HERMITE-HADAMARD TYPE INEQUALITIES FOR HYPERBOLIC TYPE CONVEX FUNCTIONS

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Abstract. Several Hermite-Hadamard type inequalities will be presented in this work for functions whose second derivative in absolute value at certain power is hyperbolic type convex.

1. Introduction

The classical inequality of Hermite-Hadamard was extended and generalized in many directions by many authors, like for example, [6, 5, 10, 1, 13, 15, 11, 12] and the references therein.

Using a recent concept of hyperbolic type convex functions given in [12] some Hermite-Hadamard type inequalities will be presented for this new kind of functions.

We begin by recalling below the classical definition for the convex functions.

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$. The function f is said to be concave on I if the inequality (1) takes place in reversed direction.

Definition 2. *(see [12]) The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called hyperbolic type convex function if for every $a, b \in I$ and $t \in [0, 1]$,*

$$f(ta + (1-t)b) \leq \left(\frac{\sinh t}{\sinh 1}\right)f(a) + \left(\frac{\sinh 1 - \sinh t}{\sinh 1}\right)f(b). \tag{2}$$

The class of all hyperbolic type convex functions on interval I is indicated by HC(I).

Basic properties of this kind of functions and Hermite-Hadamard type inequalities for functions whose first derivative in absolute value at certain power is hyperbolic type convex are presented in [12].

Date: July 25, 2020.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. Hermite-Hadamard inequality, hyperbolic type convex functions, Holder's inequality .

RGMIA Res. Rep. Coll. 23 (2020), Art. 99, 12 pp. Received 31/07/20

Definition 3. *([18]) Let $h : J \to \mathbf{R}$ be a nonnegative function and $h \neq 0$. We say that $f : I \to \mathbf{R}$ is an h-convex function, or that f belongs to the class SX(h,I), if is nonnegative and for all $m, n \in I,\ k \in [0,1]$ we have*

$$f(km + (1-k)n) \leq h(k)f(m) + h(1-k)f(n).$$

When previous inequality is reversed then f is said to be a h-concave function, i.e. $f \in SV(h, I)$. It is obvious that when $h(u) = u$ then the h-convexity becomes convexity.

For other type of convexity see also [17, 14].

The classical Hermite-Hadamard's inequality for convex functions is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

Lemma 1. *(see [3]) Let $f : I^o \to \mathbf{R},\ : I^o \subset [0,\infty)$ be a twice differentiable function on $: I^o$ where $a, b \in I,\ a < b$. If $f^{''} \in L[a,b]$, then the following equality holds:*

$$-f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)dx =$$

$$= \frac{(b-a)^2}{16}[\int_0^1 t^2 f^{''}\left(t\frac{a+b}{2} + (1-t)a\right)dt + \int_0^1 (t-1)^2 f^{''}\left(tb + (1-t)\frac{a+b}{2}\right)].$$

Lemma 2. *(see [3]) Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be a twice differentiable function on I^o, the interior of I where $a, b \in I,\ a < b$. If $f^{''} \in [a,b]$ then,*

$$-\frac{f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)}{2} + \frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{128}[\int_0^1 t^2 f^{''}\left(t\frac{3a+b}{4} + (1-t)a\right)dt +$$

$$+\int_0^1 (t-1)^2 f^{''}\left(t\frac{a+b}{2} + (1-t)\frac{3a+b}{4}\right)dt + \int_0^1 t^2 f^{''}\left(t\frac{a+3b}{4} + (1-t)\frac{a+b}{2}\right)dt +$$

$$+\int_0^1 (t-1)^2 f^{''}\left(tb + (1-t)\frac{a+3b}{4}\right)dt].$$

We will recall here the wel-known definition of the fractional integrals. Let $f \in L[a,b]$. The Riemann-Liouville integrals $J^{\alpha}_{a+}\mathrm{f}$ and $J^{\alpha}_{b-}\mathrm{f}$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x > a$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

The following result is a generalization of Lemma 1 from [4] when $\alpha > n-1$ and $n \in \mathbb{N}$.

Lemma 3. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a,b \in I^0$, $0 < a < b$. Then for any $x \in [a,b]$, we have:*

$$I(f,x,a,b,\alpha,n) = (x-a)\int_0^1 t^\alpha f^{(n)}(tx+(1-t)a)dt + (b-x)\int_0^1 (1-t)^\alpha f^{(n)}(tb+(1-t)x)dt =$$

$$= \sum_{k=2}^{n} \alpha(\alpha-1)...(\alpha-k+2) f^{(n-k)}(x)\left(\frac{(-1)^{k-1}}{(x-a)^{k-1}} - \frac{1}{(b-x)^{k-1}}\right) +$$

$$+\Gamma(\alpha+1)[\frac{(-1)^n}{(x-a)^\alpha} J_{x^-}^{\alpha-n+1} f(a) + \frac{1}{(b-x)^\alpha} J_{x^+}^{\alpha-n+1} f(b)],$$

where $\alpha > n-1$.

Theorem 1. *(Holder-Iscan integral inequality, [12]) Let $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. If f and g are real functions defined on the interval $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|dx \leq$$

$$\leq \frac{1}{b-a}\{(\int_a^b (b-x)|f(x)|dx)^{\frac{1}{p}} (\int_a^b (b-x)|f(x)||g(x)|^q dx)^{\frac{1}{q}} +$$

$$+(\int_a^b (x-a)|f(x)|dx)^{\frac{1}{q}} (\int_a^b (x-a)|f(x)||g(x)|^q dx)^{\frac{1}{q}}\}.$$

Theorem 2. *(Improved power-mean integral inequality, [12]) Let $q \geq 1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|$, $|f||g|^q$ are integrable functions on $[a,b]$ then*

$$\int_a^b |f(x)g(x)|dx \leq$$

$$\leq \frac{1}{b-a}\{(\int_a^b (b-x)|f(x)|dx)^{\frac{1}{p}} (\int_a^b (b-x)|f(x)||g(x)|^q dx)^{\frac{1}{q}} +$$

$$+(\int_a^b (x-a)|f(x)|dx)^{\frac{1}{p}} (\int_a^b (x-a)|f(x)||g(x)|^q dx)^{\frac{1}{q}}\}.$$

Some new Hermite-Hadamard type inequalities will be given in this work in the following theorems for functions whose second derivative in absolute value at certain power is hyperbolic type convex.

2. **New Hermite-Hadamard type inequalities for hyperbolic type convex functions**

The aim of this section is to present new inequalities that refine Hermite-Hadamard inequality for functions whose second derivative in absolute value at certain power is hyperbolic type convex. Moreover we use two new recent inequalities in some demonstration, Holder-Iscan integral inequality and improved power-mean integral inequality.

Theorem 3. *Let $f : I \to \mathbf{R}$, $I^o \subset \mathbf{R}$ be a twice differentiable function on I^o such that $f^{''} \in L[a, b]$, where $a, b \in I,\ a < b$. If $|f^{''}|$ is a hyperbolic type convex function on $[a, b]$ then the following inequality*

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{16\sinh 1}[(\cosh 1-\frac{5}{3}\sinh 1+1)|f^{''}\left(\frac{a+b}{2}\right)|+(\frac{7}{3}\sinh 1-3\cosh 1+2)|f^{''}(a)|+$$

$$+(2\cosh 1-3)|f^{''}(b)|]$$

takes place.

Proof. By using Lemma 1 and the properties of the modulus we will find,

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{16}\int_0^1 t^2|f^{''}\left(t\frac{a+b}{2}+(1-t)a\right)|dt+\int_0^1 (t-1)^2|f^{''}\left(tb+(1-t)\frac{a+b}{2}\right)|dt \leq$$

$$\leq \frac{(b-a)^2}{16\sinh 1}\{\int_0^1 t^2[\sinh t|f^{''}\left(\frac{a+b}{2}\right)|+(\sinh 1-\sinh t)|f^{''}(a)|]dt+$$

$$+\int_0^1 (t-1)^2[\sinh t|f^{''}(b)|+(\sinh 1-\sinh t)|f^{''}\left(\frac{a+b}{2}\right)|]dt\},$$

where we also used the definition of the hyperbolic type convex functions in last inequality.

By calculus, we get:

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{16\sinh 1}\{|f^{''}\left(\frac{a+b}{2}\right)|\int_0^1[t^2\sinh t dt+|f^{''}(a)|\int_0^1 t^2(\sinh 1-\sinh t)dt+$$

$$+|f^{''}(b)|\int_0^1 (t-1)^2(\sinh t)dt+|f^{''}\left(\frac{a+b}{2}|\right)\int_a^b (t-1)^2(\sinh 1-\sinh t)dt\}.$$

We see now that $\int_0^1[t^2\sinh t dt = 3\cosh 1-2\sinh 1-2,\quad \int_0^1(t-1)^2\sinh t dt = 2\cosh 1-3$.

Thus, by calculus, we obtain the desired inequailty by replacing these expressions in the above inequality.

$\blacksquare$

Theorem 4. *Let $f : I \to \mathbf{R}$, $I^o \subset \mathbf{R}$ be a twice differentiable function on I^o such that $f^{''} \in L[a,b]$, where $a, b \in I^o$, $a < b$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f^{''}|$ is a hyperbolic type convex function on $[a,b]$ then the following inequality*

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{128\sinh 1}\{|f^{''}(a)|(\frac{7}{3}\sinh 1 - 3\cosh 1 + 2) + |f^{''}(b)|(2\cosh 1 - 3)+$$

$$+|f^{''}\left(\frac{a+b}{2}\right)|(-\cosh 1 - 1 + \frac{7}{3}\sinh 1) + |f^{''}\left(\frac{3a+b}{4}\right)|(\cosh 1 + 1 - \frac{5}{3}\sinh 1)+$$

$$+|f^{''}\left(\frac{a+3b}{4}\right)|(\cosh 1 + 1 - \frac{5}{3}\sinh 1)\}.$$

holds.

Proof. Using Lemma 2 and then the modulus properties we have:

$$|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \leq \frac{(b-a)^2}{128}[\int_0^1 t^2|f^{''}\left(t\frac{3a+b}{4}+(1-t)a\right)|dt+$$

$$+\int_0^1 (t-1)^2|f^{''}\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)|dt+\int_0^1 t^2|f^{''}\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)|dt+$$

$$+\int_0^1 (t-1)^2|f^{''}\left(tb+(1-t)\frac{a+3b}{4}\right)|dt].$$

In this point we apply the definition of the hyperbolic convex type function and we have,

$$|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{128\sinh 1}[\int_0^1 t^2\sinh t dt|f^{''}\left(\frac{3a+b}{4}\right)|+\int_0^1 t^2(\sinh 1 - \sinh t)dt|f^{''}(a)|+$$

$$+\int_0^1 (t-1)^2\sinh t dt|f^{''}\left(\frac{a+b}{2}\right)+\int_0^1 (t-1)^2(\sinh 1 - \sinh t)dt|f^{''}\left(\frac{3a+b}{4}\right)|+$$

$$+\int_0^1 t^2\sinh t dt|f^{''}\left(\frac{a+3b}{4}\right)|+\int_0^1 t^2(\sinh 1 - \sinh t)dt|f^{''}\left(\frac{a+b}{2}\right)|+$$

$$+\int_0^1 (t-1)^2\sinh t dt|f^{''}(b)|+\int_0^1 (t-1)^2(\sinh 1 - \sinh t)dt|f^{''}\left(\frac{a+3b}{4}\right)||].$$

But these integrals are calculated in previous theorem so we obtain,

$$|\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}-\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{128\sinh 1}[(3\cosh 1 - 2\sinh 1 - 2)|f^{''}\left(\frac{3a+b}{4}\right)|+(\frac{\sinh 1}{3} - 3\cosh 1 + 2\sinh 1 + 2)|f^{''}(a)|+$$

$$+(2\cosh 1 - 3)|f^{''}\left(\frac{a+b}{2}\right)+(\frac{\sinh 1}{3} - 2\cosh 1 + 3)|f^{''}\left(\frac{3a+b}{4}\right)|+$$

$$+(3\cosh 1-2\sinh 1-2)|f^{''}\left(\frac{a+3b}{4}\right)|+(\frac{\sinh 1}{3}-3\cosh 1+2\sinh 1+2)|f^{''}\left(\frac{a+b}{2}\right)|+$$

$$+(2\cosh 1-3)|f^{''}(b)|+(\frac{\sinh 1}{3}-2\cosh 1+3)|f^{''}\left(\frac{a+3b}{4}\right)|]$$

and from here the inequality from this theorem.

∎

Theorem 5. *Let $f: I \to \mathbf{R}$, $I^o \subset \mathbf{R}$ be a twice differentiable function on I^o such that $f^{''} \in L^1[a,b]$, where $a, b \in I^o$, $a < b$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|f^{''}|^q$ is a hyperbolic type convex function on $[a,b]$ then the following inequality*

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{16}\frac{1}{(\sinh 1)^{\frac{1}{q}}(2p+1)^{\frac{1}{p}}}\{[(\cosh 1-1)|f^{''}\left(\frac{a+b}{2}\right)|^q+(\sinh 1-\cosh 1+1)|f^{''}(a)|^q]^{\frac{1}{q}}+$$

$$+[(\cosh 1-1)|f^{''}(b)|^q+(\sinh 1-\cosh 1+1)|f^{''}\left(\frac{a+b}{2}\right)|^q]^{\frac{1}{q}}\},$$

takes place.

Proof. We use the same tools like before, i.e. Lemma 1, the modulus properties and the definition of the hyperbolic type convex functions for the function $|f^{''}|^q$, but here we use the Holder's inequality like below:

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{16}\int_0^1 t^2|f^{''}\left(t\frac{a+b}{2}+(1-t)a\right)|dt+\int_0^1(t-1)^2|f^{''}\left(tb+(1-t)\frac{a+b}{2}\right)|dt \leq$$

$$\leq \frac{(b-a)^2}{16}[\left(\int_0^1 t^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^2|f^{''}\left(t\frac{a+b}{2}+(1-t)a\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1(1-t)^2dt\right)^{1-\frac{1}{q}}\left(\int_0^1(t-1)^2|f^{''}\left(tb+(1-t)\frac{a+b}{2}\right)|^q dt\right)^{\frac{1}{q}}] \leq$$

$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{1-\frac{1}{q}}\{[|f^{''}\left(\frac{a+b}{2}\right)|^q\int_0^1\frac{\sinh t}{\sinh 1}dt+|f^{''}(a)|^q\int_0^1\frac{\sinh 1-\sinh t}{\sinh 1}dt]^{\frac{1}{q}}+$$

$$+[|f^{''}(b)|^q\int_0^1\frac{\sinh t}{\sinh 1}dt+|f^{''}\left(\frac{a+b}{2}\right)|^q\int_0^1\frac{\sinh 1-\sinh t}{\sinh 1}dt]^{\frac{1}{q}}\}$$

and from here, by calculus, we get the desired inequality.

∎

Theorem 6. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a,b \in I^0,\ 0 < a < b$. If $|f^{(n)}|^q$ is a hyperbolic type convex function on $[a,b]$ for some fixed $q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$ then the following inequality*

$$|I(f,x,a,b,\alpha,n)| = |\sum_{k=2}^{n} \alpha(\alpha-1)...(\alpha-k+2)f^{(n-k)}(x)\left(\frac{(-1)^{k-1}}{(x-a)^{k-1}} - \frac{1}{(b-x)^{k-1}}\right) +$$

$$+\Gamma(\alpha+1)[\frac{(-1)^n}{(x-a)^\alpha} J_{x^-}^{\alpha-n+1} f(a) + \frac{1}{(b-x)^\alpha} J_{x^+}^{\alpha-n+1} f(b)]| \le$$

$$\le \frac{4^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}(e+1)^{\frac{1}{q}}}[(x-a)A^{\frac{1}{q}}(\frac{e-1}{2}|f^{(n)}(x)|^q, |f^{(n)}(a)|^q) +$$

$$+(b-x)A^{\frac{1}{q}}(\frac{e-1}{2}|f^{(n)}(b)|^q, |f^{(n)}(x)|^q)],$$

takes place, where $A(u,v)$ is the arithmetic mean of u and v.

Proof. Using now Lemma 3, the properties of modulus and Holder's inequality, we have,

$$|I(f,x,a,b,\alpha,n)| = |\sum_{k=2}^{n} \alpha(\alpha-1)...(\alpha-k+2)f^{(n-k)}(x)\left(\frac{(-1)^{k-1}}{(x-a)^{k-1}} - \frac{1}{(b-x)^{k-1}}\right) +$$

$$+\Gamma(\alpha+1)[\frac{(-1)^n}{(x-a)^\alpha} J_{x^-}^{\alpha-n+1} f(a) + \frac{1}{(b-x)^\alpha} J_{x^+}^{\alpha-n+1} f(b)]| \le$$

$$\le (x-a)\int_0^1 t^\alpha |f^{(n)}(tx+(1-t)a)|dt + (b-x)\int_0^1 (1-t)^\alpha |f^{(n)}(tb+(1-t)x)|dt \le$$

$$\le (x-a)\left(\int_0^1 t^{\alpha p} dt\right)^{\frac{1}{p}} \left(\int_0^1 |f^{(n)}(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}} +$$

$$+(b-x)\left(\int_0^1 (1-t)^{\alpha p} dt\right)^{\frac{1}{p}} \left(\int_0^1 |f^{(n)}(tb+(1-t)x)|^q dt\right)^{\frac{1}{q}}.$$

Now, taking into account that $|f^{(n)}|^q$ is a hyperbolic type convex function on $[a,b]$, we get by calculus,

$$|I(f,x,a,b,\alpha,n)| \le$$

$$\le \frac{1}{(\alpha p+1)^{\frac{1}{p}}(\sinh 1)^{\frac{1}{q}}}[(x-a)\left(\int_0^1 \sinh t dt |f^{(n)}(x)|^q + \int_0^1 (\sinh 1 - \sinh t)dt |f^{(n)}(a)|^q]\right)^{\frac{1}{q}} +$$

$$+(b-x)\left(\int_0^1 \sinh t dt |f^{(n)}(b)|^q + \int_0^1 (\sinh 1 - \sinh t)dt |f^{(n)}(x)|^q]\right)^{\frac{1}{q}}$$

i.e. the desired inequality, taking into account that $\int_0^1 \sinh t dt = \frac{(e-1)^2}{2e}$ and $\int_0^1 (\sinh 1 - \sinh t)dt = \frac{e-1}{e}$.

∎

Theorem 7. *Let $f : I \to \mathbf{R}$, $I^o \subset \mathbf{R}$ be a twice differentiable function on I^o such that $f^{''} \in L^1[a,b]$, where $a, b \in I^o$, $a < b$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f^{''}|^q$ is a hyperbolic type convex function on $[a,b]$ then the following inequality*

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{16(\sinh 1)^{\frac{1}{q}}(2p+2)^{\frac{1}{p}}}\{[\frac{1}{(2p+1)^{\frac{1}{p}}}\left(\frac{e^2-2e-1}{2e}|f^{''}\left(\frac{a+b}{2}\right)|^q+\frac{-e^2+4e+1}{4e}|f^{''}(a)|^q\right)^{\frac{1}{q}}+$$

$$+\left(\frac{1}{e}|f^{''}\left(\frac{a+b}{2}\right)|^q+\frac{e^2-5}{4e}|f^{''}(a)|^q\right)^{\frac{1}{q}}]+$$

$$+[\left(\frac{e^2-2e-1}{2e}|f^{''}(b)|^q+\frac{-e^2+4e+1}{4e}|f^{''}(\frac{a+b}{2})|^q\right)^{\frac{1}{q}}+$$

$$+\frac{1}{(2p+1)^{\frac{1}{p}}}\left(\frac{1}{e}|f^{''}(b)|^q+\frac{e^2-5}{4e}|f^{''}(\frac{a+b}{2})|^q\right)^{\frac{1}{q}}]\}$$

holds.

Proof. We follow the same steps like ibefore, but we use now. Holder-Iscan inequality instead of Holder inequality.Thus we have,

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{16}\int_0^1 t^2|f^{''}\left(t\frac{a+b}{2}+(1-t)a\right)|dt+\int_0^1(t-1)^2|f^{''}\left(tb+(1-t)\frac{a+b}{2}\right)|dt \leq$$

$$\leq \frac{(b-a)^2}{16}\{[\left(\int_0^1(1-t)t^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)|f^{''}\left(t\frac{a+b}{2}+(1-t)a\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 tt^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 t|f^{''}\left(t\frac{a+b}{2}+(1-t)a\right)|^q dt\right)^{\frac{1}{q}}]+$$

$$+[\left(\int_0^1(1-t)(1-t)^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)|f^{''}\left(tb+(1-t)\frac{a+b}{2}\right)|^q dt\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 t(1-t)^{2p}dt\right)^{\frac{1}{p}}\left(\int_0^1 t|f^{''}\left(tb+(1-t)\frac{a+b}{2}\right)|^q dt\right)^{\frac{1}{q}}]\} \leq$$

$$\leq \frac{(b-a)^2}{16(\sinh 1)^{\frac{1}{q}}(2p+2)^{\frac{1}{p}}}\{[\frac{1}{(2p+1)^{\frac{1}{p}}}(\int_0^1(1-t)\sinh t dt|f^{''}(\frac{a+b}{2})|^q+$$

$$+\int_0^1(1-t)(\sinh 1-\sinh t)dt|f^{''}(a)|^q)^{\frac{1}{q}}+(\int_0^1 t\sinh t dt|f^{''}(\frac{a+b}{2})|^q+$$

$$+\int_0^1 t(\sinh 1-\sinh t)dt|f^{''}(a)|^q)^{\frac{1}{q}}]+[(\int_0^1(1-t)\sinh t dt|f^{''}(b)|^q+$$

$$+\int_0^1(1-t)(\sinh 1-\sinh t)dt|f^{''}(\frac{a+b}{2})|^q)^{\frac{1}{q}}+\frac{1}{(2p+1)^{\frac{1}{p}}}(\int_0^1 t\sinh t dt|f^{''}(b)|^q+$$

$$+\int_0^1 t(\sinh 1-\sinh t)dt|f^{''}(\frac{a+b}{2})|^q)^{\frac{1}{q}}]\}.$$

Using the integrals $\int_0^1(1-t)\sinh t dt = \frac{e^2-2e-1}{2e}$, $\int_0^1(1-t)(\sinh 1-\sinh t)dt = \frac{-e^2+4e+1}{4e}$, $\int_0^1 t\sinh t dt = \frac{1}{e}$ and $\int_0^1 t(\sinh 1-\sinh t)dt = \frac{e^2-5}{4e}$ in previous inequality we get the desired inequality.

∎

Theorem 8. *Let $f: I\to \mathbf{R}$, $I^o\subset \mathbf{R}$ be a twice differentiable function on I^o such that $f^{''}\in L^1[a,b]$, where $a,b\in I^o$, $a<b$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $|f^{''}|^q$ is a hyperbolic type convex function on $[a,b]$ then next inequality*

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx|\le$$

$$\le \frac{(b-a)^2}{16\cdot 3^{\frac{1}{p}}(\sinh 1)^{\frac{1}{q}}}\{[(3\cosh 1-2\sinh 1-2)|f^{''}\left(\frac{a+b}{2}\right)|^q+(\frac{7}{3}\sinh 1-3\cosh 1+2)|f^{''}(a)|^q]^{\frac{1}{q}}+$$

$$+[(2\cosh 1-3)|f^{''}(b)|^q+(\frac{1}{3}\sinh 1-2\cosh 1+3)|f^{''}\left(\frac{a+b}{2}\right)|^q]^{\frac{1}{q}}\}$$

is satisfied.

Proof. Now we use instead of Holder's inequality, the power mean inequality, the rest of steps will be the same. Therefore we continue the demonstration after application of the power mean inequality,

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx|\le$$

$$\le \frac{(b-a)^2}{16}\{[\left(\int_0^1 t^2dt\right)^{\frac{1}{p}}\left(\int_0^1 t^2|f^{''}(t\frac{a+b}{2}+(1-t)a)|^qdt\right)^{\frac{1}{q}}]+$$

$$+[\left(\int_0^1(1-t)^2dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^2|f^{''}(tb+(1-t)\frac{a+b}{2})|^qdt\right)^{\frac{1}{q}}]\}$$

$$\le \frac{(b-a)^2}{16(\sinh 1)^{\frac{1}{q}}\cdot 3^{\frac{1}{p}}}\{[\left(\int_0^1 t^2\sinh t dt|f^{''}(\frac{a+b}{2})|^q+\int_0^1 t^2(\sinh 1-\sinh t)dt|f^{''}(a)|^qdt\right)^{\frac{1}{q}}]+$$

$$+[\left(\int_0^1(1-t)^2\sinh t dt|f^{''}(b)|^q+\int_0^1(1-t)^2(\sinh 1-\sinh t)dt|f^{''}(\frac{a+b}{2})|^qdt\right)^{\frac{1}{q}}]\}$$

and using that $\int_0^1 t^2\sinh t dt = 3\cosh 1-2\sinh 1-2$, $\int_0^1(1-t)^2\sinh t dt = 2\cosh 1-3$ we will obtain the inequality from the conclusion.

∎

Theorem 9. *Under the same hypothesis as in previous theorem, we obtain the following inequality:*

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx|\le$$

$$\le \frac{(b-a)^2}{4^{\frac{1}{p}+2}(\sinh 1)^{\frac{1}{q}}}\{\frac{1}{3^{\frac{1}{p}}}[(7\cosh 1-9\sinh 1+2)|f^{''}(\frac{a+b}{2})|^q+(4\cosh 1-\frac{83}{12}\sinh 1+2)|f^{''}(a)|^q]^{\frac{1}{q}}+$$

$$+[(7\cosh 1-9\sinh 1)|f^{''}(\frac{a+b}{2})|^q+(\frac{37}{4}\sinh 1-7\cosh 1)|f^{''}(a)|^q]^{\frac{1}{q}}+$$

$$+[(6\sinh 1-7)|f^{''}(b)|^q+(7-\frac{23}{4}\sinh 1)|f^{''}(\frac{a+b}{2})|^q]^{\frac{1}{q}}+$$

$$+\frac{1}{3^{\frac{1}{p}}}[(2\cosh 1-6\sinh 1+4)|f^{''}(b)|^q+(\frac{73}{12}\sinh 1-2\cosh 1-4)|f^{''}(\frac{a+b}{2})|^q]^{\frac{1}{q}}\}.$$

Proof. Like before by the properties of modulus and the improved power-mean integral inequality, we have

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx|\leq$$

$$\frac{(b-a)^2}{16}[\left(\int_0^1(1-t)t^2dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)t^2|f^{''}(t\frac{a+b}{2}+(1-t)a)|^q\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1 tt^2dt\right)^{\frac{1}{p}}\left(\int_0^1 tt^2|f^{''}(t\frac{a+b}{2}+(1-t)a)|^q\right)^{\frac{1}{q}}+$$

$$+\left(\int_0^1(1-t)(1-t)^2dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)(1-t)^2|f^{''}(tb+(1-t)\frac{a+b}{2})|^q\right)^{\frac{1}{q}}]+$$

$$+\left(\int_0^1 t(1-t)^2dt\right)^{\frac{1}{p}}\left(\int_0^1 t(1-t)^2|f^{''}(tb+(1-t)\frac{a+b}{2})|^q\right)^{\frac{1}{q}}.$$

We apply here the definition of the hyperbolic type convex functions and we get,

$$|-f\left(\frac{a+b}{2}\right)+\frac{1}{b-a}\int_a^b f(x)dx|\leq$$

$$\frac{(b-a)^2}{16(\sinh 1)^{\frac{1}{q}}}[\frac{1}{12^{\frac{1}{p}}}\left(\int_0^1 t^2(1-t)\sinh t dt|f^{''}(\frac{a+b}{2})|^q+\int_0^1(1-t)t^2(\sinh 1-\sinh t)dt|f^{''}(a)|^q\right)^{\frac{1}{q}}+$$

$$+\frac{1}{4^{\frac{1}{p}}}\left(\int_0^1 t^3\sinh t dt|f^{''}(\frac{a+b}{2})|^q+\int_0^1 t^3(\sinh 1-\sinh t)dt|f^{''}(a)|^q\right)^{\frac{1}{q}}+$$

$$+\frac{1}{4^{\frac{1}{p}}}\left(\int_0^1(1-t)^3\sinh t dt|f^{''}(b)|^q+\int_0^1(1-t)^3(\sinh 1-\sinh t)dt|f^{''}(\frac{a+b}{2})|^q\right)^{\frac{1}{q}}+$$

$$+\frac{1}{12^{\frac{1}{p}}}\left(\int_0^1 t(1-t)^2\sinh t dt|f^{''}(b)|^q+\int_0^1(1-t)^2t(\sinh 1-\sinh t)dt|f^{''}(\frac{a+b}{2})|^q\right)^{\frac{1}{q}}].$$

Taking into account that $J_1=\int_0^1 t^2(1-t)\sinh t dt=7\sinh 1-4\cosh 1-2,$, $J_2=\int_0^1 t^2(1-t)(\sinh 1-\sinh t)dt=4\cosh 1--\frac{83}{12}\sinh 1+2,$

$J_3=\int_0^1 t^3\sinh t dt=7\cosh 1-9\sinh 1,$ $J_4=\int_0^1 t^3(\sinh 1-\sinh t)dt=\frac{37}{4}\sinh 1-7\cosh 1,$

$J_5=\int_0^1(1-t)^3\sinh t dt=6\sinh 1-7,$ $J_6=\int_0^1(1-t)^3(\sinh 1-\sinh t)dt=7-\frac{23}{4}\sinh 1,$

$J_7=\int_0^1 t(1-t)^2\sinh t dt=2\cosh 1-6\sinh 1+4,$ $J_8=\int_0^1 t^2(1-t)(\sinh 1-\sinh t)dt=\frac{73}{12}\sinh 1-2\cosh 1-4$ we will obtain the desired inequality.

■

We use now below the Holder-Iscan inequality instead of Holder's inequality and we will obtain,

Theorem 10. *Let $f : I \to \mathbf{R}$, $I^o \subset \mathbf{R}$ be a twice differentiable function on I^o such that $f^{''} \in L[a,b]$, where $a, b \in I^o$, $a < b$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $|f^{''}|$ is a hyperbolic type convex function on $[a,b]$ then the following inequality*

$$|-\frac{f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)}{2}+\frac{1}{b-a}\int_a^b f(x)dx| \leq$$

$$\leq \frac{(b-a)^2}{256(p+1)^{\frac{1}{p}}(\sinh 1)^{\frac{1}{q}}}\{\frac{1}{(2p+1)^{\frac{1}{p}}}A^{\frac{1}{q}}\left((\sinh 1-1)|f^{''}(\frac{3a+b}{4})|^q,(1-\frac{\sinh 1}{2})|f^{''}(a)|^q\right)+$$

$$+A^{\frac{1}{q}}\left((\cosh 1-\sinh 1)|f^{''}(\frac{3a+b}{4})|^q,(\frac{3}{2}\sinh 1-\cosh 1)|f^{''}(a)|^q\right)+$$

$$+A^{\frac{1}{q}}\left((\sinh 1-1)|f^{''}(\frac{a+b}{2})|^q,(1-\frac{1}{2}\sinh 1)|f^{''}(\frac{3a+b}{4})|^q\right)+$$

$$+\frac{1}{(2p+1)^{\frac{1}{p}}}A^{\frac{1}{q}}\left((\cosh 1-\sinh 1)|f^{''}(\frac{a+b}{2})|^q,(\frac{3}{2}\sinh 1-\cosh 1)|f^{''}(\frac{3a+b}{4})|^q\right)+$$

$$+\frac{1}{(2p+1)^{\frac{1}{p}}}A^{\frac{1}{q}}\left((\sinh 1-1)|f^{''}(\frac{a+3b}{4})|^q,(1-\frac{1}{2}\sinh 1)|f^{''}(\frac{a+b}{2})|^q\right)+$$

$$+A^{\frac{1}{q}}\left((\cosh 1-\sinh 1)|f^{''}(\frac{a+3b}{4})|^q,(\frac{3}{2}\sinh 1-\cosh 1)|f^{''}(\frac{a+b}{2})|^q\right)+$$

$$+A^{\frac{1}{q}}\left((\sinh 1-1)|f^{''}(b)|^q,(1-\frac{1}{2}\sinh 1)|f^{''}(\frac{a+3b}{4})|^q\right)+$$

$$+\frac{1}{(2p+1)^{\frac{1}{p}}}A^{\frac{1}{q}}\left((\cosh 1-\sinh 1)|f^{''}(b)|^q,(\frac{3}{2}\sinh 1-\cosh 1)|f^{''}(\frac{a+3b}{4})|^q\right)\}.$$

holds.

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