SUPERADDITIVITY OF CONVEX INTEGRAL TRANSFORM FOR POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *convex integral transform*

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among other that, for all A, $B > 0$ with $BA + AB \geq 0$,

$$\mathcal{C}(w,\mu)(A+B) \geq \mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B).$$

In particular, we have for $r \in (0,1]$, the power inequality

$$(A+B)^{r+1} \geq A^{r+1} + B^{r+1}$$

and the logarithmic inequality

$$(A+B)\ln(A+B) \geq A \ln A + B \ln B.$$

Some examples for operator monotone and operator convex functions and integral transforms $\mathcal{C}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

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which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t),\ t>0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, we have the representation

$$\ln t=(t-1)\mathcal{D}(w_{\ln})(t),\ t>0. \tag{1.6}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{1.8}$$

for $T>0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

Theorem 1. *A function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t)=a+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.9}$$

where $a\in R$, $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a=f(0)$ in (1.9).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left(\left(1-\lambda \right) A+\lambda B\right) \leq \left(\geq \right) \left(1-\lambda \right) f\left(A\right) +\lambda f\left(B\right) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty)\to\mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f\left(t\right) =a+bt+ct^{2}+\int_{0}^{\infty }\frac{t^{2}\lambda }{t+\lambda }d\mu \left(\lambda \right) , \tag{1.11}$$

where a, $b\in\mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f(0)$ and $b=f'_{+}(0)$, the right derivative, in (1.11).

Assume that A, $B\geq 0$. In the recent paper [9], Moslehian and Najafi showed that $AB+BA$ is positive if and only if the following *operator subadditivity property* holds

$$f\left(A+B\right) \leq f\left(A\right) +f\left(B\right) \tag{1.12}$$

for all nonnegative operator monotone functions f on $[0,\infty)$. For some interesting consequences of this result see [9].

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call the *convex integral transform,*

$$\mathcal{C}(w,\mu)\left(t\right) :=t^{2}\mathcal{D}(w,\mu)\left(t\right) ,\ t>0. \tag{1.13}$$

For $t>0$ we have

$$\begin{aligned}
\mathcal{C}(w,\mu)\left(t\right) &:=\int_{0}^{\infty }w\left(\lambda \right) t^{2}\left(t+\lambda \right) ^{-1}d\mu \left(\lambda \right) \\
&=\int_{0}^{\infty }w\left(\lambda \right) \left(t+\lambda -\lambda \right) ^{2}\left(t+\lambda \right) ^{-1}d\mu \left(\lambda \right) \\
&=\int_{0}^{\infty }w\left(\lambda \right) \left[\left(t+\lambda \right) ^{2}-2\lambda \left(t+\lambda \right) +\lambda ^{2}\right] \left(t+\lambda \right) ^{-1}d\mu \left(\lambda \right) \\
&=\int_{0}^{\infty }w\left(\lambda \right) \left[\left(t+\lambda \right) -2\lambda +\lambda ^{2}\left(t+\lambda \right) ^{-1}\right] d\mu \left(\lambda \right) \\
&=\int_{0}^{\infty }w\left(\lambda \right) \left[t-\lambda +\lambda ^{2}\left(t+\lambda \right) ^{-1}\right] d\mu \left(\lambda \right) .
\end{aligned} \tag{1.14}$$

If $\int_{0}^{\infty }w\left(\lambda \right) d\mu \left(\lambda \right) <\infty$ and $\int_{0}^{\infty }w\left(\lambda \right) \lambda d\mu \left(\lambda \right) <\infty$, then we conclude that

$$\mathcal{C}(w,\mu)\left(t\right) =-\int_{0}^{\infty }w\left(\lambda \right) \lambda d\mu \left(\lambda \right) +t\int_{0}^{\infty }w\left(\lambda \right) d\mu \left(\lambda \right) +\mathcal{D}(\ell ^{2}w,\mu)\left(t\right) , \tag{1.15}$$

where $\ell(t)=t$, $t>0$.

Consider, for instance, the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. After some calculations, we obtain

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda} d\lambda = E_1(at)\exp(at), \ t \geq 0$$

where the *exponential integral* is given by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

We also have

$$\int_0^\infty w(\lambda)\lambda d\lambda = \int_0^\infty \exp(-a\lambda)\lambda d\lambda = \frac{1}{a^2}$$

and

$$\int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a}.$$

Therefore

$$\mathcal{C}(e_{-a})(t) := t^2\mathcal{D}(e_{-a},\mu)(t) = t^2 E_1(at)\exp(at), \ t > 0. \tag{1.16}$$

Since

$$\mathcal{D}(\ell^2 e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda^2\exp(-a\lambda)}{t+\lambda} d\lambda$$

then by (1.15) we get

$$t^2 E_1(at)\exp(at) = -\frac{1}{a^2} + \frac{t}{a} + \mathcal{D}(\ell^2 w,\mu)(t),$$

which gives

$$\mathcal{D}(\ell^2 w,\mu)(t) = t^2 E_1(at)\exp(at) - \frac{t}{a} + \frac{1}{a^2}, \ t > 0, a > 0.$$

If we take $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then $\int_0^\infty w_r(\lambda) d\lambda = \infty$ and the equality (1.15) does not hold in this case.

In this paper, we show among other that, for all A, $B > 0$ with $BA + AB \geq 0$,

$$\mathcal{C}(w,\mu)(A+B) \geq \mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B).$$

In particular, we have for $r \in (0,1]$, the power inequality

$$(A+B)^{r+1} \geq A^{r+1} + B^{r+1}$$

and the logarithmic inequality

$$(A+B)\ln(A+B) \geq A\ln A + B\ln B,$$

Some examples for power function and integral transforms $\mathcal{M}(\cdot,\cdot)$ related to the exponential and logarithmic functions are also provided.

2. Main Results

We start with the following result regarding the operator quasi-superadditivity property of $\mathcal{C}(w,\mu)$:

Theorem 3. *For all A, $B>0$ we have*

$$\begin{aligned} &\mathcal{C}(w,\mu)(A+B)-\mathcal{C}(w,\mu)(A)-\mathcal{C}(w,\mu)(B) \\ &\geq \int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda). \end{aligned} \tag{2.1}$$

Proof. Assume that A, $B>0$. Define

$$K_\lambda := (A+\lambda)^{-1}+(B+\lambda)^{-1}-(A+B+\lambda)^{-1},$$

where $\lambda \geq 0$.

Therefore

$$\begin{aligned} &(A+B+\lambda)K_\lambda(A+B+\lambda) \\ &=(A+B+\lambda)(A+\lambda)^{-1}(A+B+\lambda) \\ &+(A+B+\lambda)(B+\lambda)^{-1}(A+B+\lambda)-A-B-\lambda \\ &=\left(1+B(A+\lambda)^{-1}\right)(A+\lambda+B) \\ &+\left(A(B+\lambda)^{-1}+1\right)(A+B+\lambda)-A-B-\lambda \\ &=A+\lambda+B+B+B(A+\lambda)^{-1}B \\ &+A(B+\lambda)^{-1}A+A+A+B+\lambda-A-B-\lambda \\ &=B(A+\lambda)^{-1}B+A(B+\lambda)^{-1}A+2(A+B)+\lambda =: L_\lambda. \end{aligned} \tag{2.2}$$

If A, B, $\lambda>0$, then $L_\lambda \geq 0$, and by multiplying both sides of (2.2) with $(A+B+\lambda)^{-1}$ we get

$$K_\lambda=(A+B+\lambda)^{-1}L_\lambda(A+B+\lambda)^{-1}.$$

Further, define for $\lambda>0$

$$W_\lambda := 1-\lambda K_\lambda.$$

Then

$$\begin{aligned} &(A+B+\lambda)W_\lambda(A+B+\lambda) \\ &=(A+B+\lambda)(1-\lambda K_\lambda)(A+B+\lambda) \\ &=(A+B+\lambda)^2-\lambda(A+B+\lambda)K_\lambda(A+B+\lambda) \\ &=(A+B+\lambda)(A+B+\lambda) \\ &-\lambda\left[B(A+\lambda)^{-1}B+A(B+\lambda)^{-1}A+2(A+B)+\lambda\right] \end{aligned}$$

$$
\begin{aligned}
&= A^2 + BA + \lambda A + AB + B^2 + \lambda B + \lambda A + \lambda B + \lambda^2 \\
&- \lambda B (A+\lambda)^{-1} B - \lambda A (B+\lambda)^{-1} A - 2\lambda (A+B) - \lambda^2 \\
&= A^2 + B^2 + BA + AB - \lambda B (A+\lambda)^{-1} B - \lambda A (B+\lambda)^{-1} A \\
&= A (B+\lambda)^{-1} (B+\lambda) A - \lambda A (B+\lambda)^{-1} A \\
&+ B (A+\lambda)^{-1} (A+\lambda) B - \lambda B (A+\lambda)^{-1} B \\
&+ BA + AB = A (B+\lambda)^{-1} BA + B (A+\lambda)^{-1} AB + BA + AB,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
W_\lambda &= (A+B+\lambda)^{-1} \left[A (B+\lambda)^{-1} BA + B (A+\lambda)^{-1} AB + BA + AB \right] \\
&\times (A+B+\lambda)^{-1}
\end{aligned}
$$

We also have the representation

$$
\begin{aligned}
&\mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A+B) \\
&= \int_0^\infty w(\lambda) \left[A - \lambda + \lambda^2 (A+\lambda)^{-1} \right] d\mu(\lambda) \\
&+ \int_0^\infty w(\lambda) \left[B - \lambda + \lambda^2 (B+\lambda)^{-1} \right] d\mu(\lambda) \\
&- \int_0^\infty w(\lambda) \left[A + B - \lambda + \lambda^2 (A+B+\lambda)^{-1} \right] d\mu(\lambda) \\
&= \int_0^\infty w(\lambda) \left[\lambda^2 (A+\lambda)^{-1} + \lambda^2 (B+\lambda)^{-1} - \lambda - \lambda^2 (A+B+\lambda)^{-1} \right] d\mu(\lambda) \\
&= \int_0^\infty \lambda^2 w(\lambda) \left[(A+\lambda)^{-1} + (B+\lambda)^{-1} - \lambda^{-1} - (A+B+\lambda)^{-1} \right] d\mu(\lambda) \\
&= \int_0^\infty \lambda^2 w(\lambda) \left(K_\lambda - \lambda^{-1} \right) d\mu(\lambda).
\end{aligned} \tag{2.3}
$$

Put

$$
\begin{aligned}
Y_\lambda &:= K_\lambda - \lambda^{-1} = \lambda^{-1} (\lambda K_\lambda - 1) = -\lambda^{-1} (1 - \lambda K_\lambda) = -\lambda^{-1} W_\lambda \\
&= -\lambda^{-1} (A+B+\lambda)^{-1} \left[A (B+\lambda)^{-1} BA + B (A+\lambda)^{-1} AB + BA + AB \right] \\
&\times (A+B+\lambda)^{-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A+B) \\
&= \int_0^\infty \lambda^2 w(\lambda) Y_\lambda d\mu(\lambda) \\
&= -\int_0^\infty \lambda w(\lambda) (A+B+\lambda)^{-1} \\
&\times \left[A (B+\lambda)^{-1} BA + B (A+\lambda)^{-1} AB + BA + AB \right] (A+B+\lambda)^{-1} d\mu(\lambda),
\end{aligned}
$$

which gives the following identity of interest:

$$\begin{aligned}(2.4)\quad & \mathcal{C}(w,\mu)(A+B)-\mathcal{C}(w,\mu)(A)-\mathcal{C}(w,\mu)(B)\\ &=\int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-1}\\ &\times\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB+BA+AB\right](A+B+\lambda)^{-1}d\mu(\lambda)\\ &=\int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-1}\\ &\times\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\right](A+B+\lambda)^{-1}d\mu(\lambda)\\ &+\int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda).\end{aligned}$$

Since for all $A,\ B>0$,

$$(B+\lambda)^{-1}B>0,\ \ (A+\lambda)^{-1}A>0$$

for $\lambda\geq 0$, then

$$A(B+\lambda)^{-1}BA,\ B(A+\lambda)^{-1}AB\geq 0$$

that gives

$$A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\geq 0,$$

which implies that

$$(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\right](A+B+\lambda)^{-1}\geq 0$$

for$\lambda\geq 0$.

If we multiply this inequality by $\lambda w(\lambda)\geq 0$ and integrate on $[0,\infty)$ over μ, we get

$$\begin{aligned}&\int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-1}\\ &\times\left[A(B+\lambda)^{-1}BA+B(A+\lambda)^{-1}AB\right](A+B+\lambda)^{-1}d\mu(\lambda)\\ &\geq 0\end{aligned}$$

for all $A,\ B>0$.

Then

$$\begin{aligned}&\mathcal{C}(w,\mu)(A+B)-\mathcal{C}(w,\mu)(A)-\mathcal{C}(w,\mu)(B)\\ &\geq\int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda),\end{aligned}$$

for all $A,\ B>0$ and the inequality (2.1) is proved. □

The following superadditivity property is valid.

Corollary 1. *For all* $A,\ B>0$ *with* $BA+AB\geq 0$, *we have*

$$\mathcal{C}(w,\mu)(A+B)\geq\mathcal{C}(w,\mu)(A)+\mathcal{C}(w,\mu)(B). \tag{2.5}$$

Proof. If $BA+AB\geq 0$, then for all $\lambda\geq 0$ we have

$$(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}.$$

If we multiply by $\lambda w(\lambda) \geq 0$ and integrate, we get

$$\int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1} d\mu(\lambda) \geq 0$$

and by (2.1) we obtain (2.5). □

Corollary 2. *If the kernel* w *satisfies the condition* $E_2(w,\mu) := \int_0^\infty \lambda w(\lambda) d\mu(\lambda) < \infty$, *then*

$$E_2(w,\mu) + \mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B) \geq \mathcal{C}(w,\mu)(A+B) \tag{2.6}$$

for all $A,\ B > 0$.

Proof. Assume that $A,\ B > 0$. If $\int_0^\infty \lambda w(\lambda) d\mu(\lambda) < \infty$, then

$$\int_0^\infty \lambda^2 w(\lambda)\left(K_\lambda - \lambda^{-1}\right) d\mu(\lambda) = \int_0^\infty \lambda^2 w(\lambda) K_\lambda d\mu(\lambda) - \int_0^\infty \lambda w(\lambda) d\mu(\lambda)$$

and by (2.3) we derive

$$\begin{aligned}&\mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A+B)\\&= \int_0^\infty \lambda^2 w(\lambda) K_\lambda d\mu(\lambda) - \int_0^\infty \lambda w(\lambda) d\mu(\lambda),\end{aligned}$$

which gives

$$\begin{aligned}&\int_0^\infty \lambda w(\lambda) d\mu(\lambda) + \mathcal{C}(w,\mu)(A) + \mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A+B)\\&= \int_0^\infty \lambda^2 w(\lambda) K_\lambda d\mu(\lambda) \geq 0\end{aligned}$$

since $K_\lambda = (A+B+\lambda)^{-1} L_\lambda (A+B+\lambda)^{-1} \geq 0$ for all $\lambda > 0$. □

Remark 1. *If we write inequality (2.6) for the kernel* $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ *and* $a > 0$, *then we get for all* $A,\ B > 0$,

$$\begin{aligned}&\frac{1}{a^2} + A^2 E_1(aA)\exp(aA) + B^2 E_1(aB)\exp(aB)\\&\geq (A+B)^2 E_1(a(A+B))\exp(a(A+B)).\end{aligned} \tag{2.7}$$

In particular, we have

$$1 + A^2 E_1(A)\exp(A) + B^2 E_1(B)\exp(B) \geq (A+B)^2 E_1(A+B)\exp(A+B) \tag{2.8}$$

for all $A,\ B > 0$.

The case of operator monotone functions is as follows:

Proposition 1. *Assume that the function* $f : (0,\infty) \to \mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.1). For all* $A,\ B > 0$ *we have*

$$\begin{aligned}&(A+B) f(A+B) - Af(A) - Bf(B)\\&\geq b(BA+AB)\\&+ \int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1} d\mu(\lambda).\end{aligned} \tag{2.9}$$

If $BA+AB$, *then*

$$(A+B) f(A+B) \geq Af(A) + Bf(B). \tag{2.10}$$

Proof. By (1.9) we get by multiplying with $t>0$ that

$$tf\left(t\right)=at+bt^{2}+\mathcal{C}(\ell,\mu)\left(t\right),$$

where $\ell\left(\lambda\right)=\lambda$, $a\in R$, $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

Since

$$\begin{aligned}&\mathcal{C}(w,\mu)\left(A+B\right)-\mathcal{C}(w,\mu)\left(A\right)-\mathcal{C}(w,\mu)\left(B\right)\\&=\left(A+B\right)f\left(A+B\right)-a\left(A+B\right)-b\left(A+B\right)^{2}\\&-Af\left(A\right)+aA+bA^{2}-Bf\left(B\right)+aB+bB^{2}\\&=\left(A+B\right)f\left(A+B\right)-Af\left(A\right)-Bf\left(B\right)-b\left(BA+AB\right),\end{aligned}$$

hence by (2.1) we get (2.9). □

Remark 2. *Let $r\in(0,1]$. If A, $B>0$ with $BA+AB\geq 0$, then by writing the inequality (2.10) for the power function $f\left(t\right)=t^{r}$, which is operator monotone, we have the power inequality*

$$\left(A+B\right)^{r+1}\geq A^{r+1}+B^{r+1}. \tag{2.11}$$

If we write the inequality (2.10) for the operator monotonic function $f\left(t\right)=\ln t$, then we get

$$\left(A+B\right)\ln\left(A+B\right)\geq A\ln A+B\ln B, \tag{2.12}$$

if A, $B>0$ with $BA+AB\geq 0$.

Proposition 2. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.11). For all A, $B>0$ we have*

$$\begin{aligned}&f\left(A+B\right)+f\left(0\right)-f\left(A\right)-f\left(B\right)\\&\geq c\left(BA+AB\right)\\&+\int_{0}^{\infty}\lambda w\left(\lambda\right)\left(A+B+\lambda\right)^{-1}\left(BA+AB\right)\left(A+B+\lambda\right)^{-1}d\mu\left(\lambda\right).\end{aligned} \tag{2.13}$$

If $BA+AB\geq 0$, then

$$f\left(A+B\right)+f\left(0\right)\geq f\left(A\right)+f\left(B\right). \tag{2.14}$$

Proof. By (1.11) we have

$$f\left(t\right)=f\left(0\right)+f_{+}^{\prime}\left(0\right)t+ct^{2}+\mathcal{C}(\ell,\mu)\left(t\right),$$

where $c\geq 0$ and μ is a positive measure on $(0,\infty)$.

We have

$$\begin{aligned}&\mathcal{C}(w,\mu)\left(A+B\right)-\mathcal{C}(w,\mu)\left(A\right)-\mathcal{C}(w,\mu)\left(B\right)\\&=f\left(A+B\right)-f\left(0\right)-f_{+}^{\prime}\left(0\right)\left(A+B\right)-c\left(A+B\right)^{2}\\&-f\left(A\right)+f\left(0\right)+f_{+}^{\prime}\left(0\right)A+cA^{2}-f\left(B\right)+f\left(0\right)+f_{+}^{\prime}\left(0\right)B+cB^{2}\\&=f\left(A+B\right)-f\left(A\right)-f\left(B\right)+f\left(0\right)-c\left(BA+AB\right)\end{aligned}$$

and by (2.1) we derive

$$\begin{aligned}&f\left(A+B\right)-f\left(A\right)-f\left(B\right)+f\left(0\right)-c\left(BA+AB\right)\\&\geq\int_{0}^{\infty}\lambda w\left(\lambda\right)\left(A+B+\lambda\right)^{-1}\left(BA+AB\right)\left(A+B+\lambda\right)^{-1}d\mu\left(\lambda\right),\end{aligned}$$

which is equivalent to (2.13). □

Remark 3. *Let $a>0$ and $f(t)=(t+a)^p$ with $p\in[-1,0)\cup[1,2]$. This function is operator convex and $f(0)=a^p$. Then for all A, $B>0$ with $BA+AB\geq 0$,*

$$(A+B+a)^p+a^p\geq (A+a)^p+(B+a)^p. \tag{2.15}$$

In particular,

$$(A+B+a)^{-1}+a^{-1}\geq (A+a)^{-1}+(B+a)^{-1}. \tag{2.16}$$

The function $f(t)=-\ln(t+a)$, $a>0$ is operator convex and by (2.14) we get

$$\ln(A+B+a)+\ln a\leq \ln(A+a)+\ln(B+a), \tag{2.17}$$

if A, $B>0$ with $BA+AB\geq 0$. In particular, for $a=1$, we get

$$\ln(A+B+1)\leq \ln(A+1)+\ln(B+1). \tag{2.18}$$

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t),\ t>0. \tag{2.19}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{M}(w,\mu)(t):=t\mathcal{D}(w,\mu)(t)&=\int_0^\infty w(\lambda)\,t\,(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)(t+\lambda-\lambda)(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[1-\lambda(t+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned} \tag{2.20}$$

If we take the derivative in (2.20), then we get

$$\mathcal{M}'(w,\mu)(t)=\int_0^\infty \lambda w(\lambda)(t+\lambda)^{-2}d\mu(\lambda).$$

Corollary 3. *For all A, $B>0$ with $BA+AB\geq k$ for some real constant k, we have*

$$\mathcal{C}(w,\mu)(A+B)-\mathcal{C}(w,\mu)(A)-\mathcal{C}(w,\mu)(B)\geq k\mathcal{M}'(w,\mu)(A+B). \tag{2.21}$$

Proof. If $BA+AB\geq k$ then by multiplying both sides by $(A+B+\lambda)^{-1}$ with $\lambda\geq 0$, we get

$$(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}\geq k(A+B+\lambda)^{-2}$$

and by multiplying with $\lambda w(\lambda)\geq 0$ and integrating we obtain

$$\begin{aligned}&\int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-1}(BA+AB)(A+B+\lambda)^{-1}d\mu(\lambda)\\&\geq k\int_0^\infty \lambda w(\lambda)(A+B+\lambda)^{-2}d\mu(\lambda)=k\mathcal{M}'(w,\mu)(A+B).\end{aligned}$$

By making use of (2.1) we derive the desired result (2.21). □

Remark 4. *The symmetrized product of two operators A, $B\in B(H)$ is defined by $S(A,B)=AB+BA$. In general, the symmetrized product of two operators*

A, B is not positive (see for instance [10]*). Also Gustafson* [6] *showed that if $0\leq m\leq A\leq M$ and $0\leq n\leq B\leq N$, then we have the lower bound*

$$S(A,B)\geq 2mn-\frac{1}{4}(M-m)(N-n)=:k, \tag{2.22}$$

which can take positive or negative values depending on the parameters m, M, n, N.

The case of operator monotone functions is as follows:

Proposition 3. *Assume that the function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$. For all A, $B>0$ with $BA+AB\geq k$ for some real constant k,*

$$(A+B)f(A+B)-Af(A)-Bf(B)\geq kf'(A+B). \tag{2.23}$$

Proof. From (1.9) we have

$$f(t)=a+bt+\mathcal{M}(\ell,\mu)(t),$$

where $\ell(\lambda)=\lambda$, $a\in R$, $b\geq 0$ and μ is a positive measure on $(0,\infty)$.

If we take the derivative, we obtain

$$\mathcal{M}'(w,\mu)(t)=f'(t)-b,\ t>0.$$

By (2.21) we then obtain

$$(A+B)f(A+B)-Af(A)-Bf(B)-b(BA+AB)\geq k(f'(A+B)-b)$$

namely

$$\begin{aligned}(A+B)f(A+B)-Af(A)-Bf(B)&\geq b(BA+AB)+k(f'(A+B)-b)\\&\geq bk+k(f'(A+B)-b)=kf'(A+B)\end{aligned}$$

and the inequality (2.23) is proved. □

Remark 5. *Let $r\in(0,1]$. If A, $B>0$ with $BA+AB\geq k$, then by writing the inequality (2.23) for the power function $f(t)=t^r$, which is operator monotone, we have the power inequality*

$$(A+B)^{r+1}-A^{r+1}-B^{r+1}\geq rk(A+B)^{r-1}. \tag{2.24}$$

If we write the inequality (2.23) for the operator monotonic function $f(t)=\ln t$, then we get

$$(A+B)\ln(A+B)-A\ln A-B\ln B\geq k(A+B)^{-1} \tag{2.25}$$

if A, $B>0$ with $BA+AB\geq k$.

Proposition 4. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.11). For all A, $B>0$ with $BA+AB\geq k$ for some real constant k,*

$$\begin{aligned}&f(A+B)-f(A)-f(B)+f(0)\\&\geq k\left[(A+B)f'(A+B)-f(A+B)+f(0)\right](A+B)^{-2}.\end{aligned} \tag{2.26}$$

Proof. From (1.11) we have

$$\frac{f(t)-f(0)}{t}-f'_{+}(0)-ct=\mathcal{M}(\ell,\mu)(t),$$

where $c \geq 0$ and μ is a positive measure on $(0, \infty)$, which give by derivation that

$$\mathcal{M}'(\ell, \mu)(t) = \frac{tf'(t) - f(t) + f(0)}{t^2} - c, \ t > 0.$$

By (2.21) we have

$$f(A+B) - f(A) - f(B) + f(0) - c(BA + AB)$$
$$\geq k\left([(A+B) f'(A+B) - f(A+B) + f(0)] ((A+B))^{-2} - c \right),$$

which implies that

$$f(A+B) - f(A) - f(B) + f(0)$$
$$\geq c(BA+AB) + k\left([(A+B) f'(A+B) - f(A+B) + f(0)] (A+B)^{-2} - c \right)$$
$$\geq ck + k\left([(A+B) f'(A+B) - f(A+B) + f(0)] ((A+B))^{-2} - c \right)$$
$$= k[(A+B) f'(A+B) - f(A+B) + f(0)] (A+B)^{-2}$$

and the inequality (2.26) is proved. □

Remark 6. *Let $a > 0$ and $f(t) = (t+a)^p$ with $p \in [-1, 0) \cup [1, 2]$. This function is operator convex and $f(0) = a^p$. Then for all A, $B > 0$ with $BA + AB \geq k$,*

$$(A + B + a)^p + a^p - (A+a)^p - (B+a)^p$$
$$\geq k[(p-1)(A+B+a)^p + a^p](A+B)^{-2}. \tag{2.27}$$

In particular, for $p = -1$, we derive

$$(A+B+a)^{-1} + a^{-1} - (A+a)^{-1} - (B+a)^{-1}$$
$$\geq k\left[a^{-1} - 2(A+B+a)^{-1}\right](A+B)^{-2}. \tag{2.28}$$

3. More Examples

We define the *upper incomplete Gamma function* as [12]

$$\Gamma(a, z) := \int_z^\infty t^{a-1} e^{-t} dt,$$

which for $z = 0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1} e^{-t} dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [13]

$$\Gamma(a, z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a} e^{-t}}{z+t} dt \tag{3.1}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a} e^{-\lambda}$ for $\lambda > 0$. Then by (3.1) we obtain

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a} e^{-\lambda}}{t+\lambda} d\lambda = \Gamma(1-a) t^{-a} e^t \Gamma(a, t) \tag{3.2}$$

for $a < 1$ and $t > 0$.

Define

$$E_2\left(w_{\cdot^{-a}e^{-\cdot}}\right) := \int_0^\infty \lambda w_{\cdot^{-a}e^{-\cdot}}\left(\lambda\right) d\mu\left(\lambda\right) = \int_0^\infty \lambda^{1-a}e^{-\lambda}d\lambda = \Gamma(2-a) \tag{3.3}$$

for $a < 2$.

For $a = 0$ in (3.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^tE_1\left(t\right) \tag{3.4}$$

for $t > 0$, where the *exponential integral* E_1 is defined by

$$E_1\left(t\right) := \int_t^\infty \frac{e^{-u}}{u}du.$$

For $a = 0$ in (3.3) we get

$$E_2\left(w_{e^{-\cdot}}\right) := \int_0^\infty \lambda e^{-\lambda}d\lambda = \Gamma(2) = 1. \tag{3.5}$$

Let $a = 1-n$, with n a natural number with $n \geq 0$, then by (3.2) we have

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) \\ &= (n-1)!t^{n-1}e^t\Gamma(1-n,t).\end{aligned} \tag{3.6}$$

If we define the generalized exponential integral [14] by

$$E_p\left(z\right) := z^{p-1}\Gamma\left(1-p,z\right) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n\left(t\right)$$

for $n \geq 1$ and $t > 0$.

Using the identity [14, Eq 8.19.7], for $n \geq 2$

$$E_n\left(z\right) = \frac{\left(-z\right)^{n-1}}{(n-1)!}E_1\left(z\right) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\left(-z\right)^k,$$

we then obtain

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= (n-1)!e^tE_n\left(t\right) \\ &= (n-1)!e^t \\ &\quad\times\left[\frac{\left(-t\right)^{n-1}}{(n-1)!}E_1\left(t\right) + \frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\left(-t\right)^k\right] \\ &= \sum_{k=0}^{n-2}\left(-1\right)^k(n-k-2)!t^k + \left(-1\right)^{n-1}t^{n-1}e^tE_1\left(t\right)\end{aligned} \tag{3.7}$$

for $n \geq 2$ and $t > 0$.

For $n \geq 2$ we get

$$E_2\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) = \int_0^\infty \lambda^n e^{-\lambda}d\lambda = \Gamma(n+1) = n!.$$

For $n=2$, we derive by (3.7) that

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(t)=\int_0^\infty \lambda e^{-\lambda}\left(t+\lambda\right)^{-1}d\lambda = 1-t\exp\left(t\right)E_1\left(t\right) \tag{3.8}$$

for $t>0$. We also have

$$E_2\left(w_{\cdot e^{-\cdot}}\right):=\int_0^\infty \lambda^2 e^{-\lambda}d\lambda=\Gamma(3)=2. \tag{3.9}$$

Proposition 5. *For all* A, $B>0$,

$$\begin{aligned} &1-a+A^{2-a}\exp\left(A\right)\Gamma(a,A)+B^{2-a}\exp\left(B\right)\Gamma(a,B) \\ &\geq \left(A+B\right)^{2-a}\exp\left(A+B\right)\Gamma(a,A+B), \end{aligned} \tag{3.10}$$

where $1>a$.

In particular,

$$\begin{aligned} &1+A^2\exp\left(A\right)E_1\left(A\right)+B^2\exp\left(B\right)E_1\left(B\right) \\ &\geq \left(A+B\right)^2\exp\left(A+B\right)E_1(A+B). \end{aligned} \tag{3.11}$$

Proof. Observe that

$$\mathcal{C}(w_{\cdot^{-a}e^{-\cdot}})\left(t\right)=t^2\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=\Gamma(1-a)t^{2-a}\exp\left(t\right)\Gamma(a,t)$$

for $1-a>0$.

It follows by (2.6) that

$$\begin{aligned} &\Gamma(2-a)+\Gamma(1-a)A^{2-a}\exp\left(A\right)\Gamma(a,A)+\Gamma(1-a)B^{2-a}\exp\left(B\right)\Gamma(a,B) \\ &\geq \Gamma(1-a)\left(A+B\right)^{2-a}\exp\left(A+B\right)\Gamma(a,A+B) \end{aligned}$$

and since

$$\Gamma(2-a)=\Gamma(1+1-a)=\left(1-a\right)\Gamma(1-a)$$

for $1-a>0$, then we obtain the desired result (3.10). □

The interested reader can obtain other similar results by employing the examples of operator monotone/convex functions from [2]-[5], [10]-[11] and the references therein.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.