REVERSES OF DAVIS-CHOI-JENSEN'S INEQUALITY FOR THE CONVEX INTEGRAL TRANSFORM

SILVESTRU SEVER DRAGOMIR[1,2]

ABSTRACT. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *convex integral transform*

$$\mathcal{C}(w, \mu)(t) := \int_0^\infty w(\lambda) t^2 (\lambda + t)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for $t > 0$.

Let H and K be to Hilbert spaces. In this paper we show among others that, if the linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive, preserves the operator order and is normalised while A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$ for some scalars $m < M$, then

$$\begin{aligned} 0 &\leq \Phi[\mathcal{C}(w, \mu)(A)] - \mathcal{C}(w, \mu)(\Phi(A)) \\ &\leq \frac{\mathcal{C}(w, \mu)(m) + \mathcal{C}(w, \mu)(M)}{2} - \mathcal{C}(w, \mu)\left(\frac{M + m}{2}\right). \end{aligned}$$

Some applications for operator monotone and operator convex functions are also provided. In particular, we obtain for $r \in (0, 1]$,

$$0 \leq \Phi(A^{r+1}) - \Phi^{r+1}(A) \leq \frac{m^{r+1} + M^{r+1}}{2} - \left(\frac{M + m}{2}\right)^{r+1}$$

and

$$\begin{aligned} 0 &\leq \Phi(A \ln A) - \Phi(A) \ln(\Phi(A)) \\ &\leq \frac{m \ln m + M \ln M}{2} - \frac{M + m}{2} \ln\left(\frac{M + m}{2}\right) \end{aligned}$$

provided that $M \geq A \geq m > 0$.

1. Introduction

Let H be a complex Hilbert space and $\mathcal{B}(H)$, the Banach algebra of bounded linear operators acting on H. We denote by $\mathcal{B}_h(H)$ the semi-space of all selfadjoint operators in $\mathcal{B}(H)$. We denote by $\mathcal{B}^+(H)$ the convex cone of all positive operators on H and by $\mathcal{B}^{++}(H)$ the convex cone of all positive definite operators on H.

Let H, K be complex Hilbert spaces. Following [2] (see also [7, p. 18]) we can introduce the following definition:

Definition 1. *A map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear if it is additive and homogeneous, namely*

$$\Phi(\lambda A + \mu B) = \lambda \Phi(A) + \mu \Phi(B)$$

for any λ, $\mu \in \mathbb{C}$ and A, $B \in \mathcal{B}(H)$. The linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is positive if it preserves the operator order, i.e., if $A \in \mathcal{B}^+(H)$ then $\Phi(A) \in \mathcal{B}^+(K)$. We

---

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Davis-Choi-Jensen's Inequality, Operator monotone functions, Operator convex functions, Operator inequalities, Löwner-Heinz inequality, Logarithmic operator inequalities.

RGMIA Res. Rep. Coll. 23 (2020), Art. 98, 15 pp. Received 24/07/20

write $\Phi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$. *The linear map* $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ *is normalised if it preserves the identity operator, i.e.,* $\Phi(1) = 1$. *We write* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$.

We observe that a positive linear map Φ preserves the *order relation*, namely

$$A \leq B \text{ implies } \Phi(A) \leq \Phi(B)$$

and preserves the adjoint operation $\Phi(A^*) = \Phi(A)^*$. If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and $\alpha 1 \leq A \leq \beta 1$, then $\alpha 1 \leq \Phi(A) \leq \beta 1$.

If the map $\Psi : \mathcal{B}(H) \to \mathcal{B}(K)$ is linear, positive and $\Psi(1_H) \in \mathcal{B}^{++}(K)$ then by putting $\Phi = \Psi^{-1/2}(1)\Psi\Psi^{-1/2}(1)$ we get that $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, namely it is also normalised.

A real valued continuous function f on an interval I is said to be *operator convex (concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)(1-\lambda) f(A) + \lambda f(B)$$

for all $\lambda \in [0,1]$ and for every selfadjoint operators A, $B \in \mathcal{B}(H)$ whose spectra are contained in I.

The following Jensen's type result is well known [2]:

Theorem 1 (Davis-Choi-Jensen's Inequality). *Let* $f : I \to \mathbb{R}$ *be an operator convex function on the interval* I *and* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$, *then for any selfadjoint operator* A *whose spectrum is contained in* I *we have*

$$f(\Phi(A)) \leq \Phi(f(A)). \tag{1.1}$$

We observe that if $\Psi \in \mathfrak{P}[\mathcal{B}(H), \mathcal{B}(K)]$ with $\Psi(1) \in \mathcal{B}^{++}(K)$, then by taking $\Phi = \Psi^{-1/2}(1)\Psi\Psi^{-1/2}(1)$ in (1.1) we get

$$f\left(\Psi^{-1/2}(1)\Psi(A)\Psi^{-1/2}(1)\right) \leq \Psi^{-1/2}(1)\Psi(f(A))\Psi^{-1/2}(1).$$

If we multiply both sides of this inequality by $\Psi^{1/2}(1_H)$ we get the following *Davis-Choi-Jensen's inequality for general positive linear maps:*

$$\Psi^{1/2}(1) f\left(\Psi^{-1/2}(1)\Psi(A)\Psi^{-1/2}(1)\right)\Psi^{1/2}(1) \leq \Psi(f(A)). \tag{1.2}$$

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following integral representation for the power function when $t > 0$, $r \in (0,1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.3}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t-1} + \frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.4}$$

for all $t > 0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda), \ t > 0, \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.5) exists for all $t > 0$.

For μ, the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda, \ t > 0. \tag{1.6}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \ t > 0. \tag{1.7}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda+1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\,\mathcal{D}(w_{\ln})(t), \ t > 0. \tag{1.8}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\mu(\lambda), \tag{1.9}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda+T)^{-1} d\lambda, \tag{1.10}$$

for $T > 0$.

A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

We have the following representation of operator monotone functions [10], see for instance [1, p. 144-145]:

Theorem 2. *A function $f : (0,\infty) \to \mathbb{R}$ is operator monotone in $(0,\infty)$ if and only if it has the representation*

$$f(t) = a + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{1.11}$$

where $a \in R$, $b \geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{1.12}$$

If f is operator monotone in $[0,\infty)$, then $a = f(0)$ in (1.11).

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A + \lambda B) \leq (\geq)\,(1-\lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [1, p. 147]:

Theorem 3. *A function* $f:(0,\infty)\to\mathbb{R}$ *is operator convex in* $(0,\infty)$ *if and only if it has the representation*

$$f(t)=a+bt+ct^2+\int_0^\infty \frac{t^2\lambda}{t+\lambda}d\mu(\lambda), \tag{1.13}$$

where $a,\ b\in\mathbb{R}$, $c\geq 0$ *and a positive measure* μ *on* $(0,\infty)$ *such that (1.4) holds. If* f *is operator convex in* $[0,\infty)$, *then* $a=f(0)$ *and* $b=f'_+(0)$, *the right derivative, in (1.13).*

For a continuous and positive function $w(\lambda)$, $\lambda>0$ and a positive measure μ on $(0,\infty)$, we can define the following mapping, which we call the *convex integral transform,*

$$\mathcal{C}(w,\mu)(t):=t^2\mathcal{D}(w,\mu)(t),\ t>0. \tag{1.14}$$

For $t>0$ we have

$$\begin{aligned}\mathcal{C}(w,\mu)(t)&:=\int_0^\infty w(\lambda)t^2(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)(t+\lambda-\lambda)^2(t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[(t+\lambda)^2-2\lambda(t+\lambda)+\lambda^2\right](t+\lambda)^{-1}d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[(t+\lambda)-2\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda)\\&=\int_0^\infty w(\lambda)\left[t-\lambda+\lambda^2(t+\lambda)^{-1}\right]d\mu(\lambda).\end{aligned} \tag{1.15}$$

If $\int_0^\infty w(\lambda)d\mu(\lambda)<\infty$ and $\int_0^\infty w(\lambda)\lambda d\mu(\lambda)<\infty$, then we conclude that

$$\mathcal{C}(w,\mu)(t)=-\int_0^\infty w(\lambda)\lambda d\mu(\lambda)+t\int_0^\infty w(\lambda)d\mu(\lambda)+\mathcal{D}(\ell^2w,\mu)(t), \tag{1.16}$$

where $\ell(t)=t$, $t>0$.

Consider, for instance, the kernel $e_{-a}(\lambda):=\exp(-a\lambda)$, $\lambda\geq 0$ and $a>0$. After some calculations, we obtain

$$\mathcal{D}(e_{-a})(t)=\int_0^\infty\frac{\exp(-a\lambda)}{t+\lambda}d\lambda=E_1(at)\exp(at),\ t\geq 0$$

where the *exponential integral* is given by

$$E_1(t):=\int_t^\infty\frac{e^{-u}}{u}du.$$

We also have

$$\int_0^\infty w(\lambda)\lambda d\lambda=\int_0^\infty\exp(-a\lambda)\lambda d\lambda=\frac{1}{a^2}$$

and

$$\int_0^\infty w(\lambda)d\lambda=\int_0^\infty\exp(-a\lambda)d\lambda=\frac{1}{a}.$$

Therefore

$$\mathcal{C}(e_{-a})(t):=t^2\mathcal{D}(e_{-a},\mu)(t)=t^2E_1(at)\exp(at),\ t>0. \tag{1.17}$$

Since

$$\mathcal{D}(\ell^2 e_{-a},\mu)(t) = \int_0^\infty \frac{\lambda^2 \exp(-a\lambda)}{t+\lambda} d\lambda$$

then by (1.16) we get

$$t^2 E_1(at)\exp(at) = -\frac{1}{a^2} + \frac{t}{a} + \mathcal{D}(\ell^2 w,\mu)(t),$$

which gives

$$\mathcal{D}(\ell^2 w,\mu)(t) = t^2 E_1(at)\exp(at) - \frac{t}{a} + \frac{1}{a^2},\ t>0, a>0.$$

If we take $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then $\int_0^\infty w_r(\lambda)\, d\lambda = \infty$ and the equality (1.16) does not hold in this case.

Using the continuous functional calculus for selfadjoint operators in Hilbert spaces we can introduce the following *convex integral transform* of the positive operator T

$$\mathcal{C}(w,\mu)(T) := \int_0^\infty w(\lambda)\, T^2 (\lambda+T)^{-1} d\mu(\lambda),$$

provided the integral exist.

In this paper we show among others that, if the operator A satisfies condition $M \geq A \geq m > 0$ for some scalars $m < M$, then

$$\begin{aligned} 0 &\leq \Phi[\mathcal{C}(w,\mu)(A)] - \mathcal{C}(w,\mu)(\Phi(A)) \\ &\leq \frac{\mathcal{C}(w,\mu)(m) + \mathcal{C}(w,\mu)(M)}{2} - \mathcal{C}(w,\mu)\left(\frac{M+m}{2}\right). \end{aligned}$$

Some applications for operator monotone and operator convex functions are also provided. In particular, we obtain for $r \in (0,1]$,

$$0 \leq \Phi\left(A^{r+1}\right) - \Phi^{r+1}(A) \leq \frac{m^{r+1}+M^{r+1}}{2} - \left(\frac{M+m}{2}\right)^{r+1}$$

and

$$\begin{aligned} 0 &\leq \Phi(A\ln A) - \Phi(A)\ln(\Phi(A)) \\ &\leq \frac{m\ln m + M\ln M}{2} - \frac{M+m}{2}\ln\left(\frac{M+m}{2}\right) \end{aligned}$$

provided that $M \geq A \geq m > 0$.

2. Main Results

We recall the following reverse inequality for the inverse function [7, p. 29]:

Lemma 1. *Let $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and A a positive operator on H satisfying the condition $M \geq A \geq m > 0$ for some scalars $m < M$. Then*

$$0 \leq \Phi\left(A^{-1}\right) - [\Phi(A)]^{-1} \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}. \tag{2.1}$$

We have the following main result:

Theorem 4. *Let $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and A a positive operator on H satisfying the condition $M \geq A \geq m > 0$. Then*

$$\begin{aligned} 0 &\leq \Phi[\mathcal{C}(w,\mu)(A)] - \mathcal{C}(w,\mu)(\Phi(A)) \\ &\leq \frac{\mathcal{C}(w,\mu)(m) + \mathcal{C}(w,\mu)(M)}{2} - \mathcal{C}(w,\mu)\left(\frac{M+m}{2}\right). \end{aligned} \tag{2.2}$$

Proof. We have, by the properties of $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ and of Bochner integral, that

$$\Phi[\mathcal{C}(w,\mu)(A)] = \int_0^\infty w(\lambda)\left[\Phi(A) - \lambda + \lambda^2 \Phi\left[(A+\lambda)^{-1}\right]\right] d\mu(\lambda)$$

and

$$\mathcal{C}(w,\mu)(\Phi(A)) = \int_0^\infty w(\lambda)\left[\Phi(A) - \lambda + \lambda^2 (\Phi(A)+\lambda)^{-1}\right] d\mu(\lambda).$$

These imply that

$$\begin{aligned} &\Phi[\mathcal{C}(w,\mu)(A)] - \mathcal{C}(w,\mu)(\Phi(A)) \\ &= \int_0^\infty \lambda^2 w(\lambda)\left(\Phi\left[(\lambda+A)^{-1}\right] - (\lambda+\Phi(A))^{-1}\right) d\mu(\lambda) \end{aligned} \tag{2.3}$$

for all $A > 0$.

Since the function $f(t) = t^{-1}$ is operator convex, then by (1.1) we have

$$\Phi\left[(\lambda+A)^{-1}\right] - (\lambda+\Phi(A))^{-1} \geq 0$$

for all $\lambda \geq 0$, which by multiplication with $w(\lambda) \geq 0$ and integration gives by (2.3) the first inequality in (2.2).

Since $M + \lambda \geq A + \lambda \geq m + \lambda > 0$ for all $\lambda \geq 0$, then by (2.1) we get

$$\begin{aligned} 0 &\leq \Phi\left((\lambda+A)^{-1}\right) - [\Phi(\lambda+A)]^{-1} \leq \frac{\left(\sqrt{M+\lambda} - \sqrt{m+\lambda}\right)^2}{(m+\lambda)(M+\lambda)} \\ &= \frac{(M-m)^2}{\left(\sqrt{M+\lambda} + \sqrt{m+\lambda}\right)^2 (m+\lambda)(M+\lambda)} \end{aligned} \tag{2.4}$$

for all $\lambda \geq 0$. Using the elementary inequality $\sqrt{a} + \sqrt{b} \geq \sqrt{a+b}$, a, $b \geq 0$ we deduce that

$$\left(\sqrt{M+\lambda} + \sqrt{m+\lambda}\right)^2 \geq M + m + 2\lambda$$

for $\lambda \geq 0$, which implies that

$$\frac{(M-m)^2}{\left(\sqrt{M+\lambda} + \sqrt{m+\lambda}\right)^2 (m+\lambda)(M+\lambda)} \leq \frac{(M-m)^2}{(M+m+2\lambda)(m+\lambda)(M+\lambda)}.$$

We observe that, by performing the calculations, one has the equality

$$\begin{aligned} &\frac{1}{\left(\frac{M+m}{2} + \lambda\right)(m+\lambda)(M+\lambda)} \\ &= \frac{1}{(M-m)^2}\left(\frac{1}{m+\lambda} + \frac{1}{M+\lambda} - \frac{2}{\lambda + \frac{m+M}{2}}\right), \end{aligned}$$

for $\lambda \geq 0$.

Therefore

$$\begin{aligned} (2.5) \qquad & \frac{(M-m)^2}{\left(\sqrt{M+\lambda}+\sqrt{m+\lambda}\right)^2 (m+\lambda)(M+\lambda)} \\ & \leq \frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}-\frac{2}{\lambda+\frac{m+M}{2}}\right) \\ & = \frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}\right)-\frac{1}{\lambda+\frac{m+M}{2}} \end{aligned}$$

for $\lambda \geq 0$.

If we use (2.3), then by (2.4) and (2.5) we get

$$\begin{aligned} (2.6) \qquad & \Phi\left[\mathcal{C}(w,\mu)(A)\right]-\mathcal{C}(w,\mu)(\Phi(A)) \\ & = \int_0^\infty \lambda^2 w(\lambda)\left(\Phi\left[(\lambda+T)^{-1}\right]-(\lambda+\Phi(A))^{-1}\right)d\mu(\lambda) \\ & \leq \int_0^\infty \lambda^2 w(\lambda)\left(\frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}\right)-\frac{1}{\lambda+\frac{m+M}{2}}\right)d\mu(\lambda). \end{aligned}$$

Now, observe that

$$\mathcal{C}(w,\mu)(m) = \int_0^\infty w(\lambda)\left[m-\lambda+\lambda^2(m+\lambda)^{-1}\right]d\mu(\lambda),$$

$$\mathcal{C}(w,\mu)(M) = \int_0^\infty w(\lambda)\left[M-\lambda+\lambda^2(M+\lambda)^{-1}\right]d\mu(\lambda),$$

and

$$\mathcal{C}(w,\mu)\left(\frac{M+m}{2}\right) = \int_0^\infty w(\lambda)\left[\frac{M+m}{2}-\lambda+\lambda^2\left(\frac{M+m}{2}+\lambda\right)^{-1}\right]d\mu(\lambda),$$

which gives that

$$\begin{aligned} (2.7) \qquad & \frac{\mathcal{C}(w,\mu)(m)+\mathcal{C}(w,\mu)(M)}{2}-\mathcal{C}(w,\mu)\left(\frac{M+m}{2}\right) \\ & = \int_0^\infty \lambda^2 w(\lambda)\left(\frac{1}{2}\left(\frac{1}{m+\lambda}+\frac{1}{M+\lambda}\right)-\frac{1}{\lambda+\frac{m+M}{2}}\right)d\mu(\lambda). \end{aligned}$$

By utilising (2.6) and (2.7) we derive (2.2). □

Corollary 1. *Assume that function $f:(0,\infty)\to\mathbb{R}$ is operator monotone in $(0,\infty)$ and has the representation (1.11). If $\Phi \in \mathfrak{P}_N[\mathcal{B}(H),\mathcal{B}(K)]$ and A is a positive operator on H satisfying the condition $M \geq A \geq m > 0$, then*

$$\begin{aligned} (2.8) \qquad & 0 \leq b\left[\Phi\left(A^2\right)-[\Phi(A)]^2\right] \leq \Phi[Af(A)]-\Phi(A)f(\Phi(A)) \\ & \leq \frac{mf(m)+Mf(M)}{2}-\frac{M+m}{2}f\left(\frac{M+m}{2}\right) \\ & + b\left[\Phi\left(A^2\right)-[\Phi(A)]^2\right]-\frac{1}{4}b(M-m)^2 \\ & \leq \frac{mf(m)+Mf(M)}{2}-\frac{M+m}{2}f\left(\frac{M+m}{2}\right). \end{aligned}$$

Proof. From (1.11) we get

$$tf(t) = at + bt^2 + \mathcal{C}(\ell,\mu)(t), \ t > 0,$$

namely

$$\mathcal{C}(\ell,\mu)(t) = tf(t) - at - bt^2, \ t > 0,$$

where $\ell(\lambda) = \lambda$, $a \in R$, $b \geq 0$ and μ a positive measure on $(0,\infty)$.

We have

$$\begin{aligned}
&\Phi[\mathcal{C}(\ell,\mu)(A)] - \mathcal{C}(\ell,\mu)(\Phi(A)) \\
&= \Phi[Af(A)] - a\Phi(A) - b\Phi\left(A^2\right) - \Phi(A)f(\Phi(A)) + a\Phi(A) + b[\Phi(A)]^2 \\
&= \Phi[Af(A)] - \Phi(A)f(\Phi(A)) - b\left[\Phi\left(A^2\right) - [\Phi(A)]^2\right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\mathcal{C}(w,\mu)(m) + \mathcal{C}(w,\mu)(M)}{2} - \mathcal{C}(w,\mu)\left(\frac{M+m}{2}\right) \\
&= \frac{1}{2}\left[mf(m) - am - bm^2 + Mf(M) - aM - bM^2\right] \\
&- \left[\frac{M+m}{2}f\left(\frac{M+m}{2}\right) - a\frac{M+m}{2} - b\left(\frac{M+m}{2}\right)^2\right] \\
&= \frac{1}{2}(mf(m) + Mf(M)) - \frac{M+m}{2}f\left(\frac{M+m}{2}\right) \\
&- b\left[\frac{M^2+m^2}{2} - \left(\frac{M+m}{2}\right)^2\right] \\
&= \frac{mf(m) + Mf(M)}{2} - \frac{M+m}{2}f\left(\frac{M+m}{2}\right) - \frac{1}{4}b(M-m)^2
\end{aligned}$$

and by (2.2) we get

$$\begin{aligned}
0 &\leq \Phi[Af(A)] - \Phi(A)f(\Phi(A)) - b\left[\Phi\left(A^2\right) - [\Phi(A)]^2\right] \\
&\leq \frac{mf(m) + Mf(M)}{2} - \frac{M+m}{2}f\left(\frac{M+m}{2}\right) - \frac{1}{4}b(M-m)^2
\end{aligned}$$

namely

$$\begin{aligned}
0 &\leq b\left[\Phi\left(A^2\right) - [\Phi(A)]^2\right] \leq \Phi[Af(A)] - \Phi(A)f(\Phi(A)) \\
&\leq \frac{mf(m) + Mf(M)}{2} - \frac{M+m}{2}f\left(\frac{M+m}{2}\right) \\
&+ b\left[\Phi\left(A^2\right) - [\Phi(A)]^2\right] - \frac{1}{4}b(M-m)^2.
\end{aligned}$$

Since, see for instance [7, p. 29],

$$\left[\Phi\left(A^2\right) - [\Phi(A)]^2\right] - \frac{1}{4}(M-m)^2 \leq 0,$$

then the last part of (2.8) is also proved. □

Remark 1. *If we write the inequality (2.8) for the power function* $f(t) = t^r$, $r \in (0,1]$, *then we have for* $M \geq A \geq m > 0$ *that*

$$0 \leq \Phi\left(A^{r+1}\right) - \Phi^{r+1}(A) \leq \frac{m^{r+1} + M^{r+1}}{2} - \left(\frac{M+m}{2}\right)^{r+1}. \tag{2.9}$$

The same inequality written for the operator monotone function $f(t) = \ln t$ *gives*

$$\begin{aligned} 0 &\leq \Phi(A \ln A) - \Phi(A) \ln(\Phi(A)) \\ &\leq \frac{m \ln m + M \ln M}{2} - \frac{M+m}{2} \ln\left(\frac{M+m}{2}\right) \end{aligned} \tag{2.10}$$

provided that $M \geq A \geq m > 0$.

Corollary 2. *Assume that function* $f : (0,\infty) \to \mathbb{R}$ *is operator convex in* $(0,\infty)$ *and has the representation (1.13). If* $\Phi \in \mathfrak{P}_N[\mathcal{B}(H), \mathcal{B}(K)]$ *and* A *is a positive operator on* H *satisfying the condition* $M \geq A \geq m > 0$, *then*

$$\begin{aligned} 0 &\leq c\left[\Phi\left(A^2\right) - [\Phi(A)]^2\right] \leq \Phi[f(A)] - f(\Phi(A)) \\ &\leq \frac{f(m) + f(M)}{2} - f\left(\frac{M+m}{2}\right) \\ &+ c\left[\Phi\left(A^2\right) - [\Phi(A)]^2\right] - \frac{1}{4}c(M-m)^2 \\ &\leq \frac{f(m) + f(M)}{2} - f\left(\frac{M+m}{2}\right). \end{aligned} \tag{2.11}$$

Proof. From (1.13) we have

$$f(t) = a + bt + ct^2 + \mathcal{C}(\ell,\mu)(t),\ t > 0,$$

namely

$$\mathcal{C}(\ell,\mu)(t) = f(t) - a - bt - ct^2,\ t > 0,$$

where a, $b \in \mathbb{R}$, $c \geq 0$ and a positive measure μ on $(0,\infty)$.

From this we get

$$\begin{aligned} &\Phi[\mathcal{C}(\ell,\mu)(A)] - \mathcal{C}(\ell,\mu)(\Phi(A)) \\ &= \Phi[f(A)] - a - b\Phi(A) - c\Phi\left(A^2\right) - f(\Phi(A)) + a + b\Phi(A) + c[\Phi(A)]^2 \\ &= \Phi[f(A)] - f(\Phi(A)) - c\left[\Phi\left(A^2\right) - [\Phi(A)]^2\right] \end{aligned}$$

and

$$\begin{aligned} &\frac{\mathcal{C}(w,\mu)(m) + \mathcal{C}(w,\mu)(M)}{2} - \mathcal{C}(w,\mu)\left(\frac{M+m}{2}\right) \\ &= \frac{1}{2}\left[f(m) - a - bm - cm^2 + Mf(M) - a - bM - cM^2\right] \\ &- \left[f\left(\frac{M+m}{2}\right) - a - b\frac{M+m}{2} - c\left(\frac{M+m}{2}\right)^2\right] \\ &= \frac{1}{2}(f(m) + f(M)) - f\left(\frac{M+m}{2}\right) - c\left[\frac{M^2+m^2}{2} - \left(\frac{M+m}{2}\right)^2\right] \\ &= \frac{f(m) + f(M)}{2} - f\left(\frac{M+m}{2}\right) - \frac{1}{4}c(M-m)^2. \end{aligned}$$

By (2.2) we get

$$\begin{aligned} 0 &\leq \Phi\left[f\left(A\right)\right]-f\left(\Phi\left(A\right)\right)-c\left[\Phi\left(A^{2}\right)-\left[\Phi\left(A\right)\right]^{2}\right] \\ &\leq \frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{M+m}{2}\right)-\frac{1}{4}c\left(M-m\right)^{2}. \end{aligned}$$

namely

$$\begin{aligned} 0 &\leq c\left[\Phi\left(A^{2}\right)-\left[\Phi\left(A\right)\right]^{2}\right] \leq \Phi\left[f\left(A\right)\right]-f\left(\Phi\left(A\right)\right) \\ &\leq \frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{M+m}{2}\right) \\ &\quad + c\left[\Phi\left(A^{2}\right)-\left[\Phi\left(A\right)\right]^{2}\right]-\frac{1}{4}c\left(M-m\right)^{2} \end{aligned}$$

and since

$$c\left[\Phi\left(A^{2}\right)-\left[\Phi\left(A\right)\right]^{2}\right]-\frac{1}{4}c\left(M-m\right)^{2}\leq 0,$$

hence (2.11) is proved. □

Remark 2. *Let $a>0$ and $f\left(t\right)=\left(t+a\right)^{p}$ with $p\in\left[-1,0\right)\cup\left[1,2\right]$. From (2.11) we get*

$$\begin{aligned} 0 &\leq \Phi\left[\left(A+a\right)^{p}\right]-\left(\Phi\left(A\right)+a\right)^{p} \\ &\leq \frac{\left(m+a\right)^{p}+\left(M+a\right)^{p}}{2}-\left(\frac{M+m}{2}+a\right)^{p}, \end{aligned} \tag{2.12}$$

provided that $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$ and A is a positive operator on H satisfying the condition $M\geq A\geq m>0$.

From (2.12) we get for $p=-1$ that

$$\begin{aligned} 0 &\leq \Phi\left[\left(A+a\right)^{-1}\right]-\left(\Phi\left(A\right)+a\right)^{-1} \\ &\leq \frac{\left(m+a\right)^{-1}+\left(M+a\right)^{-1}}{2}-\left(\frac{M+m}{2}+a\right)^{-1}. \end{aligned} \tag{2.13}$$

3. Further Results

We also have:

Theorem 5. *Let $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}\left(H\right),\mathcal{B}\left(K\right)\right]$ and A a positive operator on H satisfying the condition $M\geq A\geq m>0$. If $\int_{0}^{\infty}w\left(\lambda\right)d\mu\left(\lambda\right)<\infty$, then*

$$\begin{aligned} 0 &\leq \Phi\left[\mathcal{C}\left(w,\mu\right)\left(A\right)\right]-\mathcal{C}\left(w,\mu\right)\left(\Phi\left(A\right)\right) \\ &\leq \frac{\left(M-m\right)^{2}}{\left(\sqrt{M}+\sqrt{m}\right)^{2}}\left[\int_{0}^{\infty}w\left(\lambda\right)d\mu\left(\lambda\right)-\frac{\mathcal{C}\left(w,\mu\right)\left(M\right)-\mathcal{C}\left(w,\mu\right)\left(m\right)}{M-m}\right]. \end{aligned} \tag{3.1}$$

Proof. From (2.4) we get Since $M+\lambda \geq A+\lambda \geq m+\lambda > 0$ for all $\lambda \geq 0$, then by (2.1) we get

$$0 \leq \Phi\left((\lambda + A)^{-1}\right) - [\Phi(\lambda + A)]^{-1} \leq \frac{\left(\sqrt{M+\lambda} - \sqrt{m+\lambda}\right)^2}{(m+\lambda)(M+\lambda)} \tag{3.2}$$

$$= \frac{(M-m)^2}{\left(\sqrt{M+\lambda} + \sqrt{m+\lambda}\right)^2 (m+\lambda)(M+\lambda)}$$

$$\leq \frac{(M-m)^2}{\left(\sqrt{M} + \sqrt{m}\right)^2 (m+\lambda)(M+\lambda)}.$$

If we multiply by $\lambda^2 w(\lambda) \geq 0$ and integrate, then we get

$$0 \leq \int_0^\infty \lambda^2 w(\lambda) \left(\Phi\left((\lambda + A)^{-1}\right) - [\Phi(\lambda + A)]^{-1}\right) d\mu(\lambda) \tag{3.3}$$

$$\leq \int_0^\infty \frac{(M-m)^2 \lambda^2 w(\lambda) \, d\mu(\lambda)}{\left(\sqrt{M} + \sqrt{m}\right)^2 (m+\lambda)(M+\lambda)}$$

$$= \frac{(M-m)^2}{\left(\sqrt{M} + \sqrt{m}\right)^2} \int_0^\infty \frac{\lambda^2 w(\lambda) \, d\mu(\lambda)}{(m+\lambda)(M+\lambda)}$$

which gives, by (2.3), that

$$\Phi[\mathcal{C}(w,\mu)(A)] - \mathcal{C}(w,\mu)(\Phi(A)) \tag{3.4}$$

$$\leq \frac{(M-m)^2}{\left(\sqrt{M} + \sqrt{m}\right)^2} \int_0^\infty \frac{\lambda^2 w(\lambda) \, d\mu(\lambda)}{(m+\lambda)(M+\lambda)}.$$

Now, observe that

$$\frac{\lambda^2}{(m+\lambda)(M+\lambda)} = 1 - \frac{(m+M)\lambda + mM}{(m+\lambda)(M+\lambda)}$$

$$= 1 + \frac{m^2}{M-m}\frac{1}{\lambda+m} - \frac{M^2}{M-m}\frac{1}{\lambda+M}$$

for $\lambda \geq 0$.

If we multiply this equality and integrate, then we get

$$\int_0^\infty \frac{\lambda^2 w(\lambda) \, d\mu(\lambda)}{(m+\lambda)(M+\lambda)}$$

$$= \int_0^\infty w(\lambda) \, d\mu(\lambda) + \frac{m^2}{M-m} \int_0^\infty \frac{w(\lambda) \, d\mu(\lambda)}{\lambda+m} - \frac{M^2}{M-m} \int_0^\infty \frac{w(\lambda) \, d\mu(\lambda)}{\lambda+M}$$

$$= \int_0^\infty w(\lambda) \, d\mu(\lambda) + \frac{m^2}{M-m}\mathcal{D}(w,\mu)(m) - \frac{M^2}{M-m}\mathcal{D}(w,\mu)(M)$$

$$= \int_0^\infty w(\lambda) \, d\mu(\lambda) + \frac{1}{M-m}\mathcal{C}(w,\mu)(m) - \frac{1}{M-m}\mathcal{C}(w,\mu)(M)$$

$$= \int_0^\infty w(\lambda) \, d\mu(\lambda) - \frac{\mathcal{C}(w,\mu)(M) - \mathcal{C}(w,\mu)(m)}{M-m},$$

then by (3.4) we get (3.1). □

We define the *upper incomplete Gamma function* as [14]

$$\Gamma(a,z) := \int_z^\infty t^{a-1}e^{-t}dt,$$

which for $z=0$ gives *Gamma function*

$$\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt \text{ for } \operatorname{Re} a > 0.$$

We have the integral representation [15]

$$\Gamma(a,z) = \frac{z^a e^{-z}}{\Gamma(1-a)}\int_0^\infty \frac{t^{-a}e^{-t}}{z+t}dt \tag{3.5}$$

for $\operatorname{Re} a < 1$ and $|\operatorname{ph} z| < \pi$.

Now, we consider the weight $w_{\cdot^{-a}e^{-\cdot}}(\lambda) := \lambda^{-a}e^{-\lambda}$ for $\lambda > 0$. Then by (3.5) we obtain

$$\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t) = \int_0^\infty \frac{\lambda^{-a}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1-a)t^{-a}e^t\Gamma(a,t) \tag{3.6}$$

for $a<1$ and $t>0$.

Define

$$C\left(w_{\cdot^{-a}e^{-\cdot}}\right) := \int_0^\infty \lambda^{-a}e^{-\lambda}d\lambda = \int_0^\infty \lambda^{1-a-1}e^{-\lambda}d\lambda = \Gamma(1-a) \tag{3.7}$$

for $a<1$.

For $a=0$ in (3.6) we get

$$\mathcal{D}\left(w_{e^{-\cdot}}\right)(t) = \int_0^\infty \frac{e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(1)e^t\Gamma(0,t) = e^tE_1(t) \tag{3.8}$$

for $t>0$, where the *exponential integral* E_1 is defined by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u}du.$$

For $a=0$ in (3.7) we get

$$C\left(w_{e^{-\cdot}}\right) := \int_0^\infty e^{-\lambda}d\lambda = \Gamma(1) = 1. \tag{3.9}$$

Let $a = 1-n$, with n a natural number with $n \geq 0$, then by (3.6) we have

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= \int_0^\infty \frac{\lambda^{n-1}e^{-\lambda}}{t+\lambda}d\lambda = \Gamma(n)t^{n-1}e^t\Gamma(1-n,t) \\ &= (n-1)!t^{n-1}e^t\Gamma(1-n,t).\end{aligned} \tag{3.10}$$

If we define the generalized exponential integral [16] by

$$E_p(z) := z^{p-1}\Gamma(1-p,z) = z^{p-1}\int_z^\infty \frac{e^{-t}}{t^p}dt$$

then

$$t^{n-1}\Gamma(1-n,t) = E_n(t)$$

for $n \geq 1$ and $t > 0$.

Using the identity [16, Eq 8.19.7], for $n \geq 2$

$$E_n(z) = \frac{(-z)^{n-1}}{(n-1)!}E_1(z) + \frac{e^{-z}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!(-z)^k,$$

we then obtain

$$\begin{aligned}\mathcal{D}\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t) &= (n-1)!e^{t}E_{n}(t) \\ &= (n-1)!e^{t} \\ &\times\left[\frac{(-t)^{n-1}}{(n-1)!}E_{1}(t)+\frac{e^{-t}}{(n-1)!}\sum_{k=0}^{n-2}(n-k-2)!\,(-t)^{k}\right] \\ &= \sum_{k=0}^{n-2}(-1)^{k}(n-k-2)!t^{k}+(-1)^{n-1}t^{n-1}e^{t}E_{1}(t)\end{aligned} \tag{3.11}$$

for $n\geq 2$ and $t>0$.

For $n\geq 2$ we get

$$C\left(w_{\cdot^{n-1}e^{-\cdot}}\right)(t)=\int_{0}^{\infty}\lambda^{n-1}e^{-\lambda}d\lambda=\Gamma(n)=(n-1)!.$$

For $n=2$, we derive by (3.11) that

$$\mathcal{D}\left(w_{\cdot e^{-\cdot}}\right)(t)=\int_{0}^{\infty}\lambda e^{-\lambda}(t+\lambda)^{-1}d\lambda=1-t\exp(t)E_{1}(t) \tag{3.12}$$

for $t>0$. We also have

$$C\left(w_{\cdot e^{-\cdot}}\right):=\int_{0}^{\infty}\lambda e^{-\lambda}d\lambda=\Gamma(0)=1. \tag{3.13}$$

By making use of Corollary 1 we can then state:

Proposition 1. *Let $\Phi\in\mathfrak{P}_{N}\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and A a positive operator on H satisfying the condition $M\geq A\geq m>0$. If $a<1$, then*

$$\begin{aligned}0&\leq\Phi\left[A^{2-a}\exp(A)\Gamma(a,A)\right]-(\Phi(A))^{2-a}\exp(\Phi(A))\Gamma(a,\Phi(A)) \\ &\leq\frac{(M-m)^{2}}{\left(\sqrt{M}+\sqrt{m}\right)^{2}}\left[1-\frac{M^{2-a}e^{M}\Gamma(a,M)-m^{2-a}e^{m}\Gamma(a,m)}{M-m}\right].\end{aligned} \tag{3.14}$$

In particular, we have

$$\begin{aligned}0&\leq\Phi\left[A^{2}\exp(A)E_{1}(A)\right]-(\Phi(A))^{2}\exp(\Phi(A))E_{1}(\Phi(A)) \\ &\leq\frac{(M-m)^{2}}{\left(\sqrt{M}+\sqrt{m}\right)^{2}}\left[1-\frac{M^{2}e^{M}E_{1}(M)-m^{2}e^{m}E_{1}(m)}{M-m}\right].\end{aligned} \tag{3.15}$$

Proof. Consider the convex transform

$$\mathcal{C}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=t^{2}\mathcal{D}\left(w_{\cdot^{-a}e^{-\cdot}}\right)(t)=\Gamma(1-a)t^{2-a}e^{t}\Gamma(a,t).$$

By Theorem 5 we then obtain the desired □

We can also consider the weight $w_{(\cdot^{2}+a^{2})^{-1}}(\lambda):=\frac{1}{\lambda^{2}+a^{2}}$ for $\lambda>0$ and $a>0$. Then, by simple calculations, we get

$$\begin{aligned}\mathcal{D}\left(w_{(\cdot^{2}+a^{2})^{-1}}\right)(t)&:=\int_{0}^{\infty}\frac{1}{(\lambda+t)\left(\lambda^{2}+a^{2}\right)}d\lambda \\ &=\frac{1}{t^{2}+a^{2}}\left[\frac{\pi t}{2a}-\ln\left(\frac{t}{a}\right)\right]\end{aligned}$$

for $t>0$ and $a>0$. We have

$$D\left(w_{(\cdot^2+a^2)^{-1}}\right):=\int_0^\infty \frac{1}{\lambda^2+a^2}d\lambda=\frac{\pi}{2a}.$$

For $a=1$ we also have

$$\mathcal{D}\left(w_{(\cdot^2+1)^{-1}}\right)(t):=\int_0^\infty \frac{1}{(\lambda+t)\left(\lambda^2+1\right)}d\lambda=\frac{1}{t^2+1}\left(\frac{\pi t}{2}-\ln t\right)$$

for $t>0$. In this case

$$D\left(w_{(\cdot^2+1)^{-1}}\right):=\int_0^\infty \frac{1}{\lambda^2+1}d\lambda=\frac{\pi}{2}.$$

By making use of Corollary 1 we can then state:

Proposition 2. *Let $\Phi\in\mathfrak{P}_N\left[\mathcal{B}(H),\mathcal{B}(K)\right]$ and A a positive operator on H satisfying the condition $M\geq A\geq m>0$. If $a>0$, then*

$$0\leq\Phi\left(A^2\left(A^2+a^2\right)^{-1}\left[\frac{\pi A}{2a}-\ln\left(\frac{A}{a}\right)\right]\right)-(\Phi(A))^2\left((\Phi(A))^2+a^2\right)^{-1}\left[\frac{\pi\Phi(A)}{2a}-\ln\left(\frac{\Phi(A)}{a}\right)\right]\leq\frac{(M-m)^2}{\left(\sqrt{M}+\sqrt{m}\right)^2}\times\left[\frac{\pi}{2a}-\frac{\frac{M^2}{M^2+a^2}\left[\frac{\pi M}{2a}-\ln\left(\frac{M}{a}\right)\right]-\frac{m^2}{m^2+a^2}\left[\frac{\pi m}{2a}-\ln\left(\frac{m}{a}\right)\right]}{M-m}\right]. \tag{3.16}$$

In particular, we have

$$0\leq\Phi\left(A^2\left(A^2+1\right)^{-1}\left(\frac{\pi A}{2}-\ln A\right)\right)-(\Phi(A))^2\left((\Phi(A))^2+1\right)^{-1}\left[\frac{\pi\Phi(A)}{2}-\ln(\Phi(A))\right]\leq\frac{(M-m)^2}{\left(\sqrt{M}+\sqrt{m}\right)^2}\times\left[\frac{\pi}{2}-\frac{\frac{M^2}{M^2+1}\left(\frac{\pi M}{2}-\ln M\right)-\frac{m^2}{m^2+1}\left(\frac{\pi m}{2}-\ln m\right)}{M-m}\right]. \tag{3.17}$$

Proof. We have

$$\mathcal{C}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t)=t^2\mathcal{D}\left(w_{(\cdot^2+a^2)^{-1}}\right)(t)=\frac{t^2}{t^2+a^2}\left[\frac{\pi t}{2a}-\ln\left(\frac{t}{a}\right)\right].$$

By applying Theorem 5, we get (3.16). □

The interested reader can obtain other similar results by employing the examples of operator monotone/convex functions from [3]-[6], [12]-[13] and the references therein.

References

[1] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[2] M. D. Choi, Positive linear maps on C^*-algebras. *Canad. J. Math.* **24** (1972), 520–529.

[3] S. S. Dragomir, Operator monotonicity of an integral transform of positive operators in Hilbert spaces with applications, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 65, 15 pp. [Online https://rgmia.org/papers/v23/v23a65.pdf].

[4] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.

[5] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.

[6] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[7] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[8] K. Gustafson, Interaction antieigenvalues. *J. Math. Anal. Appl.* **299** (1) (2004), 174–185.

[9] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.

[10] K. Löwner, Über monotone MatrixFunktionen, *Math. Z.* **38** (1934) 177–216.

[11] M. S. Moslehian, H. Najafi, Around operator monotone functions, *Integr. Equ. Oper. Theory* **71** (2011), 575–582.

[12] M. S. Moslehian, H. Najafi, An extension of the Löwner-Heinz inequality, *Linear Algebra Appl.*, **437** (2012), 2359–2365.

[13] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

[14] Incomplete Gamma and Related Functions, Definitions, *Digital Library of Mathematical Functions*, NIST. [Online https://dlmf.nist.gov/8.2].

[15] Incomplete Gamma and Related Functions, Integral Representations, *Digital Library of Mathematical Functions,* NIST. [Online https://dlmf.nist.gov/8.6].

[16] Generalized Exponential Integral, *Digital Library of Mathematical Functions,* NIST. [Online https://dlmf.nist.gov/8.19#E1].

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.