OPERATOR CONVEXITY OF THE CONVEX INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *convex integral transform*

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that

$$\begin{aligned}&\frac{\mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B)}{2} - \mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\ &= \frac{1}{2}\int_0^\infty \left((\lambda + A)^{-1} - (\lambda + B)^{-1}\right)\left((\lambda + A)^{-1} + (\lambda + B)^{-1}\right)^{-1} \\ &\times \left((\lambda + A)^{-1} - (\lambda + B)^{-1}\right)\lambda^2 w(\lambda) d\mu(\lambda) \\ &\geq 0.\end{aligned}$$

for all $A, B > 0$, implying that $\mathcal{C}(w,\mu)(t)$ is operator convex on $(0, \infty)$. Several examples involving the exponential and logarithmic functions are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

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which gives the representation for the logarithm

$$\ln t=(t-1)\int_{0}^{\infty}\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_{0}^{\infty}\frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_{0}^{\infty}\frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_{0}^{\infty}w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_{0}^{\infty}w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_{0}^{\infty}\frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.9}$$

where $a\in R$, $b\geq0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_{0}^{\infty}\frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.9).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left(\left(1-\lambda\right)A+\lambda B\right)\leq\left(\geq\right)\left(1-\lambda\right)f\left(A\right)+\lambda f\left(B\right) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f\left(t\right)=a+bt+ct^{2}+\int_{0}^{\infty}\frac{t^{2}\lambda}{t+\lambda}d\mu\left(\lambda\right), \tag{1.11}$$

where a, $b\in\mathbb{R}$, $c\geq0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f\left(0\right)$ and $b=f_{+}^{\prime}\left(0\right)$, the right derivative, in (1.11).

For a continuous and positive function $w\left(\lambda\right)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call the *convex integral transform,*

$$\mathcal{C}(w,\mu)\left(t\right):=t^{2}\mathcal{D}(w,\mu)\left(t\right),\ t>0. \tag{1.12}$$

For $t>0$ we have

$$\begin{aligned}
\mathcal{C}(w,\mu)\left(t\right) &:=\int_{0}^{\infty}w\left(\lambda\right)t^{2}\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right) \\
&=\int_{0}^{\infty}w\left(\lambda\right)\left(t+\lambda-\lambda\right)^{2}\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right) \\
&=\int_{0}^{\infty}w\left(\lambda\right)\left[\left(t+\lambda\right)^{2}-2\lambda\left(t+\lambda\right)+\lambda^{2}\right]\left(t+\lambda\right)^{-1}d\mu\left(\lambda\right) \\
&=\int_{0}^{\infty}w\left(\lambda\right)\left[\left(t+\lambda\right)-2\lambda+\lambda^{2}\left(t+\lambda\right)^{-1}\right]d\mu\left(\lambda\right) \\
&=\int_{0}^{\infty}w\left(\lambda\right)\left[t-\lambda+\lambda^{2}\left(t+\lambda\right)^{-1}\right]d\mu\left(\lambda\right).
\end{aligned} \tag{1.13}$$

Using the continuous functional calculus for selfadjoint operators in Hilbert spaces we can introduce the following *convex integral transform* of the positive operator T

$$\mathcal{C}\left(w,\mu\right)\left(T\right):=\int_{0}^{\infty}w\left(\lambda\right)T^{2}\left(\lambda+T\right)^{-1}d\mu\left(\lambda\right),$$

provided the integral exist.

In this paper, we show among others that

$$\begin{aligned}
&\frac{\mathcal{C}\left(w,\mu\right)\left(A\right)+\mathcal{C}\left(w,\mu\right)\left(B\right)}{2}-\mathcal{C}\left(w,\mu\right)\left(\frac{A+B}{2}\right) \\
&=\frac{1}{2}\int_{0}^{\infty}\left(\left(\lambda+A\right)^{-1}-\left(\lambda+B\right)^{-1}\right)\left(\left(\lambda+A\right)^{-1}+\left(\lambda+B\right)^{-1}\right)^{-1} \\
&\times\left(\left(\lambda+A\right)^{-1}-\left(\lambda+B\right)^{-1}\right)\lambda^{2}w\left(\lambda\right)d\mu\left(\lambda\right) \\
&\geq0.
\end{aligned}$$

for all $A, B > 0$, implying that $\mathcal{C}(w,\mu)(t)$ is operator convex on $(0,\infty)$. Several examples involving the exponential and logarithmic functions are also given.

2. Some Preliminary Facts

We start with the following elementary identity that give a simple proof for the fact that the function $f(t) = t^{-1}$ is operator convex on $(0,\infty)$, see for instance [6, p. 8]:

Lemma 1. *For any A, $B > 0$ we have*

$$\frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} = \frac{\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)}{2} \geq 0. \tag{2.1}$$

If more assumptions are made for the operators A and B, then one can obtain the following lower and upper bounds:

Corollary 1. *Assume that $0 < \alpha \leq A \leq \beta$ and $0 < \gamma \leq B \leq \delta$ for some constants α, β, γ, δ. Then*

$$\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2} \leq \frac{A^{-1}+B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2}. \tag{2.2}$$

Proof. We have $\beta^{-1} \leq A^{-1} \leq \alpha^{-1}$ and $\delta^{-1} \leq B^{-1} \leq \gamma^{-1}$, which gives

$$\beta^{-1}+\delta^{-1} \leq A^{-1}+B^{-1} \leq \alpha^{-1}+\gamma^{-1}$$

namely

$$\left(\alpha^{-1}+\gamma^{-1}\right)^{-1} \leq \left(A^{-1}+B^{-1}\right)^{-1} \leq \left(\beta^{-1}+\delta^{-1}\right)^{-1}.$$

By multiplying both sides by $\left(A^{-1}-B^{-1}\right)$ and dividing by 2, we get

$$\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2} \leq \frac{\left(A^{-1}-B^{-1}\right)\left(A^{-1}+B^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)}{2} \leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2}.$$

□

We know that for $T > 0$, we have the operator inequalities

$$0 < \left\|T^{-1}\right\|^{-1} \leq T \leq \|T\|. \tag{2.3}$$

Indeed, it is well known that, if $P \geq 0$, then

$$|\langle Px, y\rangle|^{2} \leq \langle Px, x\rangle \langle Py, y\rangle$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$0 \leq \langle x, x\rangle^{2} = \left\langle T^{-1}Tx, x\right\rangle^{2} = \left\langle Tx, T^{-1}x\right\rangle^{2} \leq \langle Tx, x\rangle \left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$1 \leq \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \leq \langle Tx, x\rangle \sup_{\|x\|=1} \left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} \leq T.$$

The second inequality in (2.3) is obvious.

Remark 1. *If A, $B > 0$ and $B - A > 0$, then by taking $\alpha = \left\|A^{-1}\right\|^{-1}$, $\beta = \|A\|$, $\gamma = \left\|B^{-1}\right\|^{-1}$ and $\delta = \|B\|$ in (2.2), we get*

$$\begin{aligned} \frac{1}{2}\left(\left\|A^{-1}\right\| + \left\|B^{-1}\right\|\right)^{-1}\left(A^{-1} - B^{-1}\right)^2 & \\ \leq \frac{A^{-1} + B^{-1}}{2} - \left(\frac{A+B}{2}\right)^{-1} & \\ \leq \frac{1}{2}\left(\|A\|^{-1} + \|B\|^{-1}\right)^{-1}\left(A^{-1} - B^{-1}\right)^2. & \end{aligned} \tag{2.4}$$

A continuous function $g : \mathcal{SA}_I(H) \to \mathcal{B}(H)$ is said to be *Gâteaux differentiable* in $A \in \mathcal{SA}_I(H)$, the class of selfadjoint operators on I, along the direction $B \in \mathcal{B}(H)$ if the following limit exists in the strong topology of $\mathcal{B}(H)$

$$\nabla g_A(B) := \lim_{s\to 0} \frac{g(A+sB) - g(A)}{s} \in \mathcal{B}(H). \tag{2.5}$$

If the limit (2.5) exists for all $B \in \mathcal{B}(H)$, then we say that g is *Gâteaux differentiable* in A and we can write $g \in \mathcal{G}(A)$. If this is true for any A in an open set $\mathcal{S}$ from $\mathcal{SA}_I(H)$ we write that $g \in \mathcal{G}(\mathcal{S})$.

If g is a continuous function on I, by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators A, $B \in \mathcal{SA}_I(H)$ we consider the segment of selfadjoint operators

$$[A, B] := \{(1-t)A + tB \mid t \in [0,1]\}.$$

We observe that A, $B \in [A, B]$ and $[A, B] \subset \mathcal{SA}_I(H)$.

We have the following gradient inequalities, see for instance :

Lemma 2. *Let f be an operator convex function on I and A, $B \in \mathcal{SA}_I(H)$, with $A \neq B$. If $f \in \mathcal{G}([A, B])$, then*

$$\nabla_B f(B-A) \geq f(B) - f(A) \geq \nabla_A f(B-A). \tag{2.6}$$

Let T, $S > 0$. The function $f(t) = t^{-1}$ is operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t\to 0}\left[\frac{f(T+tS) - f(T)}{t}\right] = -T^{-1}ST^{-1} \tag{2.7}$$

for T, $S > 0$.

Using (2.7) for the operator convex function $f(t) = t^{-1}$, we get

$$-D^{-1}(D-C)D^{-1} \geq D^{-1} - C^{-1} \geq -C^{-1}(D-C)C^{-1}$$

that is equivalent to

$$D^{-1}(D-C)D^{-1} \leq C^{-1} - D^{-1} \leq C^{-1}(D-C)C^{-1} \tag{2.8}$$

for all C, $D>0$.

If

$$m\leq D-C\leq M$$

for some constants m,M, then

$$mD^{-2}\leq D^{-1}\left(D-C\right)D^{-1}$$

and

$$C^{-1}\left(D-C\right)C^{-1}\leq MC^{-2}$$

and by (2.8) we derive

$$mD^{-2}\leq C^{-1}-D^{-1}\leq MC^{-2}. \tag{2.9}$$

Moreover, if $C\geq\alpha>0$ and $D\leq\delta$, then we get

$$C^{-2}\leq\alpha^{-2}\text{ and }D^{-2}\geq\delta^{-2},$$

which implies that

$$\frac{m}{\delta^{2}}\leq C^{-1}-D^{-1}\leq\frac{M}{\alpha^{2}} \tag{2.10}$$

Corollary 2. *Assume that* $0<\alpha\leq A\leq\beta$, $0<\gamma\leq B\leq\delta$ *and* $0<m\leq B-A\leq M$ *for some constants* α, β, γ, δ, m, M. *Then*

$$\begin{aligned}0&<\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{m^{2}}{\delta^{4}}\\&\leq\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2}\leq\frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1}\\&\leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2}\leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{M^{2}}{\alpha^{4}}.\end{aligned} \tag{2.11}$$

Proof. From (2.10) we have

$$0<\frac{m}{\delta^{2}}\leq A^{-1}-B^{-1}\leq\frac{M}{\alpha^{2}},$$

which implies that

$$0<\frac{m^{2}}{\delta^{4}}\leq\left(A^{-1}-B^{-1}\right)^{2}\leq\frac{M^{2}}{\alpha^{4}}$$

and by (2.2) we get (2.11). □

Remark 2. *If the positive operators* A, B *are separated, namely* $0<\alpha\leq A\leq\beta<\gamma\leq B\leq\delta$ *for some constants* α, β, γ, δ, *then obviously* $0<\gamma-\beta\leq B-A\leq\delta-\alpha$ *and by (2.11) for* $m=\gamma-\beta$ *and* $M=\delta-\alpha$, *we get*

$$\begin{aligned}0&<\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\frac{\left(\gamma-\beta\right)^{2}}{\delta^{4}}\leq\frac{1}{2}\left(\alpha^{-1}+\gamma^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2}\\&\leq\frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1}\\&\leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2}\leq\frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\frac{\left(\delta-\alpha\right)^{2}}{\alpha^{4}}.\end{aligned} \tag{2.12}$$

If $0<\|A\|\left\|B^{-1}\right\|<1$, *then*

$$0<\left\|A^{-1}\right\|^{-1}\leq A\leq\|A\|<\left\|B^{-1}\right\|^{-1}\leq B\leq\|B\|$$

and by (2.12) we get

$$
\begin{aligned}
0 &< \frac{1}{2}\left(\left\|A^{-1}\right\|+\left\|B^{-1}\right\|\right)^{-1} \frac{\left(\left\|B^{-1}\right\|^{-1}-\|A\|\right)^{2}}{\|B\|^{4}} \\
&\leq \frac{1}{2}\left(\left\|A^{-1}\right\|+\left\|B^{-1}\right\|\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2} \\
&\leq \frac{A^{-1}+B^{-1}}{2}-\left(\frac{A+B}{2}\right)^{-1} \\
&\leq \frac{1}{2}\left(\|A\|^{-1}+\|B\|^{-1}\right)^{-1}\left(A^{-1}-B^{-1}\right)^{2} \\
&\leq \frac{1}{2}\left(\beta^{-1}+\delta^{-1}\right)^{-1}\left(\|B\|-\left\|A^{-1}\right\|^{-1}\right)^{2}\left\|A^{-1}\right\|^{4}.
\end{aligned} \tag{2.13}
$$

We can present now our main results.

3. Main Results

We have the following identity for the Jensen's difference:

Theorem 3. *For all A, $B>0$ we have*

$$
\begin{aligned}
&\frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2}-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\
&=\frac{1}{2}\int_{0}^{\infty}\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\left((\lambda+A)^{-1}+(\lambda+B)^{-1}\right)^{-1} \\
&\times\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\lambda^{2}w(\lambda)\,d\mu(\lambda) \\
&\geq 0.
\end{aligned} \tag{3.1}
$$

The function $\mathcal{D}(w,\mu)$ is an operator convex function on $(0,\infty)$

Proof. We have for all A, $B>0$

$$
\begin{aligned}
&\frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2}-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right) \\
&=\frac{1}{2}\int_{0}^{\infty} w(\lambda)\left[A-\lambda+\lambda^{2}(A+\lambda)^{-1}\right] d\mu(\lambda) \\
&+\frac{1}{2}\int_{0}^{\infty} w(\lambda)\left[B-\lambda+\lambda^{2}(B+\lambda)^{-1}\right] d\mu(\lambda) \\
&-\int_{0}^{\infty} w(\lambda)\left[\frac{A+B}{2}-\lambda+\lambda^{2}\left(\frac{A+B}{2}+\lambda\right)^{-1}\right] d\mu(\lambda) \\
&=\int_{0}^{\infty} w(\lambda)\left\{\frac{1}{2}\left[A-\lambda+\lambda^{2}(A+\lambda)^{-1}\right]+\frac{1}{2}\left[B-\lambda+\lambda^{2}(B+\lambda)^{-1}\right]\right. \\
&\left.-\left[\frac{A+B}{2}-\lambda+\lambda^{2}\left(\frac{A+B}{2}+\lambda\right)^{-1}\right]\right\} d\mu(\lambda) \\
&=\int_{0}^{\infty} \lambda^{2} w(\lambda)\left[\frac{(\lambda+A)^{-1}+(\lambda+B)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1}\right] d\mu(\lambda).
\end{aligned} \tag{3.2}
$$

Since, by (2.1)

$$\begin{aligned}
&\frac{(\lambda+A)^{-1}+(\lambda+B)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1}\\
&=\frac{1}{2}\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\left((\lambda+A)^{-1}+(\lambda+B)^{-1}\right)^{-1}\\
&\times\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\\
&\geq 0
\end{aligned}$$

for all $\lambda \geq 0$, then by (3.2) we obtain the representation (3.1).

Since $\mathcal{C}(w,\mu)$ is continuous in $\mathcal{B}(H)$ and satisfies Jensen's inequality (3.1), it follows that $\mathcal{C}(w,\mu)$ is an operator convex function on $(0,\infty)$. □

Corollary 3. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). Then for all A, $B>0$, we have*

$$\begin{aligned}
&\frac{1}{2}\left[Af(A)+Bf(B)\right]-\frac{A+B}{2}f\left(\frac{A+B}{2}\right)-\frac{1}{4}b(B-A)^2\\
&=\frac{1}{2}\int_0^\infty\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\left((\lambda+A)^{-1}+(\lambda+B)^{-1}\right)^{-1}\\
&\times\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\lambda^3 d\mu(\lambda)\geq 0
\end{aligned}\tag{3.3}$$

namely, the function $tf(t)-bt^2$ is operator convex on $(0,\infty)$.

Proof. If we multiply (1.9) by $t>0$ then we get

$$tf(t)=at+bt^2+t^2\int_0^\infty\frac{\lambda}{t+\lambda}d\mu(\lambda)=at+bt^2+\mathcal{C}(\ell,\mu)$$

for a real, $b\geq 0$ and μ a positive measure on $(0,\infty)$. This gives

$$\mathcal{C}(\ell,\mu)=tf(t)-at-bt^2,\ t>0.$$

If A, $B>0$, then

$$\begin{aligned}
&\frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2}-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\\
&=\frac{1}{2}\left(Af(A)-aA-bA^2\right)+\frac{1}{2}\left(Bf(B)-aB-bB^2\right)\\
&-\frac{A+B}{2}f\left(\frac{A+B}{2}\right)+a\frac{A+B}{2}+b\left(\frac{A+B}{2}\right)^2\\
&=\frac{1}{2}\left[Af(A)+Bf(B)\right]-\frac{A+B}{2}f\left(\frac{A+B}{2}\right)\\
&-b\left[\frac{B^2+A^2}{2}-\left(\frac{A+B}{2}\right)^2\right]\\
&=\frac{1}{2}\left[Af(A)+Bf(B)\right]-\frac{A+B}{2}f\left(\frac{A+B}{2}\right)-\frac{1}{4}b(B-A)^2
\end{aligned}$$

and the inequality (3.3) is obtained. □

Corollary 4. *Assume that function* $f:(0,\infty)\to\mathbb{R}$ *is operator convex in* $(0,\infty)$ *and has the representation (1.11). Then for all* A, $B>0$,*we have*

$$\begin{aligned}\frac{1}{2}\left[f(A)+f(B)\right]-f\left(\frac{A+B}{2}\right)-\frac{1}{4}c(B-A)^{2}\\=\frac{1}{2}\int_{0}^{\infty}\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\left((\lambda+A)^{-1}+(\lambda+B)^{-1}\right)^{-1}\\\times\left((\lambda+A)^{-1}-(\lambda+B)^{-1}\right)\lambda^{3}d\mu(\lambda)\geq 0\end{aligned}\tag{3.4}$$

namely, the function $f(t)-ct^{2}$ *is operator convex on* $(0,\infty)$.

The proof follows by Theorem 3 and the representation (1.11).

When more assumptions are imposed on the operators A and B, then the following improvement and refinement of Jensen's inequality hold:

Theorem 4. *Assume that* $0<\alpha\leq A\leq\beta$, $0<\gamma\leq B\leq\delta$ *and* $0<m\leq B-A\leq M$ *for some constants* α, β, γ, δ, m, M. *Then*

$$\begin{aligned}0&<\frac{1}{2}m^{2}\frac{\gamma\alpha}{\alpha+\gamma}\int_{0}^{\infty}\frac{\lambda^{2}w(\lambda)}{(\delta+\lambda)^{4}}d\mu(\lambda)\\&\leq\frac{\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B)}{2}-\mathcal{C}(w,\mu)\left(\frac{A+B}{2}\right)\\&\leq\frac{1}{2}\frac{M^{2}}{\beta+\delta}\int_{0}^{\infty}\frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^{4}}\lambda^{2}w(\lambda)\,d\mu(\lambda).\end{aligned}\tag{3.5}$$

Proof. We have $0<\alpha+\lambda\leq A+\lambda\leq\beta+\lambda$, $0<\gamma+\lambda\leq B+\lambda\leq\delta+\lambda$ and $0<m\leq B+\lambda-A-\lambda=B-A\leq M$ for all $\lambda\geq 0$. By (2.11) we get

$$\begin{aligned}0&<\frac{1}{2}\left(\frac{1}{\alpha+\lambda}+\frac{1}{\gamma+\lambda}\right)^{-1}\frac{m^{2}}{(\delta+\lambda)^{4}}\\&\leq\frac{(A+\lambda)^{-1}+(B+\lambda)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1}\\&\leq\frac{1}{2}\left(\frac{1}{\beta+\lambda}+\frac{1}{\delta+\lambda}\right)^{-1}\frac{M^{2}}{(\alpha+\lambda)^{4}}.\end{aligned}\tag{3.6}$$

We have that

$$\left(\frac{1}{\beta+\lambda}+\frac{1}{\delta+\lambda}\right)^{-1}=\frac{(\beta+\lambda)(\delta+\lambda)}{\beta+\delta+2\lambda}\leq\frac{(\beta+\lambda)(\delta+\lambda)}{\beta+\delta}\tag{3.7}$$

and

$$\left(\frac{1}{\alpha+\lambda}+\frac{1}{\gamma+\lambda}\right)^{-1}=\frac{(\gamma+\lambda)(\alpha+\lambda)}{\alpha+\gamma+2\lambda}=g(\lambda).$$

We have

$$g'(\lambda)=\frac{(\alpha+\gamma+2\lambda)^{2}-2(\gamma+\lambda)(\alpha+\lambda)}{(\alpha+\gamma+2\lambda)^{2}}=\frac{(\alpha+\lambda)^{2}+(\gamma+\lambda)^{2}}{(\alpha+\gamma+2\lambda)^{2}}>0,$$

which shows that g is increasing on $[0,\infty)$.

Therefore

$$g(\lambda)\geq g(0)=\frac{\gamma\alpha}{\alpha+\gamma}\text{ for all }\lambda\geq 0.\tag{3.8}$$

By (3.4)-(3.8) we derive that

$$\begin{aligned}0&<\frac{1}{2}\frac{\gamma\alpha}{\alpha+\gamma}\frac{m^2}{(\delta+\lambda)^4}\\&\le\frac{(A+\lambda)^{-1}+(B+\lambda)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1}\\&\le\frac{1}{2}\frac{(\beta+\lambda)(\delta+\lambda)}{\beta+\delta}\frac{M^2}{(\alpha+\lambda)^4},\end{aligned}$$

which implies that

$$\begin{aligned}0&<\frac{1}{2}m^2\frac{\gamma\alpha}{\alpha+\gamma}\int_0^\infty\frac{\lambda^2w(\lambda)}{(\delta+\lambda)^4}d\mu(\lambda)\\&\le\int_0^\infty\left[\frac{(A+\lambda)^{-1}+(B+\lambda)^{-1}}{2}-\left(\lambda+\frac{A+B}{2}\right)^{-1}\right]\lambda^2w(\lambda)\,d\mu(\lambda)\\&\le\frac{1}{2}\frac{M^2}{\beta+\delta}\int_0^\infty\frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^4}\lambda^2w(\lambda)\,d\mu(\lambda).\end{aligned}\tag{3.9}$$

By making use of the identity (3.1), we derive the desired result (3.5). □

Corollary 5. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.9). Then for all A, $B>0$ satisfying the conditions in Theorem 4,*

$$\begin{aligned}0&<\frac{1}{2}m^2\frac{\gamma\alpha}{\alpha+\gamma}\int_0^\infty\frac{\lambda^3}{(\delta+\lambda)^4}d\mu(\lambda)\\&\le\frac{1}{2}[Af(A)+Bf(B)]-\frac{A+B}{2}f\left(\frac{A+B}{2}\right)-\frac{1}{4}b(B-A)^2\\&\le\frac{1}{2}\frac{M^2}{\beta+\delta}\int_0^\infty\frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^4}\lambda^3d\mu(\lambda).\end{aligned}\tag{3.10}$$

Also,

Corollary 6. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ and has the representation (1.11). Then for all A, $B>0$ satisfying the conditions in Theorem 4,*

$$\begin{aligned}0&<\frac{1}{2}m^2\frac{\gamma\alpha}{\alpha+\gamma}\int_0^\infty\frac{\lambda^3}{(\delta+\lambda)^4}d\mu(\lambda)\\&\le\frac{1}{2}[f(A)+f(B)]-f\left(\frac{A+B}{2}\right)-\frac{1}{4}c(B-A)^2\\&\le\frac{1}{2}\frac{M^2}{\beta+\delta}\int_0^\infty\frac{(\beta+\lambda)(\delta+\lambda)}{(\alpha+\lambda)^4}\lambda^3d\mu(\lambda).\end{aligned}\tag{3.11}$$

Remark 3. *If we consider the kernel* $w(\lambda) = \frac{\sin(r\pi)}{\pi}\lambda^{r-1}$, $r \in (0,1]$ *in (3.5) and use the representation (1.1) then we get the power inequality*

$$0 < \frac{1}{2}m^2 \frac{\gamma\alpha}{\alpha+\gamma} \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r+1}}{(\delta+\lambda)^4} d\lambda \le \frac{A^{r+1}+B^{r+1}}{2} - \left(\frac{A+B}{2}\right)^{r+1}. \tag{3.12}$$

For $r = 1/2$ *we have*

$$\int_0^\infty \frac{\lambda^{3/2}}{(\delta+\lambda)^4} d\lambda = \frac{\pi}{16\delta^{3/2}}$$

and by (3.12) we derive

$$0 < \frac{1}{32}m^2 \frac{\gamma\alpha}{(\alpha+\gamma)\delta^{3/2}} \le \frac{A^{3/2}+B^{3/2}}{2} - \left(\frac{A+B}{2}\right)^{3/2}. \tag{3.13}$$

4. More Examples

We define the *upper incomplete Gamma function* as [11]

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z = 0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [12]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a}e^{-t}}{z+t} dt \tag{4.1}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (4.1) we obtain

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda} d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{4.2}$$

for $a < 1$ and $t > 0$.

For $a = 0$ in (4.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda} d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^t E_1(t) \tag{4.3}$$

for $t > 0$, where the *exponential integral* E_1 is defined by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

Let $a = 1-n$, with n a natural number with $n \ge 0$, then by (3.2) we have

$$\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda} d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) = (n-1)!t^{n-1}e^t\Gamma(1-n,t). \tag{4.4}$$

If we define the generalized exponential integral [13] by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [13, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}E_1(z) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-z)^k,$$

we then obtain

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= (n-1)!e^t E_n(t) \\ &= (n-1)!e^t \\ &\times \left[\frac{(-t)^{n-1}}{(n-1)!}E_1(t) + \frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-t)^k\right] \\ &= \sum_{k=0}^{n-2}(-1)^k(n-k-2)!t^k + (-1)^{n-1}t^{n-1}e^t E_1(t)\end{aligned} \tag{4.5}$$

for $n \geq 2$ and $t > 0$.

For $n = 2$, we derive by (4.5) that

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(t) = \int_0^\infty \lambda e^{-\lambda}(t+\lambda)^{-1}d\lambda = 1 - t\exp(t)E_1(t) \tag{4.6}$$

for $t > 0$.

We have, by the definition of the convex integral transform,

$$\mathcal{C}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \Gamma(1-a)t^{2-a}\exp(t)\Gamma(a,t)$$

for $a < 1$ and $t > 0$.

By utilising Theorem 3 we can state:

Proposition 1. *For $a < 1$ the function $t^{2-a}\exp(t)\Gamma(a,t)$ is operator convex on $(0,\infty)$. In particular $t^2\exp(t)E_1(t)$ and $t^2(1-t)\exp(t)E_1(t)$ are operator convex on $(0,\infty)$.*

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\begin{aligned}\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) &:= \int_0^\infty \frac{1}{(\lambda+t)(\lambda^2+a^2)}d\lambda \\ &= \frac{1}{t^2+a^2}\left[\frac{\pi t}{2a} - \ln\left(\frac{t}{a}\right)\right]\end{aligned}$$

for $t > 0$ and $a > 0$. Therefore

$$\mathcal{C}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) = \frac{t^2}{t^2+a^2}\left[\frac{\pi t}{2a} - \ln\left(\frac{t}{a}\right)\right]$$

and we can also state that:

Proposition 2. *For* $a > 0$ *the function* $\frac{t^2}{t^2+a^2}\left[\frac{\pi t}{2a} - \ln\left(\frac{t}{a}\right)\right]$ *is operator convex on* $(0,\infty)$. *In particular,* $\frac{t^2}{t^2+1}\left(\frac{\pi t}{2} - \ln t\right)$ *is operator convex on* $(0,\infty)$.

The interested reader can obtain other similar results by employing the examples of operator monotone/convex functions from [2]-[5], [9]-[10] and the references therein.

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