BOUNDS IN TERMS OF DERIVATIVE FOR THE CONVEX INTEGRAL TRANSFORM OF POSITIVE OPERATORS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *convex integral transform*

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M. In this paper we show that, if $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$, then

$$\begin{aligned} 0 &\leq m\left[\int_0^\infty w(\lambda) d\mu(\lambda) - \mathcal{C}'(w,\mu)(\delta)\right] \\ &\leq \int_0^\infty w(\lambda) d\mu(\lambda)(B-A) - [\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)] \\ &\leq M\left[\int_0^\infty w(\lambda) d\mu(\lambda) - \mathcal{C}'(w,\mu)(\alpha)\right]. \end{aligned}$$

where $\mathcal{C}'(w,\mu)(t)$ is the derivative of $\mathcal{C}(w,\mu)(t)$ as a function of $t > 0$.

Some examples for operator monotone or operator convex functions with finite expectation of the kernel as well as for integral transforms $\mathcal{C}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

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which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ, the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty\frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.9}$$

where $a\in\mathbb{R}$, $b\geq0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty\frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.9).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left((1-\lambda)A+\lambda B\right)\leq(\geq)\,(1-\lambda)\,f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty\frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.11}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_+(0)$, the right derivative, in (1.11).

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call the *convex integral transform,*

$$\mathcal{C}(w,\mu)(t):=t^2\mathcal{D}(w,\mu)(t),\ t>0. \tag{1.12}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{C}(w,\mu)(t)&:=\int_0^\infty w(\lambda)\,t^2(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)(t+\lambda-\lambda)^2(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[(t+\lambda)^2-2\lambda(t+\lambda)+\lambda^2\right](t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[(t+\lambda)-2\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[t-\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned} \tag{1.13}$$

If $\int_0^\infty w(\lambda)\,d\mu(\lambda)<\infty$ and $\int_0^\infty w(\lambda)\,\lambda d\mu(\lambda)<\infty$, then we conclude that

$$\mathcal{C}(w,\mu)(t)=-\int_0^\infty w(\lambda)\,\lambda d\mu(\lambda)+t\int_0^\infty w(\lambda)\,d\mu(\lambda)+\mathcal{D}(\ell^2w,\mu)(t), \tag{1.14}$$

where $\ell(t)=t$, $t>0$.

Consider, for instance, the kernel $e_{-a}(\lambda):=\exp(-a\lambda)$, $\lambda\geq 0$ and $a>0$. After some calculations, we obtain

$$\mathcal{D}(e_{-a})(t)=\int_0^\infty\frac{\exp(-a\lambda)}{t+\lambda}d\lambda=E_1(at)\exp(at),\ t\geq 0$$

where the *exponential integral* is given by

$$E_1(t):=\int_t^\infty\frac{e^{-u}}{u}du.$$

We also have

$$\int_0^\infty w(\lambda)\,\lambda d\lambda = \int_0^\infty \exp(-a\lambda)\,\lambda d\lambda = \frac{1}{a^2}$$

and

$$\int_0^\infty w(\lambda)\,d\lambda = \int_0^\infty \exp(-a\lambda)\,d\lambda = \frac{1}{a}.$$

Therefore

$$\mathcal{C}(e_{-a})(t) := t^2\mathcal{D}(e_{-a},\mu)(t) = t^2E_1(at)\exp(at),\ t>0. \tag{1.15}$$

Since

$$\mathcal{D}(\ell^2 e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda^2\exp(-a\lambda)}{t+\lambda}d\lambda$$

then by (1.14) we get

$$t^2E_1(at)\exp(at) = -\frac{1}{a^2} + \frac{t}{a} + \mathcal{D}(\ell^2 w,\mu)(t),$$

which gives

$$\mathcal{D}(\ell^2 w,\mu)(t) = t^2E_1(at)\exp(at) - \frac{t}{a} + \frac{1}{a^2},\ t>0, a>0.$$

If we take $w_r(\lambda) = \lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda = \infty$ and the equality (1.14) does not hold in this case.

Assume that the kernel $w \in L_1(\mu,[0,\infty))$, namely $\int_0^\infty w(\lambda)\,d\mu(\lambda) < \infty$. In [2] we obtained the equality

$$\begin{aligned}&\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)\\ &= \left(\int_0^\infty w(\lambda)\,d\mu(\lambda)\right)(B-A) + \mathcal{D}\left(\ell^2 w,\mu\right)(B) - \mathcal{D}\left(\ell^2 w,\mu\right)(A),\end{aligned}$$

where $\ell(\lambda) = \lambda$, $\lambda \geq 0$.

Moreover, if $B \geq A > 0$, then

$$\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A) \leq \left(\int_0^\infty w(\lambda)\,d\lambda\right)(B-A),$$

which shows that the function $\left(\int_0^\infty w(\lambda)\,d\mu(\lambda)\right)t - \mathcal{C}(w,\mu)(t)$ is operator monotone on $(0,\infty)$.

Assume that $A \geq \alpha > 0$, $\delta \geq B > 0$ and $0 < m \leq B - A \leq M$ for some constants α, δ, m, M. In this paper we show among others that, if $\int_0^\infty w(\lambda)\,d\mu(\lambda) < \infty$, then

$$\begin{aligned}0 &\leq m\left[\int_0^\infty w(\lambda)\,d\mu(\lambda) - \mathcal{C}'(w,\mu)(\delta)\right]\\ &\leq \int_0^\infty w(\lambda)\,d\mu(\lambda)(B-A) - [\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)]\\ &\leq M\left[\int_0^\infty w(\lambda)\,d\mu(\lambda) - \mathcal{C}'(w,\mu)(\alpha)\right].\end{aligned}$$

where $\mathcal{C}'(w,\mu)(t)$ is the derivative of $\mathcal{C}(w,\mu)(t)$ as a function of $t>0$.

Some examples for operator monotone or operator convex functions with finite expectation of the kernel as well as for integral transforms $\mathcal{C}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

Let f be an operator convex function on the interval of real numbers I. For A, $B\in\mathcal{SA}_I(H)$, the class of all selfadjoint operators with spectra in I, we consider the auxiliary function $\varphi_{(A,B)}:[0,1]\to\mathcal{B}(H)$ defined by

$$\varphi_{(A,B)}(t):=f((1-t)A+tB). \tag{2.1}$$

For $x\in H$ we can also consider the auxiliary function $\varphi_{(A,B);x}:[0,1]\to\mathbb{R}$ defined by

$$\varphi_{(A,B);x}(t):=\left\langle\varphi_{(A,B)}(t)x,x\right\rangle=\langle f((1-t)A+tB)x,x\rangle. \tag{2.2}$$

We have the following basic fact [2]:

Lemma 1. *Let f be an operator convex function on I. For any A, $B\in\mathcal{SA}_I(H)$, $\varphi_{(A,B)}$ is well defined and convex in the operator order. For any $A,B\in\mathcal{SA}_I(H)$ and $x\in H$ the function $\varphi_{(A,B);x}$ is convex in the usual sense on $[0,1]$.*

A continuous function $g:\mathcal{SA}_I(H)\to\mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A\in\mathcal{SA}_I(H)$ along the direction $B\in\mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B):=\lim_{s\to0}\frac{g(A+sB)-g(A)}{s}\in\mathcal{B}(H). \tag{2.3}$$

If the limit (2.3) exists for all $B\in\mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g\in\mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g\in\mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators A, $B\in\mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A,B]:=\{(1-t)A+tB\mid t\in[0,1]\}.$$

We observe that A, $B\in[A,B]$ and $[A,B]\subset\mathcal{SA}_I(H)$.

We also have [2]:

Lemma 2. *Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then the auxiliary function $\varphi_{(A,B)}$ is differentiable on $(0,1)$ and*

$$\varphi'_{(A,B)}(t)=\nabla f_{(1-t)A+tB}(B-A). \tag{2.4}$$

In particular,

$$\varphi'_{(A,B)}(0+)=\nabla f_A(B-A) \tag{2.5}$$

and

$$\varphi'_{(A,B)}(1-)=\nabla f_B(B-A). \tag{2.6}$$

and, see [2],

Lemma 3. *Let f be an operator convex function on I and A, $B\in\mathcal{SA}_I(H)$, with $A\neq B$. If $f\in\mathcal{G}([A,B])$, then for $0<t_1<t_2<1$*

$$\nabla f_{(1-t_1)A+t_1B}(B-A)\leq\nabla f_{(1-t_2)A+t_2B}(B-A) \tag{2.7}$$

in the operator order.

In particular,

$$\nabla f_A (B-A) \le \nabla f_{(1-t_1)A+t_1 B} (B-A) \tag{2.8}$$

and

$$\nabla f_{(1-t_2)A+t_2 B} (B-A) \le \nabla f_B (B-A). \tag{2.9}$$

Also, we have

$$\nabla f_A (B-A) \le \nabla f_{(1-t)A+tB} (B-A) \le \nabla f_B (B-A) \tag{2.10}$$

for all $t \in (0,1)$.

We have the following gradient inequalities:

Lemma 4. *Let* f *be an operator convex function on* I *and* A, $B \in \mathcal{SA}_I(H)$, *with* $A \neq B$. *If* $f \in \mathcal{G}([A,B])$, *then*

$$\nabla_B f (B-A) \ge f(B) - f(A) \ge \nabla_A f (B-A). \tag{2.11}$$

Proof. By the properties of Bochner integral, we have

$$\begin{aligned} f(B) - f(A) &= \varphi_{(A,B)}(1) - \varphi_{(A,B)}(0) = \int_0^1 \varphi'_{(A,B)}(t)\,dt \\ &= \int_0^1 \nabla f_{(1-t)A+tB} (B-A)\,dt. \end{aligned}$$

From (2.10) we have, by integration, that

$$\nabla f_A (B-A) \le \int_0^1 \nabla f_{(1-t)A+tB} (B-A)\,dt \le \nabla f_B (B-A),$$

and the inequality (2.11) is proved. □

Let T, $S > 0$. The function $f(t) = t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T (S) := \lim_{t \to 0} \left[\frac{f(T+tS) - f(T)}{t} \right] = -T^{-1} S T^{-1} \tag{2.12}$$

for T, $S > 0$.

Using (2.12) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1} \ge D^{-1} - C^{-1} \ge -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$D^{-1}(D-C)D^{-1} \le C^{-1} - D^{-1} \le C^{-1}(D-C)C^{-1} \tag{2.13}$$

for all C, $D > 0$.

Theorem 3. *Assume that* $A \ge \alpha > 0$, $\delta \ge B > 0$ *and* $0 < m \le B - A \le M$ *for some constants* α, δ, m, M. *If* $\int_0^\infty w(\lambda)\,d\mu(\lambda) < \infty$, *then*

$$\begin{aligned} 0 &\le m \left[\int_0^\infty w(\lambda)\,d\mu(\lambda) - \mathcal{C}'(w,\mu)(\delta) \right] \\ &\le \int_0^\infty w(\lambda)\,d\mu(\lambda)(B-A) - [\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)] \\ &\le M \left[\int_0^\infty w(\lambda)\,d\mu(\lambda) - \mathcal{C}'(w,\mu)(\alpha) \right]. \end{aligned} \tag{2.14}$$

Proof. For $A, B > 0$ we have, by using continuous functional calculus for selfadjoint operators and (1.13), that

$$\mathcal{C}(w,\mu)(A) = \int_0^\infty w(\lambda)\left[A - \lambda + \lambda^2 (A+\lambda)^{-1}\right] d\mu(\lambda)$$

and

$$\mathcal{C}(w,\mu)(B) = \int_0^\infty w(\lambda)\left[B - \lambda + \lambda^2 (B+\lambda)^{-1}\right] d\mu(\lambda).$$

This gives that

$$\begin{aligned}&\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)\\ &= \int_0^\infty w(\lambda)\left[B - A + \lambda^2\left((B+\lambda)^{-1} - (A+\lambda)^{-1}\right)\right] d\mu(\lambda).\end{aligned}$$

Since $\int_0^\infty w(\lambda)\,d\mu(\lambda) < \infty$, then

$$\begin{aligned}&\int_0^\infty w(\lambda)\,d\mu(\lambda)(B-A) - [\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)]\\ &= \int_0^\infty \lambda^2 w(\lambda)\left((A+\lambda)^{-1} - (B+\lambda)^{-1}\right) d\mu(\lambda).\end{aligned} \tag{2.15}$$

From (2.13) we get for $C = \lambda + A$ and $D = \lambda + B$ that

$$\begin{aligned}(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1} &\le (\lambda+A)^{-1} - (\lambda+B)^{-1}\\ &\le (\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}\end{aligned} \tag{2.16}$$

for all $\lambda \ge 0$.

If we multiply (2.16) by $\lambda^2 w(\lambda) \ge 0$ and integrate over $\mu(\lambda)$ we get by (2.15) that

$$\begin{aligned}&\int_0^\infty \lambda^2 w(\lambda)(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1} d\mu(\lambda)\\ &\le \int_0^\infty w(\lambda)\,d\mu(\lambda)(B-A) - [\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)]\\ &\le \int_0^\infty \lambda^2 w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} d\mu(\lambda).\end{aligned} \tag{2.17}$$

Since $m \le B - A \le M$, hence

$$m(\lambda+B)^{-2} \le (\lambda+B)^{-1}(B-A)(\lambda+B)^{-1},$$

which implies, by integration that

$$\begin{aligned}&m\int_0^\infty \lambda^2 w(\lambda)(\lambda+B)^{-2} d\mu(\lambda)\\ &\le \int_0^\infty \lambda^2 w(\lambda)(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1} d\mu(\lambda).\end{aligned} \tag{2.18}$$

Also

$$(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} \le M(\lambda+A)^{-2},$$

which implies, by integration, that

$$\int_0^\infty \lambda^2 w(\lambda)(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}d\mu(\lambda) \le M\int_0^\infty \lambda^2 w(\lambda)(\lambda+A)^{-2}d\mu(\lambda). \tag{2.19}$$

Since $B \le \delta$, then $\lambda + B \le \lambda + \delta$ for all $\lambda \ge 0$, which implies that $(\lambda+B)^{-1} \ge (\lambda+\delta)^{-1}$ and therefore $(\lambda+B)^{-2} \ge (\lambda+\delta)^{-2}$. Consequently

$$m\int_0^\infty \lambda^2 w(\lambda)(\lambda+B)^{-2}d\mu(\lambda) \ge m\int_0^\infty \lambda^2 w(\lambda)(\lambda+\delta)^{-2}d\mu(\lambda). \tag{2.20}$$

Also, since $A \ge \alpha > 0$, then $\lambda + A \ge \lambda + \alpha > 0$, which implies that $(\lambda+A)^{-1} \le (\lambda+\alpha)^{-1}$, therefore $(\lambda+A)^{-2} \le (\lambda+\alpha)^{-2}$ and

$$M\int_0^\infty \lambda^2 w(\lambda)(\lambda+A)^{-2}d\mu(\lambda) \le M\int_0^\infty \lambda^2 w(\lambda)(\lambda+\alpha)^{-2}d\mu(\lambda). \tag{2.21}$$

From (2.17)-(2.21) we get

$$\begin{aligned} 0 &\le m\int_0^\infty \lambda^2 w(\lambda)(\lambda+\delta)^{-2}d\mu(\lambda) \\ &\le \int_0^\infty w(\lambda)d\mu(\lambda)(B-A) - [\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)] \\ &\le M\int_0^\infty \lambda^2 w(\lambda)(\lambda+\alpha)^{-2}d\mu(\lambda). \end{aligned} \tag{2.22}$$

Taking the derivative in (1.13) over t, we get

$$\begin{aligned} \mathcal{C}'(w,\mu)(t) &= \int_0^\infty w(\lambda)\left[1-\lambda^2(t+\lambda)^{-2}\right]d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)d\mu(\lambda) - \int_0^\infty w(\lambda)\lambda^2(t+\lambda)^{-2}d\mu(\lambda). \end{aligned}$$

From this equality we obtain

$$\int_0^\infty \lambda^2 w(\lambda)(\lambda+\delta)^{-2}d\mu(\lambda) = \int_0^\infty w(\lambda)d\mu(\lambda) - \mathcal{C}'(w,\mu)(\delta)$$

and

$$\int_0^\infty \lambda^2 w(\lambda)(\lambda+\alpha)^{-2}d\mu(\lambda) = \int_0^\infty w(\lambda)d\mu(\lambda) - \mathcal{C}'(w,\mu)(\alpha).$$

From (2.22) we derive (2.14). □

We know that for $T > 0$, we have the operator inequalities

$$0 < \left\|T^{-1}\right\|^{-1} \le T \le \|T\|. \tag{2.23}$$

Indeed, it is well known that, if $P \ge 0$, then

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle\langle Py, y\rangle$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$\begin{aligned} 0 &\le \langle x, x\rangle^2 = \left\langle T^{-1}Tx, x\right\rangle^2 = \left\langle Tx, T^{-1}x\right\rangle^2 \\ &\le \langle Tx, x\rangle\left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle\left\langle x, T^{-1}x\right\rangle \end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$1 \leq \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \leq \langle Tx, x\rangle \sup_{\|x\|=1} \left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} 1_H \leq T. \tag{2.24}$$

The second inequality in (2.24) is obvious.

Corollary 1. *If* A, $B > 0$ *and* $B - A > 0$, *then*

$$\begin{aligned}(2.25)\qquad 0 &\leq \left\|(B-A)^{-1}\right\|^{-1} \left[\int_0^\infty w(\lambda)\, d\mu(\lambda) - \mathcal{C}'(w,\mu)(\|B\|)\right] \\ &\leq \int_0^\infty w(\lambda)\, d\mu(\lambda)(B-A) - \left[\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)\right] \\ &\leq \|B-A\| \left[\int_0^\infty w(\lambda)\, d\mu(\lambda) - \mathcal{C}'(w,\mu)\left(\left\|A^{-1}\right\|^{-1}\right)\right].\end{aligned}$$

Proof. Since $A \geq \left\|A^{-1}\right\|^{-1} = \alpha > 0$, $\delta = \|B\| \geq B > 0$ and

$$0 < m = \left\|(B-A)^{-1}\right\|^{-1} \leq B - A \leq \|B - A\| = M,$$

then by (2.14) we get (2.24). □

Proposition 1. *Let* $f : [0,\infty) \to \mathbb{R}$ *be operator monotone in* $[0,\infty)$ *that has the representation (1.9) with the finite expectation* $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$. *If* $A \geq \alpha > 0$, $\delta \geq B > 0$ *and* $0 < m \leq B - A \leq M$ *for some constants* α, δ, m, M, *then*

$$\begin{aligned}(2.26)\qquad 0 &\leq m \left[\int_0^\infty w(\lambda)\, d\mu(\lambda) - f(\delta) - \delta\left(f'(\delta) - 2b\right) + f(0)\right] \\ &\leq \int_0^\infty w(\lambda)\, d\mu(\lambda)(B-A) - Bf(B) + Af(A) \\ &\quad + f(0)(B-A) + b\left(B^2 - A^2\right) \\ &\leq M \left[\int_0^\infty w(\lambda)\, d\mu(\lambda) - f(\alpha) - \alpha\left(f'(\alpha) - 2b\right) + f(0)\right].\end{aligned}$$

Proof. From (1.9) we get by multiplying with $t > 0$ that

$$tf(t) = at + bt^2 + t^2 \int_0^\infty \frac{\lambda}{t+\lambda} d\mu(\lambda) = at + bt^2 + \mathcal{C}(\ell,\mu)(t),$$

namely

$$\mathcal{C}(\ell,\mu)(t) = tf(t) - f(0)\, t - bt^2.$$

Since

$$\begin{aligned}&\mathcal{C}(\ell,\mu)(B) - \mathcal{C}(\ell,\mu)(A) \\ &\quad = Bf(B) - Af(A) - f(0)(B-A) - b\left(B^2 - A^2\right)\end{aligned}$$

and

$$\mathcal{C}'(\ell,\mu)(t) = f(t) + tf'(t) - f(0) - 2bt = f(t) + t\left(f'(t) - 2b\right) - f(0),$$

hence by (2.14) we get (2.26). □

The case of operator convex functions is as follows:

Proposition 2. *LAssume that the function* $f : [0,\infty) \to \mathbb{R}$ *is operator convex in* $[0,\infty)$ *and has the representation (1.11) with* $c \geq 0$ *and a positive measure* μ *on* $(0,\infty)$ *such that the expectation* $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$. *If* $A \geq \alpha > 0$, $\delta \geq B > 0$ *and* $0 < m \leq B - A \leq M$ *for some constants* α, δ, m, M, *then*

$$
\begin{aligned}
0 &\leq m\left[\int_0^\infty w(\lambda)\,d\mu(\lambda) - f'(\delta) + f'_+(0) + 2c\delta\right] \\
&\leq \int_0^\infty w(\lambda)\,d\mu(\lambda)(B-A) - f(B) + f(A) \\
&+ f'_+(0)(B-A) + c\left(B^2 - A^2\right) \\
&\leq M\left[\int_0^\infty w(\lambda)\,d\mu(\lambda) - f'(\alpha) + f'_+(0) + 2c\alpha\right].
\end{aligned}
\tag{2.27}
$$

Proof. From (1.11) we get

$$\mathcal{C}(\ell,\mu)(t) = f(t) - f(0) - f'_+(0)\,t - ct^2.$$

Since

$$
\begin{aligned}
&\mathcal{C}(\ell,\mu)(B) - \mathcal{C}(\ell,\mu)(A) \\
&= f(B) - f(A) - f'_+(0)(B-A) - c\left(B^2 - A^2\right)
\end{aligned}
$$

and

$$\mathcal{C}'(\ell,\mu)(t) = f'(t) - f'_+(0) - 2ct,$$

hence by (2.14) we get (2.27). □

3. More Examples

We define the *upper incomplete Gamma function* as [11]

$$\Gamma(a,z) := \int_z^\infty t^{a-1} e^{-t} dt,$$

which for $z = 0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1} e^{-t} dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [12]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a} e^{-t}}{z+t} dt \tag{3.1}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a} e^{-\lambda}$ for $\lambda > 0$. Then by (3.1) we have

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a} e^{-\lambda}}{t+\lambda} d\lambda = \Gamma(1-a) t^{-a} e^t \Gamma(a,t) \tag{3.2}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (3.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda} d\lambda = \Gamma(1) e^t \Gamma(0,t) = e^t E_1(t) \tag{3.3}$$

for $t>0$, where

$$E_1(t):=\int_t^\infty \frac{e^{-u}}{u}du.$$

We also have

$$\int_0^\infty t^{-a}e^{-t}dt=\Gamma(1-a) \text{ and } \int_0^\infty e^{-t}dt=\Gamma(1)=1.$$

We then obtain

$$\mathcal{C}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(T)=\Gamma(1-a)T^{2-a}\exp(T)\,\Gamma(a,T) \tag{3.4}$$

for $a<1$ and

$$\mathcal{C}\left(w_{e^{-\cdot}}\right)(T)=T^2\exp(T)\,E_1(T) \tag{3.5}$$

for $T>0$.

Consider $\mathcal{C}\left(w_{e^{-\cdot}}\right)(t)=t^2\exp(t)\,E_1(t)$, $t>0$. Since $E_1'(t)=-\frac{e^{-t}}{t}$, $t>0$, then

$$\begin{aligned}\mathcal{C}'\left(w_{e^{-\cdot}}\right)(t) &= \left(t^2\exp(t)\,E_1(t)\right)'=\left(t^2\exp(t)\right)'E_1(t)+t^2\exp(t)\,E_1'(t)\\ &= (2+t)\,tE_1(t)\exp t-t=t\left[(2+t)\,E_1(t)\exp t-1\right].\end{aligned}$$

By Theorem 3 we get

$$\begin{aligned}0 &\le m\left[1-\delta\left[(2+\delta)\,E_1(\delta)\exp\delta-1\right]\right]\\ &\le B-A-\left[B^2\exp(B)\,E_1(B)-A^2\exp(A)\,E_1(A)\right]\\ &\le M\left[1-\alpha\left[(2+\alpha)\,E_1(\alpha)\exp\alpha-1\right]\right]\end{aligned} \tag{3.6}$$

if $A\ge\alpha>0$, $\delta\ge B>0$ and $0<m\le B-A\le M$ for some constants α, δ, m, M.

More examples of such transforms are

$$\mathcal{C}(w_{1/(\ell^2+a^2)})(t):=\int_0^\infty \frac{t^2}{(t+\lambda)\left(\lambda^2+a^2\right)}d\lambda=\frac{\pi t^3-2at^2\ln(t/a)}{2a\left(t^2+a^2\right)},\ t\ge 0$$

and

$$\mathcal{C}(w_{\ell/(\ell^2+a^2)})(t):=\int_0^\infty \frac{t^2\lambda}{(t+\lambda)\left(\lambda^2+a^2\right)}d\lambda=\frac{\pi at^2+2t^3\ln(t/a)}{2a\left(t^2+a^2\right)},\ t\ge 0$$

for $a>0$.

The interested reader may state other similar results by employing the examples of monotone operator functions provided in [4], [5], [6], [9] and [10].

References

[1] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] S. S. Dragomir, Reverses of operator Féjer's inequalities, *Tokyo J. Math.* (to appear in Volume 43, 2020).

[3] S. S. Dragomir, Operator monotonicity of the convex integral transform of positive operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art.

[4] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[5] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.

[6] T. Furuta, Precise lower bound of $f(A)-f(B)$ for $A>B>0$ and non-constant operator monotone function f on $[0,\infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[7] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[8] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[9] M. S. Moslehian, H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.

[10] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

[11] Incomplete Gamma and Related Functions, Definitions, *Digital Library of Mathematical Functions*, NIST. [Online https://dlmf.nist.gov/8.2].

[12] Incomplete Gamma and Related Functions, Integral Representations, *Digital Library of Mathematical Functions,* NIST. [Online https://dlmf.nist.gov/8.6].

[13] Generalized Exponential Integral, *Digital Library of Mathematical Functions,* NIST. [Online https://dlmf.nist.gov/8.19#E1].

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