SOME MID-POINT AND TRAPEZOID TYPE INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with spectra $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \text{ins}(\gamma)$. In this paper we show among others that

$$\left\| \int_0^1 f((1-t)x + ty)\,dt - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{24\pi} \|y-x\|^2 \int_\gamma |f(\xi)| \max\left\{ (|\xi| - \|x\|)^{-3}, (|\xi| - \|y\|)^{-3} \right\} |d\xi|$$

and

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x + ty)\,dt \right\| \leq \frac{1}{12\pi} \|y-x\|^2 \int_\gamma |f(\xi)| \max\left\{ (|\xi| - \|x\|)^{-3}, (|\xi| - \|y\|)^{-3} \right\} |d\xi|.$$

Some examples for exponential function and functions defined by power series in Banach algebras are also given.

1. Introduction

The following mid-point inequality for twice differentiable functions holds, see for instance [13] and [15]:

Theorem 1. *Assume that f is twice differentiable on (a,b) and such that $\|f''\|_{(a,b),\infty} := \sup_{u\in(a,b)} |f''(u)| < \infty$, then*

$$\left| \frac{1}{b-a} \int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{24} (b-a)^2 \|f''\|_{(a,b),\infty}. \tag{1.1}$$

The constant $\frac{1}{24}$ is best possible in (1.1).

The corresponding trapezoid inequality is as follows, see for instance [14] and [3]:

Theorem 2. *With the assumptions of Theorem 1,*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(u)\,du \right| \leq \frac{1}{12} (b-a)^2 \|f''\|_{(a,b),\infty}. \tag{1.2}$$

The constant $\frac{1}{12}$ is best possible in (1.1).

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In order to extend mid-point and trapezoid inequalities for functions defined on Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\|\leq\|a\|\|b\|$ for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a,b\in\operatorname{Inv}(\mathcal{B})$ then $ab\in\operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in\operatorname{Inv}(\mathcal{B})$;
(ii) $\{a\in\mathcal{B}:\ \|1-b\|<1\}\subset\operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B})\ni a\longmapsto a^{-1}\in\operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a:\rho(a)\to\operatorname{Inv}(\mathcal{B})$, $R_a(z):=(z-a)^{-1}$. For each $z,w\in\rho(a)$ we have the identity

$$R_a(w)-R_a(z)=(z-w)R_a(z)R_a(w).$$

We also have that

$$\sigma(a)\subset\{z\in\mathbb{C}:\ |z|\leq\|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a)=\sup\{|z|:z\in\sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda:\mathcal{B}\to\mathbb{C}$, the function $\lambda\circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n\in\mathbb{N}$ and $r>\nu(a)$, we have $a^n=\frac{1}{2\pi i}\int_{|\xi|=r}\xi^n(\xi-a)^{-1}d\xi$;
(v) We have $\nu(a)=\lim_{n\to\infty}\|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a\in\mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a)\subset G$. If $f:G\to\mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a):=\frac{1}{2\pi i}\int_{\delta}f(\xi)(\xi-a)^{-1}d\xi, \tag{1.3}$$

where $\delta\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a)\subset\operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a))=f(\sigma(a)) \tag{1.4}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [16] and [18].

For some recent norm inequalities for functions on Banach algebras, see [8], [2] and [5]-[12].

2. Preliminary Results

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \; t \in [0,1]. \tag{2.1}$$

Lemma 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f'_{x,y}(t) = D(f)((1-t)x + ty)(y - x) \tag{2.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f'_{x,y}(0+) = D(f)(x)(y-x) \text{ and } f'_{x,y}(1-) = D(f)(y)(y-x). \tag{2.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}
&\frac{f_{x,y}(t+h) - f(t)}{h} \\
&= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\
&= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
f'_{x,y}(t) &= \frac{df_{x,y}(t)}{dt} = \lim_{h \to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\
&= \lim_{h \to 0} \left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \right] \\
&= D(f)((1-t)x + ty)(y-x),
\end{aligned}$$

which proves (2.7).

The proof is similar for the lateral derivatives. □

We also have:

Lemma 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is twice differentiable on $(0,1)$ as a function of t and we have*

$$f''_{x,y}(t) = D^2(f)((1-t)x + ty)(y-x, y-x) \tag{2.4}$$

for all $t \in (0,1)$, where $D^2(f)(\cdot)(\cdot,\cdot)$ is the second Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f''_{x,y}(0+) = D^2(f)(x)(y-x, y-x), \; f''_{x,y}(1-) = D^2(f)(y)(y-x, y-x). \tag{2.5}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}
&\frac{1}{h}\left[\frac{df_{x,y}(t+h)}{dt} - \frac{df_{x,y}(t)}{dt}\right] \\
&= \frac{D(f)((1-(t+h))x + (t+h)y)(y-x) - D(f)((1-t)x+ty)(y-x)}{h} \\
&= \frac{D(f)((1-t)x + ty + h(y-x))(y-x) - D(f)((1-t)x+ty)(y-x)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
f''_{x,y}(t) = \frac{d^2 f_{x,y}(t)}{dt^2} &= \lim_{h \to 0}\left\{\frac{1}{h}\left[\frac{df_{x,y}(t+h)}{dt} - \frac{df_{x,y}(t)}{dt}\right]\right\} \\
&= D^2(f)((1-t)x+ty)(y-x, y-x).
\end{aligned}$$

The proof is similar for the lateral derivatives. □

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{2.6}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x+hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned}
&f(x+hv) - f(x) \\
&= \frac{1}{2\pi i}\left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi\right) \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} - (\xi - x)^{-1}\right] d\xi \\
&= \frac{1}{2\pi i} h \int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} v (\xi - x)^{-1}\right] d\xi,
\end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x+hv) - f(x)}{h} = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x - hv)^{-1} v (\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (2.6). □

Corollary 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f'_{x,y}(t) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1}(y-x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{2.7}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f_{x,y}^{'}(0+) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - x)^{-1}(y-x)(\xi - x)^{-1}d\xi, \tag{2.8}$$

and

$$f_{x,y}^{'}(1-) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - y)^{-1}(y-x)(\xi - y)^{-1}d\xi. \tag{2.9}$$

The proof is obvious by Lemmas 1 and 2.

Lemma 4. *Assume that* $f : G \to \mathbb{C}$ *is analytic on the domain* G *and* $x \in \mathcal{B}$, *with* $\sigma(x) \subset G$, *then for* u, $v \in \mathcal{B}$ *we have*

$$\begin{aligned} D^2(f)(x)(u,v) &= \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} u (\xi - x)^{-1} d\xi \\ &+ \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - x)^{-1} u (\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \end{aligned} \tag{2.10}$$

where $\gamma \subset G$ *is taken to be close rectifiable curve in* G *and such that* $\sigma(x) \subset \text{ins}(\gamma)$.

In particular, we have

$$D^2(f)(x)(v,v) = \frac{1}{\pi i}\int_{\gamma} f(\xi)\left((\xi - x)^{-1} v\right)^2 (\xi - x)^{-1} d\xi, \tag{2.11}$$

for $v \in \mathcal{B}$.

Proof. Let $u, v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x + hv) \subset \text{ins}(\delta) \subset G$. Then

$$D^2(f)(x)(u,v) = \lim_{h\to 0}\frac{D(f)(x+hv)(u) - D(f)(x)(u)}{h}.$$

We have by (2.6)

$$\begin{aligned} &D(f)(x+hv)(u) - D(f)(x)(u) \\ &= \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - x - hv)^{-1} u (\xi - x - hv)^{-1} d\xi \\ &- \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - x)^{-1} u (\xi - x)^{-1} d\xi \\ &= \frac{1}{2\pi i}\int_{\gamma} f(\xi) k_{\xi,h}(x,u,v) d\xi, \end{aligned}$$

where

$$k_{\xi,h}(x,u,v) := (\xi - x - hv)^{-1} u (\xi - x - hv)^{-1} - (\xi - x)^{-1} u (\xi - x)^{-1} \tag{2.12}$$

and $\xi \in \gamma$ while h is in a small interval around 0, $h \neq 0$.

If we multiply both sides of (2.12) by $\xi - x - hv$, then we get

$$\begin{aligned}(\xi - x - hv)\, k_{\xi,h}(x,u,v)\,(\xi - x - hv) \\ = u - (\xi - x - hv)(\xi - x)^{-1} u (\xi - x)^{-1} (\xi - x - hv) \\ = u - \left(1 - hv(\xi - x)^{-1}\right) u \left(1 - h(\xi - x)^{-1} v\right) \\ = u - \left(u - hv(\xi - x)^{-1} u\right)\left(1 - h(\xi - x)^{-1} v\right) \\ = u - u + hv(\xi - x)^{-1} u + hu(\xi - x)^{-1} v - h^2 v(\xi - x)^{-1} u(\xi - x)^{-1} v \\ = h\left[v(\xi - x)^{-1} u + u(\xi - x)^{-1} v - hv(\xi - x)^{-1} u(\xi - x)^{-1} v\right].\end{aligned} \tag{2.13}$$

The identity (2.13) implies that

$$\begin{aligned}\frac{1}{h} k_{\xi,h}(x,u,v) = (\xi - x - hv)^{-1} \\ \times \left[v(\xi - x)^{-1} u + u(\xi - x)^{-1} v - hv(\xi - x)^{-1} u(\xi - x)^{-1} v\right] \\ \times (\xi - x - hv)^{-1},\end{aligned}$$

which gives that

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h} k_{\xi,h}(x,u,v) \\ = (\xi - x)^{-1}\left[v(\xi - x)^{-1} u + u(\xi - x)^{-1} v\right](\xi - x)^{-1} \\ = (\xi - x)^{-1} v(\xi - x)^{-1} u(\xi - x)^{-1} + (\xi - x)^{-1} u(\xi - x)^{-1} v(\xi - x)^{-1}\end{aligned}$$

for $\xi \in \gamma$.

Therefore

$$\begin{aligned}\lim_{h\to 0} \frac{D(f)(x + hv)(u) - D(f)(x)(u)}{h} \\ = \frac{1}{2\pi i} \lim_{h\to 0} \int_\gamma f(\xi) \frac{1}{h} k_{\xi,h}(x,u,v)\, d\xi = \frac{1}{2\pi i} \int_\gamma f(\xi) \lim_{h\to 0} \frac{1}{h} k_{\xi,h}(x,u,v)\, d\xi \\ = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x)^{-1} v(\xi - x)^{-1} u(\xi - x)^{-1} d\xi \\ + \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x)^{-1} u(\xi - x)^{-1} v(\xi - x)^{-1} d\xi\end{aligned}$$

and the lemma is thus proved. □

Corollary 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}f''_{x,y}(t) \\ = \frac{1}{\pi i} \int_\gamma f(\xi)\left((\xi - (1-t)x - ty)^{-1}(y - x)\right)^2 (\xi - (1-t)x - ty)^{-1} d\xi\end{aligned} \tag{2.14}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f''_{x,y}(0+) = \frac{1}{2\pi i} \int_\gamma f(\xi)\left((\xi - x)^{-1}(y - x)\right)^2 (\xi - x)^{-1} d\xi, \tag{2.15}$$

and

$$f''_{x,y}(1-) = \frac{1}{2\pi i}\int_\gamma f(\xi)\left((\xi - y)^{-1}(y-x)\right)^2 (\xi - y)^{-1} d\xi. \tag{2.16}$$

Lemma 5. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}(2.17)\qquad &\left\|f''_{x,y}(t)\right\| \\ &\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left\|(\xi - (1-t)x - ty)^{-1}\right\|^3 |d\xi| \\ &\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| (|\xi| - \|(1-t)x + ty\|)^{-3} |d\xi| \\ &\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| [(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)]^{-3} |d\xi| \\ &\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left[(1-t)(|\xi| - \|x\|)^{-3} + t(|\xi| - \|y\|)^{-3}\right] |d\xi| \\ &\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \max\left\{(|\xi| - \|x\|)^{-3}, (|\xi| - \|y\|)^{-3}\right\} |d\xi|\end{aligned}$$

for all $t \in [0,1]$.

Proof. By taking the norm in (2.14) and using the integral's properties, we have

$$\begin{aligned}(2.18)\qquad &\left\|f''_{x,y}(t)\right\| \\ &\le \frac{1}{\pi}\int_\gamma |f(\xi)| \\ &\times \left\|\left((\xi - (1-t)x - ty)^{-1}(y-x)\right)^2 (\xi - (1-t)x - ty)^{-1}\right\| |d\xi| \\ &\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left\|(\xi - (1-t)x - ty)^{-1}\right\|^3 |d\xi| \\ &= \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| |\xi|^{-3} \left\|\left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi}\right)^{-1}\right\|^3 |d\xi|\end{aligned}$$

for all $t \in [0,1]$, which proves the first inequality in (2.17).

Since

$$\left\|(1-t)\frac{x}{\xi} + t\frac{y}{\xi}\right\| \le (1-t)\left\|\frac{x}{\xi}\right\| + t\left\|\frac{y}{\xi}\right\| < 1 - t + t = 1$$

for $\xi \in \gamma$, hence

$$\left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi}\right)^{-1} = \sum_{k=0}^{\infty}\left[(1-t)\frac{x}{\xi} + t\frac{y}{\xi}\right]^k.$$

Therefore

$$\begin{aligned}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\| &\leq \sum_{k=0}^{\infty}\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|^{k} \\ &= \left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1} \\ &= \left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1} \\ &= |\xi|\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\end{aligned}$$

for $\xi\in\gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{3} \leq |\xi|^{2}\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}$$

for $\xi\in\gamma$.

Therefore

$$\begin{aligned}&\int_{\gamma}|f(\xi)|\,|\xi|^{-3}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{3}|d\xi| \\ &\leq \int_{\gamma}|f(\xi)|\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}|d\xi|\end{aligned}$$

and we derive the second inequality in (2.17).

By the triangle inequality we have

$$\begin{aligned}|\xi|-\|(1-t)x+ty\| &\geq |\xi|-(1-t)\|x\|-t\|y\| \\ &= (1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)>0\end{aligned}$$

for $\xi\in\gamma$.

This implies that

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\leq\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-1}$$

and thus

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}\leq\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-3}$$

for $\xi\in\gamma$ and $t\in[0,1]$. This proves the third inequality in (2.17).

By the convexity of the power function $(\cdot)^{-3}$ we also have

$$\begin{aligned}&\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-3} \\ &\leq (1-t)\left(|\xi|-\|x\|\right)^{-3}+t\left(|\xi|-\|y\|\right)^{-3}\end{aligned}$$

for $t\in[0,1]$, which proves the fourth inequality in (2.17).

Finally, observe that

$$(1-t)\left(|\xi|-\|x\|\right)^{-3}+t\left(|\xi|-\|y\|\right)^{-3}\leq\max\left\{\left(|\xi|-\|x\|\right)^{-3},\left(|\xi|-\|y\|\right)^{-3}\right\}$$

and the last part of (2.17) is thus proved. □

Corollary 3. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$
\begin{aligned}
(2.19)\quad & \left\| f''_{x,y}(t) \right\| \\
& \leq 2R \|y-x\|^2 \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| \left\| \left(Re^{2\pi i s} - (1-t)x - ty\right)^{-1} \right\|^3 ds \\
& \leq 2R \|y-x\|^2 \left(R - \|(1-t)x + ty\|\right)^{-3} \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| ds \\
& \leq 2R \|y-x\|^2 \left[(1-t)(R - \|x\|) + t(R - \|y\|)\right]^{-3} \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| ds \\
& \leq 2R \|y-x\|^2 \left[(1-t)(R - \|x\|)^{-3} + t(R - \|y\|)^{-3}\right] \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| ds \\
& \leq \frac{2R \|y-x\|^2}{\min\left\{(R - \|x\|)^3, (R - \|y\|)^3\right\}} \int_0^1 \left| f\left(Re^{2\pi i s}\right) \right| ds,
\end{aligned}
$$

for all $t \in [0,1]$.

It follows by taking γ parametrized by $\xi(s) = Re^{2\pi i s}$ where $s \in [0,1]$, then $d\xi(s) = 2\pi i Re^{2\pi i s} ds$, $|d\xi(s)| = 2\pi R ds$, $|\xi| = R$ and by (2.17) we get (2.19).

Remark 1. *If $\|f\|_{\gamma,\infty} := \sup_{\xi \in \gamma} |f(\xi)| < 0$, then from (2.17) we get*

$$
\begin{aligned}
(2.20)\quad & \left\| f''_{x,y}(t) \right\| \\
& \leq \frac{1}{\pi} \|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left\| \left(\xi - (1-t)x - ty\right)^{-1} \right\|^3 |d\xi| \\
& \leq \frac{1}{\pi} \|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left(|\xi| - \|(1-t)x + ty\|\right)^{-3} |d\xi| \\
& \leq \frac{1}{\pi} \|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)\right]^{-3} |d\xi| \\
& \leq \frac{1}{\pi} \|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left[(1-t)(|\xi| - \|x\|)^{-3} + t(|\xi| - \|y\|)^{-3}\right] |d\xi| \\
& \leq \frac{1}{\pi} \|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \max\left\{(|\xi| - \|x\|)^{-3}, (|\xi| - \|y\|)^{-3}\right\} |d\xi|
\end{aligned}
$$

for all $t \in [0,1]$.

Also, if $\|f\|_{R,\infty} := \sup_{s\in[0,1]} \left|f\left(Re^{2\pi is}\right)\right| < \infty$, *then from (2.19) we get*

$$
\begin{aligned}
(2.21)\qquad & \left\|f''_{x,y}(t)\right\| \\
& \leq 2R\|y-x\|^2 \|f\|_{R,\infty} \int_0^1 \left\|\left(Re^{2\pi is} - (1-t)x - ty\right)^{-1}\right\|^3 ds \\
& \leq 2R\|y-x\|^2 (R - \|(1-t)x + ty\|)^{-3} \|f\|_{R,\infty} \\
& \leq 2R\|y-x\|^2 [(1-t)(R-\|x\|) + t(R-\|y\|)]^{-3} \|f\|_{R,\infty} \\
& \leq 2R\|y-x\|^2 \left[(1-t)(R-\|x\|)^{-3} + t(R-\|y\|)^{-3}\right] \|f\|_{R,\infty} \\
& \leq \frac{2R\|y-x\|^2 \|f\|_{R,\infty}}{\min\left\{(R-\|x\|)^3, (R-\|y\|)^3\right\}},
\end{aligned}
$$

for all $t \in [0,1]$.

3. Main Results

We have the following mid-point type inequality:

Theorem 3. *Assume that* $f : G \to \mathbb{C}$ *is analytic on the convex domain* G *and* x, $y \in \mathcal{B}$, $x \neq y$ *with* $\sigma(x)$, $\sigma(y) \subset G$. *Then we have the norm inequality*

$$
\begin{aligned}
(3.1)\qquad & \left\|\int_0^1 f((1-t)x + ty)\,dt - f\left(\frac{x+y}{2}\right)\right\| \\
& \leq \frac{1}{48}\left[\left\|f''_{x,y}\right\|_{[0,1/2],\infty} + \left\|f''_{x,y}\right\|_{[1/2,1],\infty}\right] \leq \frac{1}{24}\left\|f''_{x,y}\right\|_{[0,1],\infty}.
\end{aligned}
$$

Proof. Using the integration by parts formula twice for the Bochner integral [17], we have

$$
\begin{aligned}
(3.2)\qquad \frac{1}{2}\int_0^{1/2} t^2 f''_{x,y}(t)\,dt &= \frac{1}{2}\left(t^2 f'_{x,y}(t)\Big|_0^{1/2} - 2\int_0^{1/2} t f'_{x,y}(t)\,dt\right) \\
&= \frac{1}{2}\left(\frac{1}{4} f'_{x,y}(1/2) - 2\int_0^{1/2} t f'_{x,y}(t)\,dt\right) \\
&= \frac{1}{8} f'_{x,y}(1/2) - \left(t f_{x,y}(t)\Big|_0^{1/2} - \int_0^{1/2} f_{x,y}(t)\,dt\right) \\
&= \frac{1}{8} f'_{x,y}(1/2) - \frac{1}{2} f_{x,y}(1/2) + \int_0^{1/2} f_{x,y}(t)\,dt
\end{aligned}
$$

and

$$\begin{aligned}(3.3)\qquad &\frac{1}{2}\int_{1/2}^{1}(t-1)^2 f''_{x,y}(t)\,dt\\ &=\frac{1}{2}\left((t-1)^2 f'_{x,y}(t)\Big|_{1/2}^{1}-2\int_{1/2}^{1}(t-1)\,f'_{x,y}(t)\,dt\right)\\ &=\frac{1}{2}\left(-\frac{1}{4}f'_{x,y}\left(\frac{1}{2}\right)-2\int_{1/2}^{1}(t-1)\,f'_{x,y}(t)\,dt\right)\\ &=-\frac{1}{8}f'_{x,y}(1/2)-\left((t-1)\,f_{x,y}(t)\,dt\Big|_{1/2}^{1}-\int_{1/2}^{1}f_{x,y}(t)\,dt\right)\\ &=-\frac{1}{8}f'_{x,y}(1/2)-\frac{1}{2}f_{x,y}(1/2)+\int_{1/2}^{1}f_{x,y}(t)\,dt.\end{aligned}$$

If we add (3.2) with (3.3), then we obtain the identity of interest

$$\begin{aligned}(3.4)\qquad &\int_0^1 f((1-t)\,x+ty)\,dt-f\left(\frac{x+y}{2}\right)\\ &=\frac{1}{2}\int_0^{1/2}t^2 f''_{x,y}(t)\,dt+\frac{1}{2}\int_{1/2}^{1}(t-1)^2 f''_{x,y}(t)\,dt.\end{aligned}$$

By taking the norm in (3.4) and using the properties of the integral, we get

$$\begin{aligned}(3.5)\qquad &\left\|\int_0^1 f((1-t)\,x+ty)\,dt-f\left(\frac{x+y}{2}\right)\right\|\\ &\le\frac{1}{2}\left\|\int_0^{1/2}t^2 f''_{x,y}(t)\,dt\right\|+\frac{1}{2}\left\|\int_{1/2}^{1}(t-1)^2 f''_{x,y}(t)\,dt\right\|\\ &\le\frac{1}{2}\int_0^{1/2}t^2\left\|f''_{x,y}(t)\right\|dt+\frac{1}{2}\int_{1/2}^{1}(t-1)^2\left\|f''_{x,y}(t)\right\|dt\\ &\le\frac{1}{48}\left\|f''_{x,y}\right\|_{[0,1/2],\infty}+\frac{1}{48}\left\|f''_{x,y}\right\|_{[1/2,1],\infty}\le\frac{1}{24}\left\|f''_{x,y}\right\|_{[0,1],\infty},\end{aligned}$$

which proves the desired result (3.1). □

Corollary 4. *Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and x, $y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)$, $\sigma(y)\subset G$ while $\gamma\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y)\subset\operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}(3.6)\qquad &\left\|\int_0^1 f((1-t)\,x+ty)\,dt-f\left(\frac{x+y}{2}\right)\right\|\\ &\le\frac{1}{24\pi}\left\|y-x\right\|^2\int_\gamma|f(\xi)|\max\left\{(|\xi|-\|x\|)^{-3},(|\xi|-\|y\|)^{-3}\right\}|d\xi|.\end{aligned}$$

If $\|f\|_{\gamma,\infty}:=\sup_{\xi\in\gamma}|f(\xi)|<0$, then

$$\begin{aligned}(3.7)\qquad &\left\|\int_0^1 f((1-t)\,x+ty)\,dt-f\left(\frac{x+y}{2}\right)\right\|\\ &\le\frac{1}{24\pi}\left\|y-x\right\|^2\|f\|_{\gamma,\infty}\int_\gamma\max\left\{(|\xi|-\|x\|)^{-3},(|\xi|-\|y\|)^{-3}\right\}|d\xi|.\end{aligned}$$

The proof follows by the last inequality in Lemma 5.

Remark 2. *Assume that* $f : D \subset \mathbb{C} \to \mathbb{C}$ *is an analytic function on the convex domain* D *and* $x, y \in \mathcal{B}$ *with* $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ *where* $D(0,R)$ *is an open disk centered in* 0 *and of radius* R*, then*

$$\left\| \int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{R\|y-x\|^2}{12\min\left\{(R-\|x\|)^3, (R-\|y\|)^3\right\}} \int_0^1 \left| f\left(Re^{2\pi is}\right) \right| ds. \tag{3.8}$$

If $\|f\|_{R,\infty} := \sup_{s\in[0,1]} \left| f\left(Re^{2\pi is}\right) \right| < \infty$*, then*

$$\left\| \int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{R\|y-x\|^2 \|f\|_{R,\infty}}{12\min\left\{(R-\|x\|)^3, (R-\|y\|)^3\right\}}. \tag{3.9}$$

We have the following trapezoid type inequality:

Theorem 4. *Assume that* $f : G \to \mathbb{C}$ *is analytic on the convex domain* G *and* x*,* $y \in \mathcal{B}$*,* $x \neq y$ *with* $\sigma(x), \sigma(y) \subset G$*. Then we have the norm inequality*

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt \right\| \leq \frac{1}{24}\left[\left\|f''_{x,y}\right\|_{[0,1/2],\infty} + \left\|f''_{x,y}\right\|_{[1/2,1],\infty} \right] \leq \frac{1}{12}\left\|f''_{x,y}\right\|_{[0,1],\infty}. \tag{3.10}$$

Proof. Using integration by parts formula for Bochner integral, then we have

$$\begin{aligned} \frac{1}{2}\int_0^1 t(1-t) f''_{x,y}(t)\,dt &= \frac{1}{2}\left(t(1-t) f'_{x,y}(t)\Big|_0^1 - \int_0^1 (1-2t) f'_{x,y}(t)\,dt \right) \\ &= \int_0^1 \left(t-\frac{1}{2}\right) f'_{x,y}(t)\,dt \\ &= \frac{f_{x,y}(1)+f_{x,y}(0)}{2} - \int_0^1 f_{x,y}(t)\,dt, \end{aligned}$$

which produces the following identity of interest

$$\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt = \frac{1}{2}\int_0^1 t(1-t) f''_{x,y}(t)\,dt.$$

By taking the norm, we get

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt\right\|\\&\le\frac{1}{2}\left\|\int_0^{1/2}t(1-t)f''_{x,y}(t)\,dt\right\|+\frac{1}{2}\left\|\int_{1/2}^1 t(1-t)f''_{x,y}(t)\,dt\right\|\\&\le\frac{1}{2}\int_0^{1/2}t(1-t)\left\|f''_{x,y}(t)\right\|dt+\frac{1}{2}\int_{1/2}^1 t(1-t)\left\|f''_{x,y}(t)\right\|dt\\&\le\frac{1}{2}\left\|f''_{x,y}\right\|_{[0,1/2],\infty}\int_0^{1/2}t(1-t)\,dt+\frac{1}{2}\left\|f''_{x,y}\right\|_{[1/2,1],\infty}\int_{1/2}^1 t(1-t)\,dt\\&=\frac{1}{24}\left\|f''_{x,y}\right\|_{[0,1/2],\infty}+\frac{1}{24}\left\|f''_{x,y}\right\|_{[1/2,1],\infty}\le\frac{1}{12}\left\|f''_{x,y}\right\|_{[0,1],\infty},\end{aligned}$$

which proves (3.10). □

Corollary 5. *Assume that $f: G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt\right\|\\&\le\frac{1}{12\pi}\|y-x\|^2\int_\gamma |f(\xi)|\max\left\{(|\xi|-\|x\|)^{-3},(|\xi|-\|y\|)^{-3}\right\}|d\xi|.\end{aligned}\tag{3.11}$$

If $\|f\|_{\gamma,\infty} := \sup_{\xi\in\gamma}|f(\xi)| < 0$, then

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt\right\|\\&\le\frac{1}{12\pi}\|y-x\|^2\|f\|_{\gamma,\infty}\int_\gamma \max\left\{(|\xi|-\|x\|)^{-3},(|\xi|-\|y\|)^{-3}\right\}|d\xi|.\end{aligned}\tag{3.12}$$

Remark 3. *Assume that $f: D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt\right\|\\&\le\frac{R\|y-x\|^2}{6\min\left\{(R-\|x\|)^3,(R-\|y\|)^3\right\}}\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds.\end{aligned}\tag{3.13}$$

If $\|f\|_{R,\infty} := \sup_{s\in[0,1]}\left|f\left(Re^{2\pi is}\right)\right| < \infty$, then

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt\right\|\\&\le\frac{R\|y-x\|^2\|f\|_{R,\infty}}{6\min\left\{(R-\|x\|)^3,(R-\|y\|)^3\right\}}.\end{aligned}\tag{3.14}$$

4. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z)=\left(\frac{1}{2}z\right)^\nu\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma\left(\nu+k+1\right)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{\left(k!\right)^2}.$$

An integral formula for real number ν is

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\left(\nu\theta\right)d\theta-\frac{\sin\left(\nu\pi\right)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\left(n\theta\right)d\theta.$$

For $n=0$ we have

$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f\left(a\right)=\exp a$, $a\in\mathcal{B}$. Assume that $x,\ y\in\mathcal{B}$ and $\left\|x\right\|,\ \left\|y\right\|<R$ for some $R>0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right|=\left|\exp\left[R\left(\cos\left(2\pi t\right)+i\sin\left(2\pi t\right)\right)\right]\right|=\exp\left[R\cos\left(2\pi t\right)\right]$$

and, by changing the variable $\theta=2\pi t$, we get $dt=\frac{1}{2\pi}d\theta$ and

$$\begin{aligned}(4.1)\qquad &\int_0^1\exp\left[R\cos\left(2\pi t\right)\right]dt\\ &=\frac{1}{2\pi}\int_0^{2\pi}\exp\left[R\cos\theta\right]d\theta\\ &=\frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi\exp\left[R\cos\theta\right]d\theta+\frac{1}{\pi}\int_\pi^{2\pi}\exp\left[R\cos\theta\right]d\theta\right)\\ &=\frac{1}{2}\left(I_0(R)+I_0(-R)\right)=I_0(R).\end{aligned}$$

Assume that $x,\ y\in\mathcal{B}$ with $\sigma\left(x\right),\ \sigma\left(y\right)\subset D\left(0,R\right)\subset D$ where $D\left(0,R\right)$ is an open disk centered in 0 and of radius R. By (3.8) and (3.13) for the exponential function we get

$$\begin{aligned}(4.2)\qquad &\left\|\int_0^1\exp\left(\left(1-t\right)x+ty\right)dt-\exp\left(\frac{x+y}{2}\right)\right\|\\ &\le\frac{RI_0(R)\left\|y-x\right\|^2}{12\min\left\{\left(R-\left\|x\right\|\right)^3,\left(R-\left\|y\right\|\right)^3\right\}}.\end{aligned}$$

and

$$\begin{aligned}(4.3)\qquad &\left\| \frac{\exp x + \exp y}{2} - \int_0^1 \exp\left((1-t)x + ty\right) dt \right\| \\ &\leq \frac{RI_0(R)\left\|y-x\right\|^2}{6\min\left\{\left(R-\left\|x\right\|\right)^3, \left(R-\left\|y\right\|\right)^3\right\}}.\end{aligned}$$

Let f be an analytic functions on the open disk $D(0,\rho)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $(|\lambda| < \rho)$. If $\nu(a) < \rho$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. We define the associated function

$$\lambda \mapsto f_A(\lambda) : D(0,\rho) \to \mathbb{C}, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned}(4.4)\qquad f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda},\ \ \lambda \in D(0,1); \\ g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos\lambda,\ \ \lambda \in \mathbb{C}; \\ h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin\lambda,\ \ \lambda \in \mathbb{C}; \\ l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda},\ \ \lambda \in D(0,1);\end{aligned}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned}(4.5)\qquad f_A(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda},\ \ \lambda \in D(0,1); \\ g_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh\lambda,\ \ \lambda \in \mathbb{C};\end{aligned}$$

$$\begin{aligned}h_A(\lambda) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh\lambda,\ \ \lambda \in \mathbb{C}; \\ l_A(\lambda) &= \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda},\ \ \lambda \in D(0,1).\end{aligned}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n \qquad \lambda \in \mathbb{C}, \tag{4.6}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma>0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

Lemma 6. *Let f be an analytic functions on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty}\alpha_j\lambda^j$. We have*

$$|f(\lambda)| \le f_A(|\lambda|) \text{ for } \lambda \in D(0,\rho). \tag{4.7}$$

Proof. For $\lambda \in D(0,\rho)$ we have

$$\begin{aligned} |f(\lambda)| &= \left|\sum_{j=0}^{\infty}\alpha_j\lambda^j\right| = \lim_{n\to\infty}\left|\sum_{j=0}^{n}\alpha_j\lambda^j\right| \le \lim_{n\to\infty}\sum_{j=0}^{n}\left|\alpha_j\lambda^j\right| \\ &= \lim_{n\to\infty}\sum_{j=0}^{n}|\alpha_j||\lambda|^j = \sum_{j=0}^{\infty}|\alpha_j||\lambda|^j = f_A(|\lambda|), \end{aligned}$$

which proves the statement. □

Proposition 1. *Let f be an analytic function on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty}\alpha_j\lambda^j$. If $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$ and $0<R<\rho$, then*

$$\begin{aligned} &\left\|\int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right)\right\| \\ &\le \frac{R\|y-x\|^2 f_A(R)}{12\min\left\{(R-\|x\|)^3,(R-\|y\|)^3\right\}} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} &\left\|\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt\right\| \\ &\le \frac{R\|y-x\|^2 f_A(R)}{6\min\left\{(R-\|x\|)^3,(R-\|y\|)^3\right\}}. \end{aligned} \tag{4.9}$$

The proof follows by (3.9), (3.14) and Lemma 6. As examples, one can consider the functions f and f_A listed above.

For instance, if $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$ and $0 < R < 1$, then by Proposition 1 for the function $f(\lambda) = \ln(1 \pm \lambda)$ defined on $D(0,1)$,

$$\left\| \int_0^1 \ln\left(1 \pm [(1-t)x + ty]\right)^{-1} dt - \ln\left(1 \pm \frac{x+y}{2}\right)^{-1} \right\| \leq \frac{R\|y-x\|^2 \ln(1-R)^{-1}}{12 \min\left\{(R-\|x\|)^3, (R-\|y\|)^3\right\}} \tag{4.10}$$

and

$$\left\| \frac{\ln(1 \pm x)^{-1} + \ln(1 \pm y)^{-1}}{2} - \int_0^1 \ln\left(1 \pm [(1-t)x + ty]\right)^{-1} dt \right\| \leq \frac{R\|y-x\|^2 \ln(1-R)^{-1}}{6 \min\left\{(R-\|x\|)^3, (R-\|y\|)^3\right\}}. \tag{4.11}$$

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