\end{aligned}
$$

If $\int_0^\infty w(\lambda)\, d\mu(\lambda) < \infty$ and $\int_0^\infty w(\lambda)\, \lambda d\mu(\lambda) < \infty$, then we conclude that

$$
(1.16)\qquad \mathcal{C}(w,\mu)(t) = -\int_0^\infty w(\lambda)\, \lambda d\mu(\lambda) + t\int_0^\infty w(\lambda)\, d\mu(\lambda) + \mathcal{D}(\ell^2 w,\mu)(t),
$$

where $\ell(t)=t$, $t>0$.

Consider, for instance, the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \ge 0$ and $a>0$. After some calculations, we obtain

$$
\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda} d\lambda = E_1(at)\exp(at),\ t\ge 0
$$

where the *exponential integral* is given by

$$
E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.
$$

We also have

$$
\int_0^\infty w(\lambda)\, \lambda d\lambda = \int_0^\infty \exp(-a\lambda)\, \lambda d\lambda = \frac{1}{a^2}
$$

and

$$
\int_0^\infty w(\lambda)\, d\lambda = \int_0^\infty \exp(-a\lambda)\, d\lambda = \frac{1}{a}.
$$

Therefore

$$
(1.17)\qquad \mathcal{C}(e_{-a})(t) := t^2 \mathcal{D}(e_{-a},\mu)(t) = t^2 E_1(at)\exp(at),\ t>0.
$$

Since

$$
\mathcal{D}(\ell^2 e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda^2 \exp(-a\lambda)}{t+\lambda} d\lambda
$$

then by (1.16) we get

$$
t^2 E_1(at)\exp(at) = -\frac{1}{a^2} + \frac{t}{a} + \mathcal{D}(\ell^2 w,\mu)(t),
$$

which gives

$$
\mathcal{D}(\ell^2 w,\mu)(t) = t^2 E_1(at)\exp(at) - \frac{t}{a} + \frac{1}{a^2},\ t>0, a>0.
$$

If we take $w_r(\lambda) = \lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\, d\lambda = \infty$ and the equality (1.16) does not hold in this case.

Using the continuous functional calculus for selfadjoint operators in Hilbert spaces we can introduce the following *convex integral transform* of the positive operator T

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda)\, T^2 (\lambda + T)^{-1} d\mu(\lambda),$$

provided the integral exist.

In this paper, we show among others that, if $\int_0^\infty w(\lambda)\, d\mu(\lambda) < \infty$ and $B \geq A > 0$, then

$$\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) \leq \left(\int_0^\infty w(\lambda)\, d\lambda\right)(B - A),$$

which shows that the function $\left(\int_0^\infty w(\lambda)\, d\mu(\lambda)\right) t - \mathcal{C}(w,\mu)(t)$ is operator monotone on $(0,\infty)$. From this we derive that, if $f : [0,\infty) \to \mathbb{R}$ is an operator convex function on $[0,\infty)$ that has the representation

$$f(t) = f(0) + f'_+(0)\, t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t+\lambda} d\mu(\lambda),$$

where $c \geq 0$ and μ a positive measure on $(0,\infty)$ with finite expectation $E(\mu) := \int_0^\infty \lambda d\mu(\lambda)$, then $\left[E(\mu) + f'_+(0)\right] t + ct^2 - f(t)$ is operator monotone on $(0,\infty)$. Several examples involving the exponential and logarithmic functions are also given.

2. Main Results

In what follows we assume that the integral transform

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda),\ \ t > 0,$$

defined for a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$, exists for all $t > 0$.

Theorem 4. *For all A, $B > 0$ we have the representation*

$$\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) = \int_0^\infty w(\lambda)\Big[B - A - \lambda^2 \left(\int_0^1 (\lambda + (1-t)A + tB)^{-1} (B-A) (\lambda + (1-t)A + tB)^{-1} dt\right)\Big] d\mu(\lambda). \tag{2.1}$$

Proof. For A, $B > 0$ we have, by using continuous functional calculus for selfadjoint operators and (1.15), that

$$\mathcal{C}(w,\mu)(A) = \int_0^\infty w(\lambda)\left[A - \lambda + \lambda^2 (A+\lambda)^{-1}\right] d\mu(\lambda)$$

and

$$\mathcal{C}(w,\mu)(B) = \int_0^\infty w(\lambda)\left[B - \lambda + \lambda^2 (B+\lambda)^{-1}\right] d\mu(\lambda).$$

This gives that

$$\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) = \int_0^\infty w(\lambda)\left[B - A + \lambda^2 \left((B+\lambda)^{-1} - (A+\lambda)^{-1}\right)\right] d\mu(\lambda). \tag{2.2}$$

Let T, $S>0$. The function $f(s)=-s^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{s\to 0}\left[\frac{f(T+sS)-f(T)}{s}\right]=T^{-1}ST^{-1} \tag{2.3}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-s)C+sD,\ s\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(s):=f((1-s)C+sD),\ s\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{ds}(f_{C,D}(s))\,ds=\int_0^1\nabla f_{(1-s)C+sD}(D-C)\,ds. \tag{2.4}$$

If we write this equality for the function $f(s)=-s^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-s)C+sD)^{-1}(D-C)((1-s)C+sD)^{-1}\,ds. \tag{2.5}$$

Now, if we take in (2.5) $C=\lambda+B$, $D=\lambda+A$, then

$$\begin{aligned}&(\lambda+B)^{-1}-(\lambda+A)^{-1}\\&=\int_0^1((1-s)(\lambda+B)+s(\lambda+A))^{-1}(A-B)\\&\times((1-s)(\lambda+B)+s(\lambda+A))^{-1}\,ds\\&=\int_0^1(\lambda+(1-s)B+sA)^{-1}(A-B)(\lambda+(1-s)B+sA)^{-1}\,ds\\&=\int_0^1(\lambda+(1-t)A+tB)^{-1}(A-B)(\lambda+(1-t)A+tB)^{-1}\,dt,\end{aligned}$$

where for the last equality we used the change of variable $s=1-t$.

By utilising (2.2) we derive (2.1). □

Corollary 3. *Assume that the kernel $w\in L_1(\mu,[0,\infty))$, namely $\int_0^\infty w(\lambda)\,d\mu(\lambda)<\infty$. Then we have the equality*

$$\begin{aligned}&\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)\\&=\left(\int_0^\infty w(\lambda)\,d\mu(\lambda)\right)(B-A)+\mathcal{D}\left(\ell^2w,\mu\right)(B)-\mathcal{D}\left(\ell^2w,\mu\right)(A),\end{aligned} \tag{2.6}$$

where $\ell(\lambda)=\lambda$, $\lambda\geq 0$.

Moreover, if $B\geq A>0$, then

$$\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)\leq\left(\int_0^\infty w(\lambda)\,d\lambda\right)(B-A), \tag{2.7}$$

which shows that the function $\left(\int_0^\infty w(\lambda)\,d\mu(\lambda)\right)t-\mathcal{C}(w,\mu)(t)$ is operator monotone on $(0,\infty)$.

Remark 1. *If we consider the transform from the introduction (1.17),*

$$\mathcal{C}(e_{-a})(t) = t^2 E_1(at)\exp(at), \; t>0, \; a>0, \tag{2.8}$$

then we can conclude that the function $\frac{1}{a}t - t^2 E_1(at)\exp(at)$ *is operator monotone on* $(0,\infty)$ *for* $a>0$.

If $B \geq A > 0$, *then we also have the operator inequality*

$$B^2 E_1(aB)\exp(aB) - A^2 E_1(aA)\exp(aA) \leq \frac{1}{a}(B-A). \tag{2.9}$$

Proposition 1. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *and has the representation (1.9) with* $b \geq 0$ *and a positive measure* μ *on* $(0,\infty)$ *such that the expectation* $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$, *then for* $B \geq A > 0$

$$Bf(B) - Af(A) \leq [E(\mu) + f(0)](B-A) + b\left(B^2 - A^2\right). \tag{2.10}$$

This is equivalent to the fact that $[E(\mu)+f(0)]t + bt^2 - tf(t)$ *is operator monotone on* $(0,\infty)$.

Proof. From (1.9) we get by multiplying with $t>0$ that

$$tf(t) = at + bt^2 + t^2\int_0^\infty \frac{\lambda}{t+\lambda} d\mu(\lambda) = at + bt^2 + \mathcal{C}(\ell,\mu)(t),$$

namely

$$\mathcal{C}(\ell,\mu)(t) = tf(t) - f(0)t - bt^2.$$

Since

$$\mathcal{C}(\ell,\mu)(B) - \mathcal{C}(\ell,\mu)(A) = Bf(B) - Af(A) - f(0)(B-A) - b\left(B^2 - A^2\right)$$

hence by (2.7) we get

$$Bf(B) - Af(A) - f(0)(B-A) - b\left(B^2-A^2\right) \leq E(\mu)(B-A),$$

which is equivalent to (2.10). □

Proposition 2. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator convex in* $[0,\infty)$ *and has the representation (1.13) with* $c \geq 0$ *and a positive measure* μ *on* $(0,\infty)$ *such that the expectation* $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$, *then for* $B \geq A > 0$

$$f(B) - f(A) \leq \left[E(\mu) + f'_+(0)\right](B-A) + c\left(B^2 - A^2\right). \tag{2.11}$$

This is equivalent to the fact that $\left[E(\mu)+f'_+(0)\right]t + ct^2 - f(t)$ *is operator monotone on* $(0,\infty)$.

Proof. From (1.13) we get

$$\mathcal{C}(\ell,\mu)(t) = f(t) - f(0) - f'_+(0)t - ct^2.$$

Since

$$\mathcal{C}(\ell,\mu)(B) - \mathcal{C}(\ell,\mu)(A) = f(B) - f(A) - f'_+(0)(B-A) - c\left(B^2 - A^2\right)$$

hence by (2.7) we obtain

$$f(B) - f(A) - f'_+(0)(B-A) - c\left(B^2-A^2\right) \leq E(\mu)(B-A),$$

which is equivalent to (2.11). □

Theorem 5. *For all $B\geq A>0$, we have*

$$\int_{0}^{1}\mathcal{C}(w,\mu)\left((1-t)A+tB\right)tdt-\frac{1}{2}\int_{0}^{1}\mathcal{C}(w,\mu)\left((1-t)A+tB\right)dt \\ \leq\frac{1}{12}\left(\int_{0}^{\infty}w(\lambda)\,d\lambda\right)(B-A). \tag{2.12}$$

Proof. Let $t\in[0,1]$. Then

$$(1-t)A+tB-\frac{A+B}{2}=\left(t-\frac{1}{2}\right)(B-A),$$

which is positive for $t\in\left(\frac{1}{2},1\right]$ and negative for $t\in\left[0,\frac{1}{2}\right)$.

Let $t\in\left(\frac{1}{2},1\right]$, then by (2.7) we get

$$\mathcal{C}(w,\mu)\left((1-t)A+tB\right)-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\ \leq\left(\int_{0}^{\infty}w(\lambda)\,d\lambda\right)\left((1-t)A+tB-\frac{A+B}{2}\right) \\ =\left(\int_{0}^{\infty}w(\lambda)\,d\lambda\right)\left(t-\frac{1}{2}\right)(B-A). \tag{2.13}$$

If we multiply (2.13) by $t-\frac{1}{2}>0$, we get

$$\left(\mathcal{C}(w,\mu)\left((1-t)A+tB\right)-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\right)\left(t-\frac{1}{2}\right) \\ \leq\left(\int_{0}^{\infty}w(\lambda)\,d\lambda\right)\left(t-\frac{1}{2}\right)^{2}(B-A). \tag{2.14}$$

We observe that the inequality (2.14) also holds for $t\in\left[0,\frac{1}{2}\right)$. So, if we integrate (2.14) over $t\in[0,1]$, then we get

$$\int_{0}^{1}\left(\mathcal{C}(w,\mu)\left((1-t)A+tB\right)-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\right)\left(t-\frac{1}{2}\right)dt \\ \leq\left(\int_{0}^{\infty}w(\lambda)\,d\lambda\right)\left(\int_{0}^{1}\left(t-\frac{1}{2}\right)^{2}dt\right)(B-A). \tag{2.15}$$

Observe that

$$\int_{0}^{1}\left(\mathcal{C}(w,\mu)\left((1-t)A+tB\right)-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\right)\left(t-\frac{1}{2}\right)dt \\ =\int_{0}^{1}\mathcal{C}(w,\mu)\left((1-t)A+tB\right)\left(t-\frac{1}{2}\right)dt-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\int_{0}^{1}\left(t-\frac{1}{2}\right)dt \\ =\int_{0}^{1}\mathcal{C}(w,\mu)\left((1-t)A+tB\right)tdt-\frac{1}{2}\int_{0}^{1}\mathcal{C}(w,\mu)\left((1-t)A+tB\right)dt$$

and

$$\int_{0}^{1}\left(t-\frac{1}{2}\right)^{2}dt=\frac{1}{12}$$

and by (2.15) we get (2.12). □

Remark 2. *For all $B \geq A > 0$, we have by (2.12) and (2.8) for $a = 1$ that*

$$\int_0^1 ((1-t)A + tB)^2 E_1((1-t)A + tB)\exp((1-t)A + tB)\,tdt \tag{2.16}$$
$$-\frac{1}{2}\int_0^1 ((1-t)A + tB)^2 E_1((1-t)A + tB)\exp((1-t)A + tB)\,dt$$
$$\leq \frac{1}{12}(B - A).$$

In the case of operator monotone functions we have:

Proposition 3. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.9) with $b \geq 0$ and a positive measure μ on $(0,\infty)$ such that the expectation $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$, then for $B \geq A > 0$*

$$\int_0^1 ((1-t)A + tB) f((1-t)A + tB)\,tdt \tag{2.17}$$
$$-\frac{1}{2}\int_0^1 ((1-t)A + tB) f((1-t)A + tB)\,dt$$
$$\leq \frac{1}{12}[E(\mu) + f(0)](B - A) + \frac{1}{12}b\left(B^2 - A^2\right).$$

Proof. Using a similar argument as in Theorem 5, we have

$$\left[((1-t)A + tB) f((1-t)A + tB) - \frac{A+B}{2} f\left(\frac{A+B}{2}\right)\right]\left(t - \frac{1}{2}\right) \tag{2.18}$$
$$\leq [E(\mu) + f(0)]\left((1-t)A + tB - \frac{A+B}{2}\right)\left(t - \frac{1}{2}\right)$$
$$+ b\left(((1-t)A + tB)^2 - \left(\frac{A+B}{2}\right)^2\right)\left(t - \frac{1}{2}\right),$$

for all $t \in [0,1]$.

Integrating (2.18) over $t \in [0,1]$ and taking into account that $\int_0^1 \left(t - \frac{1}{2}\right) dt = 0$, then we obtain

$$\int_0^1 ((1-t)A + tB) f((1-t)A + tB)\left(t - \frac{1}{2}\right) dt$$
$$\leq [E(\mu) + f(0)]\int_0^1 ((1-t)A + tB)\left(t - \frac{1}{2}\right) dt$$
$$+ b\int_0^1 ((1-t)A + tB)^2\left(t - \frac{1}{2}\right) dt.$$

Observe that

$$\int_0^1 ((1-t)A + tB)\left(t - \frac{1}{2}\right) dt$$
$$= \left(\int_0^1 (1-t)\left(t - \frac{1}{2}\right) dt\right) A + \left(\int_0^1 t\left(t - \frac{1}{2}\right) dt\right) B$$
$$= -\frac{1}{12}A + \frac{1}{12}B = \frac{1}{12}(B - A)$$

and

$$\int_0^1 ((1-t)A+tB)^2\left(t-\frac{1}{2}\right)dt$$
$$=\int_0^1\left((1-t)^2A^2+(1-t)t(AB+BA)+t^2B^2\right)\left(t-\frac{1}{2}\right)dt$$
$$=\left(\int_0^1(1-t)^2\left(t-\frac{1}{2}\right)dt\right)A^2+\left(\int_0^1(1-t)t\left(t-\frac{1}{2}\right)dt\right)(AB+BA)$$
$$+\left(\int_0^1 t^2\left(t-\frac{1}{2}\right)dt\right)B^2=-\frac{1}{12}A^2+\frac{1}{12}B^2=\frac{1}{12}\left(B^2-A^2\right).$$

By employing (2.18) we derive (2.17). □

By making use of a similar argument, we can also obtain the following result for operator convex functions:

Proposition 4. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.13) with $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that the expectation $E(\mu):=\int_0^\infty\lambda d\mu(\lambda)<\infty$, then for $B\geq A>0$*

$$\int_0^1 f((1-t)A+tB)\,tdt-\frac{1}{2}\int_0^1 f((1-t)A+tB)\,dt \tag{2.19}$$
$$\leq\frac{1}{12}\left[E(\mu)+f'_+(0)\right](B-A)+\frac{1}{12}c\left(B^2-A^2\right).$$

3. More Examples of Transforms with Finite $\int_0^\infty w(\lambda)\,d\lambda$

We define the *upper incomplete Gamma function* as [10]

$$\Gamma(a,z):=\int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a):=\int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re}a>0.$$

We have the integral representation [11]

$$\Gamma(a,z)=\frac{z^ae^{-z}}{\Gamma(1-a)}\int_0^\infty\frac{t^{-a}e^{-t}}{z+t}dt \tag{3.1}$$

for $\operatorname{Re}a<1$ and $|\operatorname{ph}z|<\pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda):=\lambda^{-a}e^{-\lambda}$ for $\lambda>0$. Then by (3.1) we obtain

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=\int_0^\infty\frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{3.2}$$

for $a<1$ and $t>0$.

Define

$$C\left(w_{\cdot^{-a}e^{-\cdot}}\right):=\int_0^\infty\lambda^{-a}e^{-\lambda}d\lambda=\int_0^\infty\lambda^{1-a-1}e^{-\lambda}d\lambda=\Gamma(1-a) \tag{3.3}$$

for $a<1$.

For $a=0$ in (3.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t)=\int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(1)e^t\Gamma(0,t)=e^tE_1\left(t\right) \tag{3.4}$$

for $t>0$, where the *exponential integral* E_1 is defined by

$$E_1\left(t\right):=\int_t^\infty \frac{e^{-u}}{u}du.$$

For $a=0$ in (3.3) we get

$$C\left(w_{e^{-\cdot}}\right):=\int_0^\infty e^{-\lambda}d\lambda=\Gamma(1)=1. \tag{3.5}$$

Let $a=1-n$, with n a natural number with $n\geq 0$, then by (3.2) we have

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)&=\int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda=\Gamma(n)t^{n-1}e^t\Gamma(1-n,t)\\&=(n-1)!t^{n-1}e^t\Gamma(1-n,t).\end{aligned} \tag{3.6}$$

If we define the generalized exponential integral [12] by

$$E_p\left(z\right):=z^{p-1}\Gamma\left(1-p,z\right)=z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t)=E_n\left(t\right)$$

for $n\geq 1$ and $t>0$.

Using the identity [12, Eq 8.19.7], for $n\geq 2$

$$E_n\left(z\right)=\frac{\left(-z\right)^{n-1}}{(n-1)!}E_1\left(z\right)+\frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\left(-z\right)^k,$$

we then obtain

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)&=(n-1)!e^tE_n\left(t\right)\\&=(n-1)!e^t\\&\quad\times\left[\frac{\left(-t\right)^{n-1}}{(n-1)!}E_1\left(t\right)+\frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\left(-t\right)^k\right]\\&=\sum_{k=0}^{n-2}\left(-1\right)^k(n-k-2)!t^k+\left(-1\right)^{n-1}t^{n-1}e^tE_1\left(t\right)\end{aligned} \tag{3.7}$$

for $n\geq 2$ and $t>0$.

For $n\geq 2$ we get

$$C\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)=\int_0^\infty \lambda^{n-1}e^{-\lambda}d\lambda=\Gamma(n)=(n-1)!.$$

For $n=2$, we derive by (3.7) that

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(t)=\int_0^\infty \lambda e^{-\lambda}\left(t+\lambda\right)^{-1}d\lambda=1-t\exp\left(t\right)E_1\left(t\right) \tag{3.8}$$

for $t>0$. We also have

$$C\left(w_{\cdot e^{-\cdot}}\right):=\int_0^\infty \lambda e^{-\lambda}d\lambda=\Gamma(0)=1. \tag{3.9}$$

By making use of Corollary 3 we can then state:

Proposition 5. *For $a<1$, the function $t-t^{2-a}\exp(t)\Gamma(a,t)$ is operator monotone on $(0,\infty)$. In particular, $t-t^{2}\exp(t)E_{1}(t)$ and $t-t^{2}+t^{3}\exp(t)E_{1}(t)$ are operator monotone on $(0,\infty)$.*

Proof. Consider the convex transform

$$\mathcal{C}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=t^{2}\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=\Gamma(1-a)t^{2-a}e^{t}\Gamma(a,t).$$

By Corollary 3 we have that $\Gamma(1-a)t-\Gamma(1-a)t^{2-a}e^{t}\Gamma(a,t)$ is operator monotone, which implies the claim, since $\Gamma(1-a)>0$. □

We can also consider the weight $w_{(\cdot^{2}+a^{2})^{-1}}(\lambda):=\frac{1}{\lambda^{2}+a^{2}}$ for $\lambda>0$ and $a>0$. Then, by simple calculations, we get

$$\begin{aligned}\mathcal{D}\left(w_{(\cdot^{2}+a^{2})^{-1}}\right)(t)&:=\int_{0}^{\infty}\frac{1}{(\lambda+t)\left(\lambda^{2}+a^{2}\right)}d\lambda\\&=\frac{1}{t^{2}+a^{2}}\left[\frac{\pi t}{2a}-\ln\left(\frac{t}{a}\right)\right]\end{aligned}$$

for $t>0$ and $a>0$. We have

$$D\left(w_{(\cdot^{2}+a^{2})^{-1}}\right):=\int_{0}^{\infty}\frac{1}{\lambda^{2}+a^{2}}d\lambda=\frac{\pi}{2a}.$$

For $a=1$ we also have

$$\begin{aligned}\mathcal{D}\left(w_{(\cdot^{2}+1)^{-1}}\right)(t)&:=\int_{0}^{\infty}\frac{1}{(\lambda+t)\left(\lambda^{2}+1\right)}d\lambda\\&=\frac{1}{t^{2}+1}\left(\frac{\pi t}{2}-\ln t\right)\end{aligned}$$

for $t>0$. In this case

$$D\left(w_{(\cdot^{2}+1)^{-1}}\right):=\int_{0}^{\infty}\frac{1}{\lambda^{2}+1}d\lambda=\frac{\pi}{2}.$$

By making use of Corollary 3 we can then state:

Proposition 6. *For $a>0$, the function*

$$\frac{1}{t^{2}+a^{2}}\left[t^{2}\ln\left(\frac{t}{a}\right)+\frac{\pi}{2}ta\right]$$

is operator monotone on $(0,\infty)$. In particular

$$\frac{1}{t^{2}+1}\left(t^{2}\ln t+\frac{\pi}{2}t\right)$$

is operator monotone on $(0,\infty)$.

Proof. We have

$$\mathcal{C}\left(w_{(\cdot^{2}+a^{2})^{-1}}\right)(t)=t^{2}\mathcal{D}\left(w_{(\cdot^{2}+a^{2})^{-1}}\right)(t)=\frac{t^{2}}{t^{2}+a^{2}}\left[\frac{\pi t}{2a}-\ln\left(\frac{t}{a}\right)\right].$$

By Corollary 3 we derive that

$$\begin{aligned}\frac{\pi}{2a}t-\frac{t^2}{t^2+a^2}\left[\frac{\pi t}{2a}-\ln\left(\frac{t}{a}\right)\right]&=\frac{t^2}{t^2+a^2}\ln\left(\frac{t}{a}\right)+\frac{\pi}{2a}t\left(\frac{a^2}{t^2+a^2}\right)\\&=\frac{1}{t^2+a^2}\left[t^2\ln\left(\frac{t}{a}\right)+\frac{\pi}{2}ta\right]\end{aligned}$$

is operator monotone on $(0,\infty)$. □

The interested reader can obtain other similar results by employing the examples of operator monotone/convex functions from [2]-[5], [8]-[9] and the references therein.

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