OPERATOR SYNCHRONICITY RELATED TO THE CONVEX INTEGRAL TRANSFORM OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function \( w(\lambda), \lambda > 0 \) and a positive measure on \((0, \infty)\), we consider the following convex integral transform

\[
C(w, \mu)(T) := \int_0^\infty w(\lambda) T^2(\lambda + T)^{-1} d\mu(\lambda),
\]

where the integral is assumed to exist for a positive operator on a complex Hilbert space \( H \).

We show among others that, for all \( A, B > 0 \), we have the representation

\[
[C(w, \mu)(B) - C(w, \mu)(A)](B - A) = \int_0^\infty w(\lambda) \left( (B - A)^2 - \lambda^2 \left( \int_0^1 \left[ (\lambda + (1 - t)A + tB)^{-1} (B - A)^2 \right] dt \right) \right) d\mu(\lambda).
\]

Assume that \( \int_0^\infty w(\lambda) d\mu(\lambda) < \infty \). We also give some sufficient conditions for the operators \( A, B > 0 \) such that the inequality

\[
[C(w, \mu)(B) - C(w, \mu)(A)](B - A) \leq \left( \int_0^\infty w(\lambda) d\mu(\lambda) \right) (B - A)^2
\]

holds. Some inequalities for operator monotone and operator convex functions are also given.

1. Introduction

Consider a complex Hilbert space \((H, \langle \cdot, \cdot \rangle)\). An operator \( T \) is said to be positive (denoted by \( T \geq 0 \)) if \( \langle Tx, x \rangle \geq 0 \) for all \( x \in H \) and also an operator \( T \) is said to be strictly positive (denoted by \( T > 0 \)) if \( T \) is positive and invertible.

We have the following integral representation for the power function when \( t > 0, r \in (0, 1] \), see for instance [1, p. 145]

\[
t^r = \frac{\sin(r \pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1} \lambda^r}{\lambda + t} \lambda^r d\lambda.
\]

(1.1)

Observe that for \( t > 0, t \neq 1 \), we have

\[
\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln \left( \frac{u + t}{u + 1} \right)
\]

for all \( u > 0 \).
By taking the limit over $u \to \infty$ in this equality, we derive
\[
\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)} ,
\]
which gives the representation for the logarithm
(1.2)
\[
\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)}
\]
for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda), \lambda > 0$, the following \textit{integral transform}
(1.3)
\[
\mathcal{D}(w, \mu) (t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \quad t > 0,
\]
where $\mu$ is a positive measure on $(0, \infty)$ and the integral (1.3) exists for all $t > 0$.

For $\mu$ the Lebesgue usual measure, we put
(1.4)
\[
\mathcal{D}(w) (t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \quad t > 0.
\]

If we take $\mu$ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then
(1.5)
\[
t^{r-1} = \frac{\sin (r\pi)}{\pi} \mathcal{D}(w_r) (t), \quad t > 0.
\]

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, we have the representation
(1.6)
\[
\ln t = (t-1) \mathcal{D}(w_{\ln}) (t), \quad t > 0.
\]

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator
(1.7)
\[
\mathcal{D}(w, \mu) (T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\mu(\lambda),
\]
where $w$ and $\mu$ are as above. Also, when $\mu$ is the usual Lebesgue measure, then
(1.8)
\[
\mathcal{D}(w) (T) := \int_0^\infty w(\lambda) (\lambda + T)^{-1} d\lambda,
\]
for $T > 0$.

A real valued continuous function $f$ on $(0, \infty)$ is said to be operator monotone if $f(A) \succeq f(B)$ holds for any $A \succeq B > 0$.

We have the following representation of operator monotone functions [8], see for instance [1, p. 144-145]:

\textbf{Theorem 1.} A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation
(1.9)
\[
f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda),
\]
where $a \in \mathbb{R}$, $b \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ such that
(1.10)
\[
\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty.
\]

If $f$ is operator monotone in $[0, \infty)$, then $a = f(0)$ in (1.9).
A real valued continuous function \( f \) on an interval \( I \) is said to be \textit{operator convex (operator concave)} on \( I \) if

\[(OC) \quad f ((1 - \lambda) A + \lambda B) \leq (\geq) (1 - \lambda) f (A) + \lambda f (B)\]

in the operator order, for all \( \lambda \in [0, 1] \) and for every selfadjoint operator \( A \) and \( B \) on a Hilbert space \( H \) whose spectra are contained in \( I \). Notice that a function \( f \) is operator concave if \( -f \) is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

**Theorem 2.** A function \( f : (0, \infty) \to \mathbb{R} \) is operator convex in \((0, \infty)\) if and only if it has the representation

\[(1.11) \quad f (t) = a + bt + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t + \lambda} d\mu (\lambda),\]

where \( a, b \in \mathbb{R}, c \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that (1.2) holds. If \( f \) is operator convex in \([0, \infty)\), then \( a = f (0) \) and \( b = f_+ (0), \) the right derivative, in (1.11).

For a continuous and positive function \( w (\lambda), \lambda > 0 \) and a positive measure \( \mu \) on \((0, \infty)\), we can define the following mapping, which we call the \textit{convex integral transform},

\[(1.12) \quad \mathcal{C}(w, \mu) (t) := t^2 \mathcal{D}(w, \mu) (t), \quad t > 0.\]

For \( t > 0 \) we have

\[(1.13) \quad \mathcal{C}(w, \mu) (t) := \int_0^\infty w (\lambda) t^2 (t + \lambda)^{-1} d\mu (\lambda)
\]

\[= \int_0^\infty w (\lambda) (t + \lambda - \lambda)^2 (t + \lambda)^{-1} d\mu (\lambda)
\]

\[= \int_0^\infty w (\lambda) \left[ (t + \lambda)^2 - 2 \lambda (t + \lambda) + \lambda^2 \right] (t + \lambda)^{-1} d\mu (\lambda)
\]

\[= \int_0^\infty w (\lambda) \left[ (t + \lambda) - 2 \lambda + \lambda^2 (t + \lambda)^{-1} \right] d\mu (\lambda)
\]

\[= \int_0^\infty w (\lambda) \left[ t - \lambda + \lambda^2 (t + \lambda)^{-1} \right] d\mu (\lambda).
\]

If \( \int_0^\infty w (\lambda) d\mu (\lambda) < \infty \) and \( \int_0^\infty w (\lambda) \lambda d\mu (\lambda) < \infty \), then we conclude that

\[(1.14) \quad \mathcal{C}(w, \mu) (t) = -\int_0^\infty w (\lambda) \lambda d\mu (\lambda) + t \int_0^\infty w (\lambda) d\mu (\lambda) + \mathcal{D}(\ell^2 w, \mu) (t),\]

where \( \ell (t) = t, \ t > 0. \)

Consider, for instance, the kernel \( e_{-a} (\lambda) := \exp (-a \lambda), \lambda \geq 0 \) and \( a > 0. \) After some calculations, we obtain

\[\mathcal{D}(e_{-a}) (t) = \int_0^\infty \frac{\exp (-a \lambda)}{t + \lambda} d\lambda = E_1 (at) \exp (at), \ t \geq 0\]

where the \textit{exponential integral} is given by

\[E_1 (t) := \int_t^\infty \frac{e^{-u}}{u} du.\]
We also have
\[ \int_0^\infty w(\lambda)\lambda d\lambda = \int_0^\infty \exp(-a\lambda)\lambda d\lambda = \frac{1}{a^2} \]
and
\[ \int_0^\infty w(\lambda) d\lambda = \int_0^\infty \exp(-a\lambda) d\lambda = \frac{1}{a}. \]
Therefore
\[ (1.15) \quad C(e_{-a}) (t) := t^2 D(e_{-a}, \mu) (t) = t^2 E_1 (at) \exp (at), \quad t > 0. \]
Since
\[ D(t^2 e_{-a}, \mu) (t) = \int_0^\infty \lambda^2 \exp (-a\lambda) \frac{d\lambda}{t + \lambda} \]
them by (1.14) we get
\[ t^2 E_1 (at) \exp (at) = -\frac{1}{a^2} + \frac{t}{a} + D(t^2 w, \mu) (t), \]
which gives
\[ D(t^2 w, \mu) (t) = t^2 E_1 (at) \exp (at) - \frac{t}{a} + \frac{1}{a^2}, \quad t > 0, a > 0. \]
If we take \( w_r (\lambda) = \lambda^{r-1}, r \in (0, 1], \) then \( \int_0^\infty w_r (\lambda) d\lambda = \infty \) and the equality (1.14) does not hold in this case.

Using the continuous functional calculus for selfadjoint operators in Hilbert spaces we can introduce the following **convex integral transform** of the positive operator \( T \)
\[ C(w, \mu) (T) := \int_0^\infty w(\lambda) T^2 (\lambda + T)^{-1} d\mu (\lambda), \]
provided the integral exist.

2. **Main Results**

In what follows we assume that the integral transform
\[ D(w, \mu) (t) := \int_0^\infty w(\lambda) \frac{d\mu (\lambda)}{\lambda + t}, \quad t > 0, \]
defined for a continuous and positive function \( w(\lambda), \lambda > 0 \) and \( \mu \) a positive measure on \( (0, \infty), \) exists for all \( t > 0. \)

**Theorem 3.** For all \( A, B > 0 \) we have the representation
\[ (2.1) \quad \[ C(w, \mu) (B) - C(w, \mu) (A) \] (B – A) \]
\[ = \int_0^\infty w(\lambda) \left[ (B – A)^2 \right. \]
\[ - \lambda^2 \left( \int_0^1 \left[ (\lambda + (1 - t) A + t B)^{-1} (B – A) \right]^2 dt \right) \] \[ \left. \right] d\mu (\lambda) \]
and

\[(2.2) \quad (B - A) [C(w, \mu)(B) - C(w, \mu)(A)] \]
\[= \int_0^\infty w(\lambda) \left[ (B - A)^2 \right. \]
\[\left. - \lambda^2 \left( \int_0^1 [(B - A)(\lambda + (1 - t)A + tB)^{-1}]^2 dt \right) \right] d\mu(\lambda). \]

**Proof.** For \(A, B > 0\) we have, by using continuous functional calculus for selfadjoint operators and (1.13), that

\[C(w, \mu)(A) = \int_0^\infty w(\lambda) \left[ A - \lambda + \lambda^2 (A + \lambda)^{-1} \right] d\mu(\lambda) \]

and

\[C(w, \mu)(B) = \int_0^\infty w(\lambda) \left[ B - \lambda + \lambda^2 (B + \lambda)^{-1} \right] d\mu(\lambda). \]

This gives that

\[(2.3) \quad C(w, \mu)(B) - C(w, \mu)(A) \]
\[= \int_0^\infty w(\lambda) \left[ B - A + \lambda^2 \left( (B + \lambda)^{-1} - (A + \lambda)^{-1} \right) \right] d\mu(\lambda). \]

Let \(T, S > 0\). The function \(f(s) = -s^{-1}\) is operator monotone on \((0, \infty)\), operator Gâteaux differentiable and the Gâteaux derivative is given by

\[(2.4) \quad \nabla f_T(S) := \lim_{s \to 0} \left[ \frac{f(T + sS) - f(T)}{s} \right] = T^{-1}ST^{-1} \]

for \(T, S > 0\).

Consider the continuous function \(f\) defined on an interval \(I\) for which the corresponding operator function is Gâteaux differentiable on the segment \([C, D]:\{(1 - s)C + sD, s \in [0, 1]\}\) for \(C, D\) selfadjoint operators with spectra in \(I\). We consider the auxiliary function defined on \([0, 1]\) by

\[f_{C,D}(s) := f((1 - s)C + sD), \quad s \in [0, 1]. \]

Then we have, by the properties of the Bochner integral, that

\[(2.5) \quad f(D) - f(C) = \int_0^1 \frac{d}{ds} (f_{C,D}(s)) ds = \int_0^1 \nabla f_{(1-s)C+sD}(D-C) ds. \]

If we write this equality for the function \(f(s) = -s^{-1}\) and \(C, D > 0\), then we get the representation

\[(2.6) \quad C^{-1} - D^{-1} = \int_0^1 ((1 - s)C + sD)^{-1}(D - C)((1 - s)C + sD)^{-1} ds. \]
Now, if we take in (2.6) $C = \lambda + B$, $D = \lambda + A$, then
\[
(\lambda + B)^{-1} - (\lambda + A)^{-1}
\]
\[
= \int_0^1 ((1 - s) (\lambda + B) + s (\lambda + A))^{-1} (A - B)
\times ((1 - s) (\lambda + B) + s (\lambda + A))^{-1} ds
\]
\[
= \int_0^1 (\lambda + (1 - s) B + s A)^{-1} (A - B) (\lambda + (1 - s) B + s A)^{-1} ds
\]
\[
= \int_0^1 (\lambda + (1 - t) A + t B)^{-1} (A - B) (\lambda + (1 - t) A + t B)^{-1} dt,
\]
where for the last equality we used the change of variable $s = 1 - t$.

By utilising (2.3) we derive
\[
C (w, \mu) (B) - C (w, \mu) (A) = \int_0^\infty w (\lambda) [B - A
\]
\[
- \lambda^2 \left( \int_0^1 (\lambda + (1 - t) A + t B)^{-1} (A - B) (\lambda + (1 - t) A + t B)^{-1} dt \right) d\mu (\lambda).
\]

If we multiply this equality at the right with $B - A$, we obtain (2.1). If we multiply this equality at the left by $B - A$, then we obtain (2.2).

**Corollary 1.** Assume that the kernel $w \in L_1 (\mu, [0, \infty))$, namely $\int_0^\infty w (\lambda) d\mu (\lambda) < \infty$. Then we have the equality
\[
(2.7) \quad [C (w, \mu) (B) - C (w, \mu) (A)] (B - A)
\]
\[
= \left( \int_0^\infty w (\lambda) d\mu (\lambda) \right) (B - A)^2
\]
\[
+ [D (\ell^2 w, \mu) (B) - D (\ell^2 w, \mu) (A)] (B - A),
\]
and
\[
(2.8) \quad (B - A) [C (w, \mu) (B) - C (w, \mu) (A)]
\]
\[
= \left( \int_0^\infty w (\lambda) d\mu (\lambda) \right) (B - A)^2
\]
\[
+ (B - A) [D (\ell^2 w, \mu) (B) - D (\ell^2 w, \mu) (A)],
\]
where $\ell (\lambda) = \lambda, \lambda \geq 0$.

In the following, in order to simplify terminology, when we write $T \geq 0$ we automatically assume that the operator $T$ is selfadjoint.

We need the following lemmas:

**Lemma 1.** Let $A, B > 0$. The following statements are equivalent:

(i) For all $s \geq 0$,
\[
(2.9) \quad (A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) \geq 2.
\]

(ii) For all $s \geq 0$,
\[
\int_0^1 \left( ((1 - t) A + t B + s)^{-1} (B - A) \right)^2 dt \geq 0.
\]
(iii) For all $s \geq 0$,
\[
(\ell_s(B) - \ell_s(A))(B - A) \geq 0,
\]
where $\ell_s(t) = -(t + s)^{-1}, t > 0$.

Proof. From (2.14) we have, by multiplying at right with $B - A$ that
\[
\left[(A + s)^{-1} - (B + s)^{-1}\right](B - A)
= \int_0^1 ((1 - t) A + tB + s)^{-1} (B - A) ((1 - t) A + tB + s)^{-1} (B - A) \, dt
= \int_0^1 \left[((1 - t) A + tB + s)^{-1} (B - A)\right]^2 \, dt
\]
for all $s \geq 0$.

Also
\[
\left[(A + s)^{-1} - (B + s)^{-1}\right](B - A)
= \left[(A + s)^{-1} - (B + s)^{-1}\right][B + s - (A + s)]
= (A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) - 2
\]
for all $s \geq 0$.

Therefore
\[
(\ell_s(B) - \ell_s(A))(B - A)
= (A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) - 2
= \int_0^1 \left[((1 - t) A + tB + s)^{-1} (B - A)\right]^2 \, dt
\]
for all $s \geq 0$.

The identity (2.10) reveals that the statements (i), (ii) and (iii) are equivalent.

In the recent note [3] Fujii and Nakamoto proved the following inequality:

Lemma 2. If $C, D > 0$ and $CD^{-1} + DC^{-1}$ is selfadjoint, then
\[
CD^{-1} + DC^{-1} \geq 2.
\]

Indeed, as shown in [3], if we put $T = CD^{-1}$, then $V = T + T^{-1}$ is selfadjoint by the assumption. Note that the spectrum $\text{Sp}(T)$ of $T$ is included in $(0, \infty)$, because $C, D > 0$ and $\text{Sp}(T) = \text{Sp} \left( C^{1/2}D^{-1}C^{1/2}\right)$. Since $\text{Sp}(V) = \left\{t + \frac{1}{t}, \ t \in \text{Sp}(T)\right\}$ by the spectral mapping theorem for rational functions, hence we have $T + T^{-1} \geq 2$.

As a consequence, they proved that, if
\[
(i') \ Operator \ A(B + s)^{-1} + B(A + s)^{-1} \ is \ selfadjoint \ for \ all \ s \geq 0,
\]
then
\[
(B - A)(f(B) - f(A)) \geq 0
\]
for all $f$ operator monotone functions on $(0, \infty)$.

Lemma 3. Let $A, B > 0$, then the statements (i) and (i') are equivalent.
Proof. Notice that for all $s \geq 0$,
\begin{equation}
(A + s)^{-1} (B + s) + (B + s)^{-1} (A + s) = (A + s)^{-1} B + (B + s)^{-1} A + s (A + s)^{-1} + s (B + s)^{-1}.
\end{equation}

Also, the operator $s (A + s)^{-1} + s (B + s)^{-1}$ is selfadjoint for $s \geq 0$.

If the statement $(i)$ holds, then $(A + s)^{-1} (B + s) + (B + s)^{-1} (A + s)$ is selfadjoint and by (1.13) we must have that $(A + s)^{-1} B + (B + s)^{-1} A$ is selfadjoint, which shows that
\[
\left( (A + s)^{-1} B + (B + s)^{-1} A \right)^* = B (A + s)^{-1} + A (B + s)^{-1}
\]
is selfadjoint, namely $(i')$ is true.

If the statement $(i')$ holds, then by (2.12) we get
\[
(A + s)^{-1} (B + s) + (B + s)^{-1} (A + s)
\]
is selfadjoint and by (2.11) for $C = (A + s)^{-1}$, $D = (B + s)^{-1}$ we obtain the inequality (2.9), namely $(i)$ is true.

We define the class of operators
\[
\mathcal{C}_{1(0, \infty)} (H) := \{ (A, B) \mid A, B > 0 \text{ and satisfy condition } (i') \}.
\]

We observe that if $(A, B) \in \mathcal{C}_{1(0, \infty)} (H)$ then $(B, A) \in \mathcal{C}_{1(0, \infty)} (H)$.

Also if $AB = BA$, $A, B > 0$, then $U_s := (A + s)^{-1} (B + s)$ and $U^{-1}_s = (B + s)^{-1} (A + s)$ are selfadjoint and since $U_s + U^{-1}_s \geq 2$, $s \geq 0$ we derive that $(A, B) \in \mathcal{C}_{1(0, \infty)} (H)$. Therefore, if $\mathcal{C}_{0(0, \infty)} (H)$ is the class of all pairs of commutative operators $A$, $B > 0$, then we have
\begin{equation}
0 \neq \mathcal{C}_{0(0, \infty)} (H) \subset \mathcal{C}_{1(0, \infty)} (H).
\end{equation}

We have:

**Theorem 4.** Assume that the kernel $w \in L_1 (\mu, [0, \infty))$. Then for all $(A, B) \in \mathcal{C}_{1(0, \infty)} (H),
\begin{equation}
[C(w, \mu) (B) - C(w, \mu) (A)] (B - A) = (B - A) [C(w, \mu) (B) - C(w, \mu) (A)] \leq \left( \int_0^\infty w(\lambda) \, d\mu(\lambda) \right) (B - A)^2.
\end{equation}

Proof. If $(A, B) \in \mathcal{C}_{1(0, \infty)} (H)$, then by Lemma 1, then
\[
\int_0^1 \left( ((1-t) A + tB + s)^{-1} (B - A) \right)^2 \, dt \geq 0
\]
for all $s \geq 0$.

By taking the integral, we get
\[
\int_0^\infty \lambda^2 \left( \int_0^1 \left[ (\lambda + (1-t) A + tB)^{-1} (B - A) \right]^2 \, dt \right) \, d\mu(\lambda) \geq 0.
\]
By the identity (2.1)
\[
[C(w, \mu)(B) - C(w, \mu)(A)](B - A)
= \int_0^\infty w(\lambda) \left( (B - A)^2 \right)
- \int_0^1 \lambda^2 \left( \int_0^1 \left( (\lambda + (1 - t) A + tB)^{-1} (B - A) \right)^2 dt \right) d\mu(\lambda)
\leq \int_0^\infty w(\lambda) d\mu(\lambda) (B - A)^2
\]
and \( U := [C(w, \mu)(B) - C(w, \mu)(A)](B - A) \) is selfadjoint.

Since
\[
U^* := \left[ [C(w, \mu)(B) - C(w, \mu)(A)](B - A) \right]^*
= (B - A)^* [C(w, \mu)(B) - C(w, \mu)(A)]^*
= (B - A) [C(w, \mu)(B) - C(w, \mu)(A)]
\]
then \( U^* = U \) is equivalent to the identity in (2.14) and the proof is completed. \( \Box \)

**Remark 1.** If we consider the transform from the introduction (1.15),
\[
C(e_{-a})(t) = t^2 E_1(at) \exp(at), \ t > 0, \ a > 0.
\]
If \((A, B) \in \mathcal{C}_{(0, \infty)}(H)\), then we also have the operator inequality
\[
[B^2 E_1(aB) \exp(aB) - A^2 E_1(aA) \exp(aA)](B - A) \leq \frac{1}{a} (B - A)^2.
\]

The case of operator monotone functions is as follows:

**Proposition 1.** Assume that the function \( f : [0, \infty) \to \mathbb{R} \) is operator monotone in \([0, \infty)\) and has the representation (1.9) with \( b \geq 0 \) and a positive measure \( \mu \) on \((0, \infty)\) such that the expectation \( E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty\), then for \((A, B) \in \mathcal{C}_{(0, \infty)}(H)\)
\[
B^2 f(B) + A^2 f(A) - f(A) AB - f(B) BA
\leq [E(\mu) + f(0)](B - A)^2 + b (B^2 - A^2) (B - A).
\]

**Proof.** From (1.9) we get by multiplying with \( t > 0 \) that
\[
t f(t) = at + bt^2 + t^2 \int_0^\infty \frac{\lambda}{t + \lambda} d\mu(\lambda) = at + bt^2 + C(\ell, \mu)(t),
\]
namely
\[
C(\ell, \mu)(t) = t f(t) - f(0) t - bt^2.
\]

Since
\[
C(\ell, \mu)(B) - C(\ell, \mu)(A)
= B f(B) - A f(A) - f(0)(B - A) - b (B^2 - A^2)
\]
hence by (2.14) we get
\[
[B f(B) - A f(A) - f(0)(B - A) - b (B^2 - A^2)](B - A)
\leq E(\mu)(B - A)^2,
\]
which is equivalent to (2.17). \( \Box \)
The case of operator convex functions is as follows:

**Proposition 2.** Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ and has the representation (1.11) with $c \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ such that the expectation $E(\mu) := \int_0^\infty \lambda d\mu(\lambda) < \infty$, then for $(A, B) \in \mathcal{C}_{(0, \infty)} (H)$

\begin{equation}
(2.18) \quad f(B)B + f(A)A - f(A)B - f(B)A \\
\leq \left[ E(\mu) + f'_+(0) \right] (B - A)^2 + c(B^2 - A^2) (B - A).
\end{equation}

**Proof.** From (1.11) we get

\[ C(\ell, \mu)(t) = f(t) - f(0) - f'_+(0) t - ct^2. \]

Since

\[ C(\ell, \mu)(B) - C(\ell, \mu)(A) = f(B) - f(A) - f'_+(0) (B - A) - c(B^2 - A^2) \]

hence by (2.14) we obtain

\[ \left[ f(B) - f(A) - f'_+(0) (B - A) - c(B^2 - A^2) \right] (B - A) \leq E(\mu) (B - A)^2, \]

which is equivalent to (2.18). \hfill \Box

### 3. Inequalities for Čebyshev’s Simple Functional

For a continuous function $f$ on $(0, \infty)$, an $n$-tuple of positive operators $A = (A_1, \ldots, A_n)$ and a probability distribution $p = (p_1, \ldots, p_n)$ we consider the Čebyshev simple functional defined by

\[ \mathcal{C}(f, A, p) = \sum_{k=1}^n p_k A_k f(A_k) - \sum_{k=1}^n p_k f(A_k) \sum_{k=1}^n p_k A_k. \]

For the identity function $\ell(t) = t, t \geq 0$ we have

\[ \mathcal{C}(\ell, A, p) := \sum_{k=1}^n p_k A_k^2 - \left( \sum_{k=1}^n p_k A_k \right)^2. \]

**Theorem 5.** Assume that the kernel $w \in L_1(\mu, [0, \infty))$. For any $n$-tuple of positive operators $A = (A_1, \ldots, A_n)$ with $(A_j, A_k) \in \mathcal{C}_{(0, \infty)} (H)$ for all $j, k = 1, \ldots, n$ and probability distribution $p = (p_1, \ldots, p_n)$ we have

\begin{equation}
(3.1) \quad \mathcal{C}(w, \mu), A, p) \leq \left( \int_0^\infty w(\lambda) d\mu(\lambda) \right) \mathcal{C}(\ell, A, p).
\end{equation}

**Proof.** From (2.14) we get

\begin{equation}
(3.2) \quad C(w, \mu)(B)B + C(w, \mu)(A)A - C(w, \mu)(B)A - C(w, \mu)(A)B \\
\leq \left( \int_0^\infty w(\lambda) d\mu(\lambda) \right) (B^2 + A^2 - BA - AB)
\end{equation}

for all $A, B > 0$.\[ \square \]
If we take $B = A_i$, $A = A_j$, $i, j \in \{1, \ldots, n\}$ in (3.2), then we get

$$C(w, \mu) (A_i) A_i + C(w, \mu) (A_j) A_j - C(w, \mu) (A_i) A_j - C(w, \mu) (A_j) A_i$$

$$\leq \left( \int_0^\infty w (\lambda) d\mu (\lambda) \right) (A_i^2 + A_j^2 - A_i A_j - A_j A_i)$$

for all $i, j \in \{1, \ldots, n\}$.

If we multiply (3.3) by $p_i p_j \geq 0$, sum over $i, j$ from 1 to $n$ and take into account that $\sum_{j=1}^n p_j = \sum_{i=1}^n p_i = 1$, we get

$$\sum_{i=1}^n p_i C(w, \mu) (A_i) A_i + \sum_{j=1}^n p_j C(w, \mu) (A_j) A_j$$

$$- \sum_{i=1}^n p_i C(w, \mu) (A_i) \sum_{j=1}^n p_j A_j - \sum_{j=1}^n p_j C(w, \mu) (A_j) \sum_{i=1}^n p_i A_i$$

$$\leq \left( \int_0^\infty w (\lambda) d\mu (\lambda) \right)$$

$$\times \left( \sum_{i=1}^n p_i A_i^2 + \sum_{j=1}^n p_j A_j^2 - \sum_{i=1}^n p_i A_i \sum_{j=1}^n p_j A_j - \sum_{j=1}^n p_j A_j \sum_{i=1}^n p_i A_i \right),$$

which is equivalent to the first inequality in (3.1).

The proof of second inequality is similar. \(\square\)

The case of operator monotone functions is as follows:

**Corollary 2.** Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ and has the representation (1.9) with $b \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ such that the expectation $E (\mu) := \int_0^\infty \lambda d\mu (\lambda) < \infty$, then for any $n$-tuple of positive operators $A = (A_1, \ldots, A_n)$ with $(A_j, A_k) \in \mathcal{C}_{(0,\infty)} (H)$ for all $j, k = 1, \ldots, n$ and probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$,

$$\mathcal{C} (\ell f, A, \mathbf{p}) \leq [E (\mu) + f (0)] \mathcal{C} (\ell, A, \mathbf{p}) + b \mathcal{C} (\ell^2, A, \mathbf{p})$$

where $\ell (\lambda) = \lambda$, $\lambda \geq 0$ and

$$\mathcal{C} (\ell^2, A, \mathbf{p}) = \sum_{k=1}^n p_k A_k^2 + \sum_{k=1}^n p_k A_k^2 \sum_{k=1}^n p_k A_k.$$

The case of operator convex functions is as follows:

**Corollary 3.** Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ and has the representation (1.11) with $c \geq 0$ and a positive measure $\mu$ on $(0, \infty)$ such that the expectation $E (\mu) := \int_0^\infty \lambda d\mu (\lambda) < \infty$, then for any $n$-tuple of positive operators $A = (A_1, \ldots, A_n)$ with $(A_j, A_k) \in \mathcal{C}_{(0,\infty)} (H)$ for all $j, k = 1, \ldots, n$ and probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$,

$$\mathcal{C} (f, A, \mathbf{p}) \leq [E (\mu) + f' (0)] \mathcal{C} (\ell, A, \mathbf{p}) + b \mathcal{C} (\ell^2, A, \mathbf{p})$$

**Remark 2.** For any $n$-tuple of positive operators $A = (A_1, \ldots, A_n)$ $(A_j, A_k) \in \mathcal{C}_{(0,\infty)} (H)$ for all $j, k = 1, \ldots, n$ and a probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$, we have

$$\mathcal{C} (\ell^2 E_1 (a \cdot) \exp (a \cdot), A, \mathbf{p}) \leq \frac{1}{a} \mathcal{C} (\ell, A, \mathbf{p}).$$
4. More Examples of Transform with Finite $\int_0^\infty w(\lambda) d\lambda$

We define the upper incomplete Gamma function as [11]

$$\Gamma(a, z) := \int_z^\infty t^{a-1} e^{-t} dt,$$

which for $z = 0$ gives Gamma function

$$\Gamma(a) := \int_0^\infty t^{a-1} e^{-t} dt \text{ for } \Re a > 0.$$

We have the integral representation [12]

$$\Gamma(a, z) = \frac{z^a e^{-z}}{\Gamma(1-a)} \int_0^\infty \frac{t^{-a} e^{-t}}{z + t} dt \quad \text{for } \Re a < 1 \text{ and } |\text{ph } z| < \pi.$$

Now, we consider the weight $w_{-a e^{-\lambda}} := \lambda^{-a} e^{-\lambda}$ for $\lambda > 0$. Then by (4.1) we obtain

$$\mathcal{D}(w_{-a e^{-\lambda}})(t) = \int_0^\infty \frac{\lambda^{-a} e^{-\lambda}}{t + \lambda} d\lambda = \Gamma(1 - a) t^{-a} e^t \Gamma(a, t)$$

for $a < 1$ and $t > 0$.

Define

$$C(w_{-a e^{-\lambda}}) := \int_0^\infty \lambda^{-a} e^{-\lambda} d\lambda = \int_0^\infty \lambda^{1-a-1} e^{-\lambda} d\lambda = \Gamma(1 - a)$$

for $a < 1$.

For $a = 0$ in (4.2) we get

$$\mathcal{D}(w_{e^{-\lambda}})(t) = \int_0^\infty \frac{e^{-\lambda}}{t + \lambda} d\lambda = \Gamma(1) e^t \Gamma(0, t) = e^t E_1(t)$$

for $t > 0$, where the exponential integral $E_1$ is defined by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

For $a = 0$ in (4.3) we get

$$C(w_{e^{-\lambda}}) := \int_0^\infty e^{-\lambda} d\lambda = \Gamma(1) = 1.$$

Let $a = 1 - n$, with $n$ a natural number with $n \geq 0$, then by (4.2) we have

$$\mathcal{D}(w_{n-1 e^{-\lambda}})(t) = \int_0^\infty \frac{\lambda^{n-1} e^{-\lambda}}{t + \lambda} d\lambda = \Gamma(n) t^{n-1} e^t \Gamma(1 - n, t)$$

$$= (n - 1)! t^{n-1} e^t \Gamma(1 - n, t).$$

If we define the generalized exponential integral [13] by

$$E_p(z) := z^{p-1} \Gamma(1 - p, z) = z^{p-1} \int_z^\infty \frac{e^{-t}}{t^p} dt$$

then

$$t^{n-1} \Gamma(1 - n, t) = E_n(t)$$

for $n \geq 1$ and $t > 0$. 
Using the identity [13, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = (-z)^{n-1} \frac{1}{(n-1)!} E_1(z) + \frac{e^{-z}}{(n-1)!} \sum_{k=0}^{n-2} (n-k-2)! (-z)^k,$$

we then obtain

$$(4.7) \quad D (w_{n-1}e^{-}) (t) = (n-1)! e^t \frac{1}{(n-1)!} E_1(t)$$

for $n \geq 2$ and $t > 0$.

For $n \geq 2$ we get

$$(4.8) \quad D (w_{n}e^{-}) (t) = \int_0^\infty \lambda^{n-1} e^{-\lambda} d\lambda = \Gamma(n) = (n-1)!.$$ 

For $n=2$, we derive by (4.7) that

$$(4.9) \quad C (w_{n}e^{-}) := \int_0^\infty \lambda e^{-\lambda} d\lambda = \Gamma(0) = 1.$$ 

**Proposition 3.** Let $a < 1$, then for all $(A, B) \in C_{(0, \infty)}(H)$,

$$(4.10) \quad [B^{2-a} \exp(B) \Gamma(a, B) - A^{2-a} \exp(A) \Gamma(a, A)] (B - A) \leq (B - A)^2.$$ 

In particular

$$(4.11) \quad [B^{2-a} \exp(B) E_1(B) - A^{2-a} \exp(A) E_1(A)] (B - A) \leq (B - A)^2.$$ 

**Proof.** It follows by Theorem 4 for the operator convex transform

$$C (w_{n-1}e^{-}) (t) = t^2 D (w_{n-1}e^{-}) (t) = \Gamma(1-a) t^{2-a} e^t \Gamma(a, t).$$

We can also consider the weight $w_{(\lambda^2 + a^2)^{-1}} (\lambda) := \frac{1}{\lambda^2 + a^2}$ for $\lambda > 0$ and $a > 0$. Then, by simple calculations, we get

$$(4.7) \quad D (w_{(\lambda^2 + a^2)^{-1}}) (t) := \int_0^\infty \frac{1}{(\lambda + t)(\lambda^2 + a^2)} d\lambda$$

$$= \frac{\pi t}{2a} \left[ \frac{\pi t}{2a} \ln \left( \frac{t}{a} \right) \right]$$

for $t > 0$ and $a > 0$. We have

$$D (w_{(\lambda^2 + a^2)^{-1}}) := \int_0^\infty \frac{1}{\lambda^2 + a^2} d\lambda = \frac{\pi}{2a}.$$
For $a = 1$ we also have

$$D \left( w_{(2a+1)^{-1}} \right) (t) := \int_{0}^{\infty} \frac{1}{(\lambda + t)(\lambda^2 + 1)} d\lambda = \frac{1}{t^2 + 1} \left( \frac{\pi t}{2} - \ln t \right)$$

for $t > 0$. In this case

$$D \left( w_{(2a+1)^{-1}} \right) := \int_{0}^{\infty} \frac{1}{\lambda^2 + 1} d\lambda = \frac{\pi}{2}.$$

By making use of Theorem 3 we can obtain some similar inequalities. The details are omitted.

The interested reader can obtain other similar results by employing the examples of operator monotone/convex functions from [2]-[6], [9]-[10] and the references therein.

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