OPERATOR SYNCHRONICITY RELATED TO THE CONVEX INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *convex integral transform*

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H.

We show among others that, for all A, $B > 0$ we have the representation

$$\begin{aligned}&[\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)](B-A)\\ &= \int_0^\infty w(\lambda) \Big[(B-A)^2\\ &-\lambda^2 \left(\int_0^1 \left[(\lambda + (1-t)A + tB)^{-1}(B-A)\right]^2 dt\right)\Big] d\mu(\lambda).\end{aligned}$$

Assume that $\int_0^\infty w(\lambda) d\mu(\lambda) < \infty$. We also give some sufficient conditions for the operators $A, B > 0$ such that the inequality

$$[\mathcal{C}(w,\mu)(B) - \mathcal{C}(w,\mu)(A)](B-A) \le \left(\int_0^\infty w(\lambda) d\mu(\lambda)\right)(B-A)^2$$

holds. Some inequalities for operator monotone and operator convex functions are also given.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x \rangle \ge 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \ne 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

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By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.2}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \ t > 0, \tag{1.3}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.3) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \ t > 0. \tag{1.4}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t), \ t > 0. \tag{1.5}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t), \ t > 0. \tag{1.6}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda), \tag{1.7}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.8}$$

for $T > 0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

Theorem 1. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.9}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.10}$$

If f is operator monotone in $[0,\infty)$, then $a = f(0)$ in (1.9).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left(\left(1-\lambda \right) A+\lambda B\right) \leq \left(\geq \right) \left(1-\lambda \right) f\left(A\right) +\lambda f\left(B\right) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 2. *A function $f:(0,\infty) \rightarrow \mathbb{R}$ is operator convex in $(0,\infty)$ if and only if it has the representation*

$$f\left(t\right) =a+bt+ct^{2}+\int_{0}^{\infty }\frac{t^{2}\lambda }{t+\lambda }d\mu \left(\lambda \right) , \tag{1.11}$$

where a, $b\in \mathbb{R}$, $c\geq 0$ and a positive measure μ on $(0,\infty)$ such that (1.2) holds. If f is operator convex in $[0,\infty)$, then $a=f\left(0\right)$ and $b=f_{+}^{\prime }\left(0\right)$, the right derivative, in (1.11).

For a continuous and positive function $w\left(\lambda \right)$, $\lambda >0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call the *convex integral transform,*

$$\mathcal{C}(w,\mu)\left(t\right) :=t^{2}\mathcal{D}(w,\mu)\left(t\right) ,\ t>0. \tag{1.12}$$

For $t>0$ we have

$$\begin{aligned} \mathcal{C}(w,\mu)\left(t\right) &:=\int_{0}^{\infty }w\left(\lambda \right) t^{2}\left(t+\lambda \right) ^{-1}d\mu \left(\lambda \right) \\ &=\int_{0}^{\infty }w\left(\lambda \right) \left(t+\lambda -\lambda \right) ^{2}\left(t+\lambda \right) ^{-1}d\mu \left(\lambda \right) \\ &=\int_{0}^{\infty }w\left(\lambda \right) \left[\left(t+\lambda \right) ^{2}-2\lambda \left(t+\lambda \right) +\lambda ^{2}\right] \left(t+\lambda \right) ^{-1}d\mu \left(\lambda \right) \\ &=\int_{0}^{\infty }w\left(\lambda \right) \left[\left(t+\lambda \right) -2\lambda +\lambda ^{2}\left(t+\lambda \right) ^{-1}\right] d\mu \left(\lambda \right) \\ &=\int_{0}^{\infty }w\left(\lambda \right) \left[t-\lambda +\lambda ^{2}\left(t+\lambda \right) ^{-1}\right] d\mu \left(\lambda \right) . \end{aligned} \tag{1.13}$$

If $\int_{0}^{\infty }w\left(\lambda \right) d\mu \left(\lambda \right) <\infty$ and $\int_{0}^{\infty }w\left(\lambda \right) \lambda d\mu \left(\lambda \right) <\infty$, then we conclude that

$$\mathcal{C}(w,\mu)\left(t\right) =-\int_{0}^{\infty }w\left(\lambda \right) \lambda d\mu \left(\lambda \right) +t\int_{0}^{\infty }w\left(\lambda \right) d\mu \left(\lambda \right) +\mathcal{D}(\ell ^{2}w,\mu)\left(t\right) , \tag{1.14}$$

where $\ell \left(t\right) =t$, $t>0$.

Consider, for instance, the kernel $e_{-a}\left(\lambda \right) :=\exp \left(-a\lambda \right)$, $\lambda \geq 0$ and $a>0$. After some calculations, we obtain

$$\mathcal{D}(e_{-a})\left(t\right) =\int_{0}^{\infty }\frac{\exp \left(-a\lambda \right) }{t+\lambda }d\lambda =E_{1}\left(at\right) \exp \left(at\right) ,\ t\geq 0$$

where the *exponential integral* is given by

$$E_{1}\left(t\right) :=\int_{t}^{\infty }\frac{e^{-u}}{u}du.$$

We also have

$$\int_0^\infty w(\lambda)\,\lambda d\lambda=\int_0^\infty \exp(-a\lambda)\,\lambda d\lambda=\frac{1}{a^2}$$

and

$$\int_0^\infty w(\lambda)\,d\lambda=\int_0^\infty \exp(-a\lambda)\,d\lambda=\frac{1}{a}.$$

Therefore

$$\mathcal{C}(e_{-a})(t):=t^2\mathcal{D}(e_{-a},\mu)(t)=t^2E_1(at)\exp(at),\ \ t>0. \tag{1.15}$$

Since

$$\mathcal{D}(\ell^2e_{-a},\mu)(t)=\int_0^\infty \frac{\lambda^2\exp(-a\lambda)}{t+\lambda}d\lambda$$

then by (1.14) we get

$$t^2E_1(at)\exp(at)=-\frac{1}{a^2}+\frac{t}{a}+\mathcal{D}(\ell^2w,\mu)(t),$$

which gives

$$\mathcal{D}(\ell^2w,\mu)(t)=t^2E_1(at)\exp(at)-\frac{t}{a}+\frac{1}{a^2},\ \ t>0,a>0.$$

If we take $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then $\int_0^\infty w_r(\lambda)\,d\lambda=\infty$ and the equality (1.14) does not hold in this case.

Using the continuous functional calculus for selfadjoint operators in Hilbert spaces we can introduce the following *convex integral transform* of the positive operator T

$$\mathcal{C}(w,\mu)(T):=\int_0^\infty w(\lambda)\,T^2(\lambda+T)^{-1}\,d\mu(\lambda),$$

provided the integral exist.

2. Main Results

In what follows we assume that the integral transform

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ \ t>0,$$

defined for a continuous and positive function $w(\lambda)$, $\lambda>0$ and μ a positive measure on $(0,\infty)$, exists for all $t>0$.

Theorem 3. *For all A, $B>0$ we have the representation*

$$\begin{aligned}&[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)](B-A)\\&=\int_0^\infty w(\lambda)\Big[(B-A)^2\\&-\lambda^2\left(\int_0^1\Big[(\lambda+(1-t)A+tB)^{-1}(B-A)\Big]^2dt\right)\Big]d\mu(\lambda)\end{aligned} \tag{2.1}$$

and

$$(2.2)\qquad \begin{aligned}&(B-A)\left[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)\right]\\&=\int_0^\infty w(\lambda)\Big[(B-A)^2\\&-\lambda^2\left(\int_0^1\left[(B-A)(\lambda+(1-t)A+tB)^{-1}\right]^2dt\right)\Big]d\mu(\lambda).\end{aligned}$$

Proof. For A, $B>0$ we have, by using continuous functional calculus for selfadjoint operators and (1.13), that

$$\mathcal{C}(w,\mu)(A)=\int_0^\infty w(\lambda)\left[A-\lambda+\lambda^2(A+\lambda)^{-1}\right]d\mu(\lambda)$$

and

$$\mathcal{C}(w,\mu)(B)=\int_0^\infty w(\lambda)\left[B-\lambda+\lambda^2(B+\lambda)^{-1}\right]d\mu(\lambda).$$

This gives that

$$(2.3)\qquad \begin{aligned}&\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)\\&=\int_0^\infty w(\lambda)\left[B-A+\lambda^2\left((B+\lambda)^{-1}-(A+\lambda)^{-1}\right)\right]d\mu(\lambda).\end{aligned}$$

Let T, $S>0$. The function $f(s)=-s^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$(2.4)\qquad \nabla f_T(S):=\lim_{s\to 0}\left[\frac{f(T+sS)-f(T)}{s}\right]=T^{-1}ST^{-1}$$

for T, $S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-s)C+sD,\ s\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(s):=f((1-s)C+sD),\ s\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$(2.5)\qquad f(D)-f(C)=\int_0^1\frac{d}{ds}(f_{C,D}(s))\,ds=\int_0^1\nabla f_{(1-s)C+sD}(D-C)\,ds.$$

If we write this equality for the function $f(s)=-s^{-1}$ and C, $D>0$, then we get the representation

$$(2.6)\qquad C^{-1}-D^{-1}=\int_0^1((1-s)C+sD)^{-1}(D-C)((1-s)C+sD)^{-1}\,ds.$$

Now, if we take in (2.6) $C=\lambda+B$, $D=\lambda+A$, then

$$\begin{aligned}&(\lambda+B)^{-1}-(\lambda+A)^{-1}\\&=\int_0^1((1-s)(\lambda+B)+s(\lambda+A))^{-1}(A-B)\\&\times((1-s)(\lambda+B)+s(\lambda+A))^{-1}ds\\&=\int_0^1(\lambda+(1-s)B+sA)^{-1}(A-B)(\lambda+(1-s)B+sA)^{-1}ds\\&=\int_0^1(\lambda+(1-t)A+tB)^{-1}(A-B)(\lambda+(1-t)A+tB)^{-1}dt,\end{aligned}$$

where for the last equality we used the change of variable $s=1-t$.

By utilising (2.3) we derive

$$\begin{aligned}&\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)=\int_0^\infty w(\lambda)\Big[B-A\\&-\lambda^2\left(\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt\right)\Big]d\mu(\lambda).\end{aligned}$$

If we multiply this equality at the right with $B-A$, the we obtain (2.1). If we multiply this equality at the left by $B-A$, then we obtain (2.2). $\square$

Corollary 1. *Assume that the kernel $w\in L_1(\mu,[0,\infty))$, namely $\int_0^\infty w(\lambda)\,d\mu(\lambda)<\infty$. Then we have the equality*

$$\begin{aligned}&[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)](B-A)\\&=\left(\int_0^\infty w(\lambda)\,d\mu(\lambda)\right)(B-A)^2\\&+\left[\mathcal{D}\left(\ell^2w,\mu\right)(B)-\mathcal{D}\left(\ell^2w,\mu\right)(A)\right](B-A),\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}&(B-A)[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)]\\&=\left(\int_0^\infty w(\lambda)\,d\mu(\lambda)\right)(B-A)^2\\&+(B-A)\left[\mathcal{D}\left(\ell^2w,\mu\right)(B)-\mathcal{D}\left(\ell^2w,\mu\right)(A)\right],\end{aligned}\tag{2.8}$$

where $\ell(\lambda)=\lambda$, $\lambda\geq 0$.

In the following, in order to simplify terminology, when we write $T\geq 0$ we automatically assume that the operator T is selfadjoint.

We need the following lemmas:

Lemma 1. *Let A, $B>0$. The following statements are equivalent:*

(i) *For all $s\geq 0$,*

$$(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)\geq 2.\tag{2.9}$$

(ii) *For all $s\geq 0$,*

$$\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2dt\geq 0.$$

(iii) *For all* $s \geq 0$,

$$(\ell_s(B) - \ell_s(A))(B - A) \geq 0,$$

where $\ell_s(t) = -(t+s)^{-1}$, $t > 0$.

Proof. From (2.14) we have, by multiplying at right with $B - A$ that

$$\begin{aligned}&\left[(A+s)^{-1} - (B+s)^{-1}\right](B-A)\\&= \int_0^1 ((1-t)A + tB + s)^{-1}(B-A)((1-t)A + tB + s)^{-1}(B-A)\,dt\\&= \int_0^1 \left[((1-t)A + tB + s)^{-1}(B-A)\right]^2 dt\end{aligned}$$

for all $s \geq 0$.

Also

$$\begin{aligned}&\left[(A+s)^{-1} - (B+s)^{-1}\right](B-A)\\&= \left[(A+s)^{-1} - (B+s)^{-1}\right][B + s - (A+s)]\\&= (A+s)^{-1}(B+s) + (B+s)^{-1}(A+s) - 2\end{aligned}$$

for all $s \geq 0$.

Therefore

$$\begin{aligned}&(\ell_s(B) - \ell_s(A))(B-A)\\&= (A+s)^{-1}(B+s) + (B+s)^{-1}(A+s) - 2\\&= \int_0^1 \left[((1-t)A + tB + s)^{-1}(B-A)\right]^2 dt\end{aligned} \tag{2.10}$$

for all $s \geq 0$.

The identity (2.10) reveals that the statements (i), (ii) and (iii) are equivalent. □

In the recent note [3] Fujii and Nakamoto proved the following inequality:

Lemma 2. *If* C, $D > 0$ *and* $CD^{-1} + DC^{-1}$ *is selfadjoint, then*

$$CD^{-1} + DC^{-1} \geq 2. \tag{2.11}$$

Indeed, as shown in [3], if we put $T = CD^{-1}$, then $V = T + T^{-1}$ is selfadjoint by the assumption. Note that the spectrum $\mathrm{Sp}(T)$ of T is included in $(0, \infty)$, because C, $D > 0$ and $\mathrm{Sp}(T) = \mathrm{Sp}\left(C^{1/2}D^{-1}C^{1/2}\right)$. Since $\mathrm{Sp}(V) = \left\{t + \frac{1}{t},\ t \in \mathrm{Sp}(T)\right\}$ by the spectral mapping theorem for rational functions, hence we have $T + T^{-1} \geq 2$.

As a consequence, they proved that, if

(i') Operator $A(B+s)^{-1} + B(A+s)^{-1}$ *is selfadjoint for all* $s \geq 0$,

then

$$(B - A)(f(B) - f(A)) \geq 0$$

for all f operator monotone functions on $(0, \infty)$.

Lemma 3. *Let* A, $B > 0$, *then the statements (i) and (i') are equivalent.*

Proof. Notice that for all $s \geq 0$,

$$\begin{aligned} & (A+s)^{-1}(B+s)+(B+s)^{-1}(A+s) \\ & =(A+s)^{-1}B+(B+s)^{-1}A+s(A+s)^{-1}+s(B+s)^{-1}. \end{aligned} \tag{2.12}$$

Also, the operator $s(A+s)^{-1}+s(B+s)^{-1}$ is selfadjoint for $s \geq 0$.

If the statement *(i)* holds, then $(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)$ is selfadjoint and by (1.13) we must have that $(A+s)^{-1}B+(B+s)^{-1}A$ is selfadjoint, which shows that

$$\left((A+s)^{-1}B+(B+s)^{-1}A\right)^{*}=B(A+s)^{-1}+A(B+s)^{-1}$$

is selfadjoint, namely *(i')* is true.

If the statement *(i')* holds, then by (2.12) we get

$$(A+s)^{-1}(B+s)+(B+s)^{-1}(A+s)$$

is selfadjoint and by (2.11) for $C=(A+s)^{-1}$, $D=(B+s)^{-1}$ we obtain the inequality (2.9), namely *(i)* is true. □

We define the class of operators

$$\mathfrak{Cl}_{(0,\infty)}(H):=\{(A,B) \mid A,\ B>0 \text{ and satisfy condition } (i')\}.$$

We observe that if $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$ then $(B,A)\in\mathfrak{Cl}_{(0,\infty)}(H)$.

Also if $AB=BA$, $A,\ B>0$, then $U_s:=(A+s)^{-1}(B+s)$ and $U_s^{-1}=(B+s)^{-1}(A+s)$ are selfadjoint and since $U_s+U_s^{-1}\geq 2$, $s\geq 0$ we derive that $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$. Therefore, if $\mathfrak{Co}_{(0,\infty)}(H)$ is the class of all pairs of commutative operators $A,\ B>0$, then we have

$$\emptyset \neq \mathfrak{Co}_{(0,\infty)}(H)\subset\mathfrak{Cl}_{(0,\infty)}(H). \tag{2.13}$$

We have:

Theorem 4. *Assume that the kernel* $w\in L_1(\mu,[0,\infty))$. *Then for all* $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$,

$$\begin{aligned} & [\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)](B-A) \\ & =(B-A)[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)]\leq\left(\int_0^\infty w(\lambda)\,d\mu(\lambda)\right)(B-A)^2. \end{aligned} \tag{2.14}$$

Proof. If $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$, then by Lemma 1, then

$$\int_0^1\left[((1-t)A+tB+s)^{-1}(B-A)\right]^2 dt\geq 0$$

for all $s\geq 0$.

By taking the integral, we get

$$\int_0^\infty \lambda^2\left(\int_0^1\left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2 dt\right)d\mu(\lambda)\geq 0.$$

By the identity (2.1)

$$\begin{aligned}
&[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)](B-A)\\
&=\int_0^\infty w(\lambda)\Big[(B-A)^2\\
&-\int_0^\infty \lambda^2\left(\int_0^1\left[(\lambda+(1-t)A+tB)^{-1}(B-A)\right]^2dt\right)\Big]d\mu(\lambda)\\
&\le\int_0^\infty w(\lambda)\,d\mu(\lambda)(B-A)^2
\end{aligned}$$

and $U:=[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)](B-A)$ is selfadjoint.

Since

$$\begin{aligned}
U^* \quad : &= \{[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)](B-A)\}^*\\
&= (B-A)^*[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)]^*\\
&= (B-A)[\mathcal{C}(w,\mu)(B)-\mathcal{C}(w,\mu)(A)]
\end{aligned}$$

then $U^*=U$ is equivalent to the identity in (2.14) and the proof is completed. □

Remark 1. *If we consider the transform from the introduction (1.15),*

$$\mathcal{C}(e_{-a})(t)=t^2E_1(at)\exp(at),\ t>0,\ a>0. \tag{2.15}$$

If $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$, *then we also have the operator inequality*

$$\left[B^2E_1(aB)\exp(aB)-A^2E_1(aA)\exp(aA)\right](B-A)\le\frac{1}{a}(B-A)^2. \tag{2.16}$$

The case of operator monotone functions is as follows:

Proposition 1. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *and has the representation (1.9) with* $b\ge0$ *and a positive measure* μ *on* $(0,\infty)$ *such that the expectation* $E(\mu):=\int_0^\infty\lambda d\mu(\lambda)<\infty$, *then for* $(A,B)\in\mathfrak{Cl}_{(0,\infty)}(H)$

$$\begin{aligned}
&B^2f(B)+A^2f(A)-f(A)AB-f(B)BA\\
&\le[E(\mu)+f(0)](B-A)^2+b\left(B^2-A^2\right)(B-A).
\end{aligned}\tag{2.17}$$

Proof. From (1.9) we get by multiplying with $t>0$ that

$$tf(t)=at+bt^2+t^2\int_0^\infty\frac{\lambda}{t+\lambda}d\mu(\lambda)=at+bt^2+\mathcal{C}(\ell,\mu)(t),$$

namely

$$\mathcal{C}(\ell,\mu)(t)=tf(t)-f(0)t-bt^2.$$

Since

$$\begin{aligned}
&\mathcal{C}(\ell,\mu)(B)-\mathcal{C}(\ell,\mu)(A)\\
&=Bf(B)-Af(A)-f(0)(B-A)-b\left(B^2-A^2\right)
\end{aligned}$$

hence by (2.14) we get

$$\begin{aligned}
&\left[Bf(B)-Af(A)-f(0)(B-A)-b\left(B^2-A^2\right)\right](B-A)\\
&\le E(\mu)(B-A)^2,
\end{aligned}$$

which is equivalent to (2.17). □

The case of operator convex functions is as follows:

Proposition 2. *Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ and has the representation (1.11) with $c \geq 0$ and a positive measure μ on $(0, \infty)$ such that the expectation $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$, then for $(A, B) \in \mathfrak{Cl}_{(0,\infty)}(H)$*

$$\begin{aligned} & f(B) B + f(A) A - f(A) B - f(B) A \\ & \leq \left[E(\mu) + f'_+(0)\right](B - A)^2 + c\left(B^2 - A^2\right)(B - A). \end{aligned} \tag{2.18}$$

Proof. From (1.11) we get

$$\mathcal{C}(\ell, \mu)(t) = f(t) - f(0) - f'_+(0)t - ct^2.$$

Since

$$\begin{aligned} & \mathcal{C}(\ell, \mu)(B) - \mathcal{C}(\ell, \mu)(A) \\ & = f(B) - f(A) - f'_+(0)(B - A) - c\left(B^2 - A^2\right) \end{aligned}$$

hence by (2.14) we obtain

$$\left[f(B) - f(A) - f'_+(0)(B - A) - c\left(B^2 - A^2\right)\right](B - A) \leq E(\mu)(B - A)^2,$$

which is equivalent to (2.18). □

3. Inequalities for Čebyšev's Simple Functional

For a continuous function f on $(0, \infty)$, an n-tuple of positive operators $\mathbf{A} = (A_1, ..., A_n)$ and a probability distribution $\mathbf{p} = (p_1, ..., p_n)$ we consider the *Čebyšev simple functional* defined by

$$\mathfrak{C}(f, \mathbf{A}, \mathbf{p}) = \sum_{k=1}^n p_k A_k f(A_k) - \sum_{k=1}^n p_k f(A_k) \sum_{k=1}^n p_k A_k.$$

For the identity function $\ell(t) = t$, $t \geq 0$ we have

$$\mathfrak{C}(\ell, \mathbf{A}, \mathbf{p}) := \sum_{k=1}^n p_k A_k^2 - \left(\sum_{k=1}^n p_k A_k\right)^2.$$

Theorem 5. *Assume that the kernel $w \in L_1(\mu, [0, \infty))$. For any n-tuple of positive operators $\mathbf{A} = (A_1, ..., A_n)$ with $(A_j, A_k) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for all $j, k = 1, ..., n$ and probability distribution $\mathbf{p} = (p_1, ..., p_n)$ we have*

$$\mathfrak{C}(\mathcal{C}(w, \mu), \mathbf{A}, \mathbf{p}) \leq \left(\int_0^\infty w(\lambda) d\mu(\lambda)\right) \mathfrak{C}(\ell, \mathbf{A}, \mathbf{p}). \tag{3.1}$$

Proof. From (2.14) we get

$$\begin{aligned} & \mathcal{C}(w, \mu)(B) B + \mathcal{C}(w, \mu)(A) A - \mathcal{C}(w, \mu)(B) A - \mathcal{C}(w, \mu)(A) B \\ & \leq \left(\int_0^\infty w(\lambda) d\mu(\lambda)\right)\left(B^2 + A^2 - BA - AB\right) \end{aligned} \tag{3.2}$$

for all A, $B > 0$.

If we take $B = A_i$, $A = A_j$, $i, j \in \{1, ..., n\}$ in (3.2), then we get

$$\mathcal{C}(w,\mu)(A_i)A_i + \mathcal{C}(w,\mu)(A_j)A_j - \mathcal{C}(w,\mu)(A_i)A_j - \mathcal{C}(w,\mu)(A_j)A_i \leq \left(\int_0^\infty w(\lambda)\,d\mu(\lambda)\right)\left(A_i^2 + A_j^2 - A_iA_j - A_jA_i\right) \tag{3.3}$$

for all $i, j \in \{1, ..., n\}$.

If we multiply (3.3) by $p_ip_j \geq 0$, sum over i, j from 1 to n and take into account that $\sum_{j=1}^n p_j = \sum_{i=1}^n p_i = 1$, we get

$$\begin{aligned} &\sum_{i=1}^n p_i\mathcal{C}(w,\mu)(A_i)A_i + \sum_{j=1}^n p_j\mathcal{C}(w,\mu)(A_j)A_j \\ &- \sum_{i=1}^n p_i\mathcal{C}(w,\mu)(A_i)\sum_{j=1}^n p_jA_j - \sum_{j=1}^n p_j\mathcal{C}(w,\mu)(A_j)\sum_{i=1}^n p_iA_i \\ &\leq \left(\int_0^\infty w(\lambda)\,d\mu(\lambda)\right) \\ &\times\left(\sum_{i=1}^n p_iA_i^2 + \sum_{j=1}^n p_jA_j^2 - \sum_{i=1}^n p_iA_i\sum_{j=1}^n p_jA_j - \sum_{j=1}^n p_jA_j\sum_{i=1}^n p_iA_i\right), \end{aligned} \tag{3.4}$$

which is equivalent to the first inequality in (3.1).

The proof of second inequality is similar. □

The case of operator monotone functions is as follows:

Corollary 2. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.9) with $b \geq 0$ and a positive measure μ on $(0,\infty)$ such that the expectation $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$, then for any n-tuple of positive operators $\mathbf{A} = (A_1, ..., A_n)$ with $(A_j, A_k) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for all $j, k = 1, ..., n$ and probability distribution $\mathbf{p} = (p_1, ..., p_n)$,*

$$\mathfrak{C}(\ell f, \mathbf{A}, \mathbf{p}) \leq [E(\mu) + f(0)]\,\mathfrak{C}(\ell, \mathbf{A}, \mathbf{p}) + b\mathfrak{C}\left(\ell^2, \mathbf{A}, \mathbf{p}\right) \tag{3.5}$$

where $\ell(\lambda) = \lambda$, $\lambda \geq 0$ and

$$\mathfrak{C}\left(\ell^2, \mathbf{A}, \mathbf{p}\right) = \sum_{k=1}^n p_kA_k^3 - \sum_{k=1}^n p_kA_k^2\sum_{k=1}^n p_kA_k.$$

The case of operator convex functions is as follows:

Corollary 3. *Assume that the function $f : [0,\infty) \to \mathbb{R}$ is operator convex in $[0,\infty)$ and has the representation (1.11) with $c \geq 0$ and a positive measure μ on $(0,\infty)$ such that the expectation $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$, then for any n-tuple of positive operators $\mathbf{A} = (A_1, ..., A_n)$ with $(A_j, A_k) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for all $j, k = 1, ..., n$ and probability distribution $\mathbf{p} = (p_1, ..., p_n)$,*

$$\mathfrak{C}(f, \mathbf{A}, \mathbf{p}) \leq \left[E(\mu) + f'_+(0)\right]\mathfrak{C}(\ell, \mathbf{A}, \mathbf{p}) + b\mathfrak{C}\left(\ell^2, \mathbf{A}, \mathbf{p}\right) \tag{3.6}$$

Remark 2. *For any n-tuple of positive operators $\mathbf{A} = (A_1, ..., A_n)$ $(A_j, A_k) \in \mathfrak{Cl}_{(0,\infty)}(H)$ for all $j, k = 1, ..., n$ and a probability distribution $\mathbf{p} = (p_1, ..., p_n)$, we have*

$$\mathfrak{C}\left(\cdot^2 E_1(a\cdot)\exp(a\cdot), \mathbf{A}, \mathbf{p}\right) \leq \frac{1}{a}\mathfrak{C}(\ell, \mathbf{A}, \mathbf{p}). \tag{3.7}$$

4. More Examples of Transform with Finite $\int_0^\infty w(\lambda)\,d\lambda$

We define the *upper incomplete Gamma function* as [11]

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [12]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{4.1}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (4.1) we obtain

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^{t}\Gamma(a,t) \tag{4.2}$$

for $a < 1$ and $t > 0$.

Define

$$C\left(w_{\cdot^{-a}e^{-\cdot}}\right) := \int_0^\infty \lambda^{-a}e^{-\lambda}d\lambda = \int_0^\infty \lambda^{1-a-1}e^{-\lambda}d\lambda = \Gamma(1-a) \tag{4.3}$$

for $a < 1$.

For $a = 0$ in (4.2) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^{t}\Gamma(0,t) = e^{t}E_1(t) \tag{4.4}$$

for $t > 0$, where the *exponential integral* E_1 is defined by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du.$$

For $a = 0$ in (4.3) we get

$$C\left(w_{e^{-\cdot}}\right) := \int_0^\infty e^{-\lambda}d\lambda = \Gamma(1) = 1. \tag{4.5}$$

Let $a = 1-n$, with n a natural number with $n \geq 0$, then by (4.2) we have

$$\begin{aligned} \mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^{t}\Gamma(1-n,t) \\ &= (n-1)!t^{n-1}e^{t}\Gamma(1-n,t). \end{aligned} \tag{4.6}$$

If we define the generalized exponential integral [13] by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [13, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}E_1(z) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-z)^k,$$

we then obtain

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= (n-1)!e^t E_n(t) \\ &= (n-1)!e^t \\ &\times \left[\frac{(-t)^{n-1}}{(n-1)!}E_1(t) + \frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-t)^k\right] \\ &= \sum_{k=0}^{n-2}(-1)^k(n-k-2)!t^k + (-1)^{n-1}t^{n-1}e^tE_1(t)\end{aligned} \tag{4.7}$$

for $n \geq 2$ and $t > 0$.

For $n \geq 2$ we get

$$C\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) = \int_0^\infty \lambda^{n-1}e^{-\lambda}d\lambda = \Gamma(n) = (n-1)!.$$

For $n = 2$, we derive by (4.7) that

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(t) = \int_0^\infty \lambda e^{-\lambda}(t+\lambda)^{-1}d\lambda = 1 - t\exp(t)E_1(t) \tag{4.8}$$

for $t > 0$. We also have

$$C\left(w_{\cdot e^{-\cdot}}\right) := \int_0^\infty \lambda e^{-\lambda}d\lambda = \Gamma(0) = 1. \tag{4.9}$$

Proposition 3. *Let $a < 1$, then for all $(A, B) \in \mathfrak{Cl}_{(0,\infty)}(H)$,*

$$\left[B^{2-a}\exp(B)\Gamma(a,B) - A^{2-a}\exp(A)\Gamma(a,A)\right](B-A) \leq (B-A)^2. \tag{4.10}$$

In particular

$$\left[B^2\exp(B)E_1(B) - A^2\exp(A)E_1(A)\right](B-A) \leq (B-A)^2. \tag{4.11}$$

Proof. It follows by Theorem 4 for the operator convex transform

$$\mathcal{C}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = t^2\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \Gamma(1-a)t^{2-a}e^t\Gamma(a,t).$$

□

We can also consider the weight $w_{(\cdot^2+a^2)^{-1}}(\lambda) := \frac{1}{\lambda^2+a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$\begin{aligned}\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t) &:= \int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+a^2\right)}d\lambda \\ &= \frac{1}{t^2+a^2}\left[\frac{\pi t}{2a} - \ln\left(\frac{t}{a}\right)\right]\end{aligned}$$

for $t > 0$ and $a > 0$. We have

$$D\left(w_{(\cdot^2+a^2)^{-1}}\right) := \int_0^\infty \frac{1}{\lambda^2+a^2}d\lambda = \frac{\pi}{2a}.$$

For $a = 1$ we also have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+1\right)} d\lambda = \frac{1}{t^2+1}\left(\frac{\pi t}{2} - \ln t\right)$$

for $t > 0$. In this case

$$D\left(w_{(\cdot^2+1)^{-1}}\right) := \int_0^\infty \frac{1}{\lambda^2+1} d\lambda = \frac{\pi}{2}.$$

By making use of Theorem 3 we can obtain some similar inequalities. The details are omitted.

The interested reader can obtain other similar results by employing the examples of operator monotone/convex functions from [2]-[6], [9]-[10] and the references therein.

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