SOME MID-POINT AND TRAPEZOID TYPE L_1-NORM INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with spectra $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. In this paper we show among others that

$$\left\| \int_0^1 f((1-t)x + ty)\, dt - f\left(\frac{x+y}{2}\right) \right\| \leq \frac{1}{8\pi} \|y - x\|^2 \begin{cases} \int_\gamma \frac{1}{(|\xi| - \|y\|)^2 (|\xi| - \|x\|)^2} \left(|\xi| - \frac{\|x\| + \|y\|}{2}\right) |f(\xi)| \, |d\xi|, \\ \left\{ \frac{1}{2} \int_\gamma |f(\xi)| \left[(|\xi| - \|y\|)^{-3} + (|\xi| - \|x\|)^{-3} \right] |d\xi| \right. \\ \left. + \int_\gamma |f(\xi)| \left(|\xi| - \left\| \frac{x+y}{2} \right\| \right)^{-3} |d\xi| \right\} \end{cases}$$

and

$$\left\| \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty)\, dt \right\| \leq \frac{1}{8\pi} \|y - x\|^2 \begin{cases} \int_\gamma \frac{1}{(|\xi| - \|y\|)^2 (|\xi| - \|x\|)^2} \left(|\xi| - \frac{\|x\| + \|y\|}{2}\right) |f(\xi)| \, |d\xi|, \\ \left\{ \frac{1}{2} \int_\gamma |f(\xi)| \left[(|\xi| - \|y\|)^{-3} + (|\xi| - \|x\|)^{-3} \right] |d\xi| \right. \\ \left. + \int_\gamma |f(\xi)| \left(|\xi| - \left\| \frac{x+y}{2} \right\| \right)^{-3} |d\xi| \right\}. \end{cases}$$

Some examples for exponential function and functions defined by power series in Banach algebras are also given.

1. Introduction

The following mid-point inequality for twice differentiable functions holds, see for instance [13] and [15]:

Theorem 1. *Assume that f is twice differentiable on (a,b) and such that $\|f''\|_{(a,b),1} := \int_a^b |f''(u)|\, du < \infty$, then*

$$\left| \frac{1}{b-a} \int_a^b f(u)\, du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{1}{8} (b-a) \|f''\|_{(a,b),1}. \tag{1.1}$$

The constant $\frac{1}{8}$ is best possible in (1.1).

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The corresponding trapezoid inequality is as follows, see for instance [14] and [3]:

Theorem 2. *With the assumptions of Theorem 1,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right|\leq\frac{1}{8}(b-a)\|f''\|_{(a,b),1}. \tag{1.2}$$

The constant $\frac{1}{8}$ *is best possible in (1.1).*

In order to extend mid-point and trapezoid inequalities for functions defined on Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\|\leq\|a\|\|b\|$ for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a,b\in\operatorname{Inv}(\mathcal{B})$ then $ab\in\operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in\operatorname{Inv}(\mathcal{B})$;
(ii) $\{a\in\mathcal{B}:\ \|1-b\|<1\}\subset\operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B})\ni a\longmapsto a^{-1}\in\operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a:\rho(a)\to\operatorname{Inv}(\mathcal{B})$, $R_a(z):=(z-a)^{-1}$. For each $z,w\in\rho(a)$ we have the identity

$$R_a(w)-R_a(z)=(z-w)R_a(z)R_a(w).$$

We also have that

$$\sigma(a)\subset\{z\in\mathbb{C}:\ |z|\leq\|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a)=\sup\{|z|:z\in\sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda:\mathcal{B}\to\mathbb{C}$, the function $\lambda\circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n\in\mathbb{N}$ and $r>\nu(a)$, we have $a^n=\frac{1}{2\pi i}\int_{|\xi|=r}\xi^n(\xi-a)^{-1}d\xi$;
(v) We have $\nu(a)=\lim_{n\to\infty}\|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.3}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.4}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [16] and [18].

For some recent norm inequalities for functions on Banach algebras, see [8], [2] and [5]-[12].

2. Preliminary Results

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \ t \in [0,1]. \tag{2.1}$$

Lemma 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f'_{x,y}(t) = D(f)((1-t)x + ty)(y - x) \tag{2.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f'_{x,y}(0+) = D(f)(x)(y-x) \text{ and } f'_{x,y}(1-) = D(f)(y)(y-x). \tag{2.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f'_{x,y}(t) &= \frac{df_{x,y}(t)}{dt} = \lim_{h \to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \right] \\ &= D(f)((1-t)x + ty)(y-x), \end{aligned}$$

which proves (2.7).

The proof is similar for the lateral derivatives. □

We also have:

Lemma 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is twice differentiable on $(0,1)$ as a function of t and we have*

$$f''_{x,y}(t) = D^2(f)((1-t)x + ty)(y-x, y-x) \tag{2.4}$$

for all $t \in (0,1)$, where $D^2(f)(\cdot)(\cdot,\cdot)$ is the second Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f''_{x,y}(0+) = D^2(f)(x)(y-x, y-x), \quad f''_{x,y}(1-) = D^2(f)(y)(y-x, y-x). \tag{2.5}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{1}{h}\left[\frac{df_{x,y}(t+h)}{dt} - \frac{df_{x,y}(t)}{dt}\right] \\ &= \frac{D(f)((1-(t+h))x + (t+h)y)(y-x) - D(f)((1-t)x+ty)(y-x)}{h} \\ &= \frac{D(f)((1-t)x + ty + h(y-x))(y-x) - D(f)((1-t)x+ty)(y-x)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f''_{x,y}(t) &= \frac{d^2 f_{x,y}(t)}{dt^2} = \lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{df_{x,y}(t+h)}{dt} - \frac{df_{x,y}(t)}{dt}\right]\right\} \\ &= D^2(f)((1-t)x + ty)(y-x, y-x). \end{aligned}$$

The proof is similar for the lateral derivatives. □

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{2.6}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x + hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned} &f(x+hv) - f(x) \\ &= \frac{1}{2\pi i}\left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi\right) \\ &= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} - (\xi - x)^{-1}\right] d\xi \\ &= \frac{1}{2\pi i} h \int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} v (\xi - x)^{-1}\right] d\xi, \end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x+hv)-f(x)}{h}=\frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi-x-hv)^{-1} v(\xi-x)^{-1} d\xi .$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (2.6). □

Corollary 1. *Assume that $f: G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f_{x,y}^{'}(t)=\frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1} d\xi \tag{2.7}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f_{x,y}^{'}(0+)=\frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi-x)^{-1}(y-x)(\xi-x)^{-1} d\xi, \tag{2.8}$$

and

$$f_{x,y}^{'}(1-)=\frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi-y)^{-1}(y-x)(\xi-y)^{-1} d\xi. \tag{2.9}$$

The proof is obvious by Lemmas 1 and 2.

Lemma 4. *Assume that $f: G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for u, $v \in \mathcal{B}$ we have*

$$\begin{aligned} D^2(f)(x)(u,v) &= \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi-x)^{-1} v(\xi-x)^{-1} u(\xi-x)^{-1} d\xi \\ &+ \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi-x)^{-1} u(\xi-x)^{-1} v(\xi-x)^{-1} d\xi, \end{aligned} \tag{2.10}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$.

In particular, we have

$$D^2(f)(x)(v,v)=\frac{1}{\pi i}\int_{\gamma} f(\xi)\left((\xi-x)^{-1} v\right)^2(\xi-x)^{-1} d\xi, \tag{2.11}$$

for $v \in \mathcal{B}$.

Proof. Let $u, v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x+hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$D^2(f)(x)(u,v)=\lim_{h\to 0}\frac{D(f)(x+hv)(u)-D(f)(x)(u)}{h}.$$

We have by (2.6)

$$\begin{aligned} &D(f)(x+hv)(u)-D(f)(x)(u) \\ &=\frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi-x-hv)^{-1} u(\xi-x-hv)^{-1} d\xi \\ &-\frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi-x)^{-1} u(\xi-x)^{-1} d\xi \\ &=\frac{1}{2\pi i}\int_{\gamma} f(\xi) k_{\xi,h}(x,u,v) d\xi, \end{aligned}$$

where

$$k_{\xi,h}(x,u,v) := (\xi - x - hv)^{-1} u (\xi - x - hv)^{-1} - (\xi - x)^{-1} u (\xi - x)^{-1} \tag{2.12}$$

and $\xi \in \gamma$ while h is in a small interval around 0, $h \neq 0$.

If we multiply both sides of (2.12) by $\xi - x - hv$, then we get

$$\begin{aligned}
&(\xi - x - hv) k_{\xi,h}(x,u,v)(\xi - x - hv) \\
&= u - (\xi - x - hv)(\xi - x)^{-1} u (\xi - x)^{-1} (\xi - x - hv) \\
&= u - \left(1 - hv(\xi - x)^{-1}\right) u \left(1 - h(\xi - x)^{-1} v\right) \\
&= u - \left(u - hv(\xi - x)^{-1} u\right) \left(1 - h(\xi - x)^{-1} v\right) \\
&= u - u + hv(\xi - x)^{-1} u + hu(\xi - x)^{-1} v - h^2 v(\xi - x)^{-1} u (\xi - x)^{-1} v \\
&= h\left[v(\xi - x)^{-1} u + u(\xi - x)^{-1} v - hv(\xi - x)^{-1} u (\xi - x)^{-1} v\right].
\end{aligned} \tag{2.13}$$

The identity (2.13) implies that

$$\begin{aligned}
\frac{1}{h} k_{\xi,h}(x,u,v) &= (\xi - x - hv)^{-1} \\
&\quad \times \left[v(\xi - x)^{-1} u + u(\xi - x)^{-1} v - hv(\xi - x)^{-1} u (\xi - x)^{-1} v\right] \\
&\quad \times (\xi - x - hv)^{-1},
\end{aligned}$$

which gives that

$$\begin{aligned}
&\lim_{h \to 0} \frac{1}{h} k_{\xi,h}(x,u,v) \\
&= (\xi - x)^{-1} \left[v(\xi - x)^{-1} u + u(\xi - x)^{-1} v\right] (\xi - x)^{-1} \\
&= (\xi - x)^{-1} v (\xi - x)^{-1} u (\xi - x)^{-1} + (\xi - x)^{-1} u (\xi - x)^{-1} v (\xi - x)^{-1}
\end{aligned}$$

for $\xi \in \gamma$.

Therefore

$$\begin{aligned}
&\lim_{h \to 0} \frac{D(f)(x + hv)(u) - D(f)(x)(u)}{h} \\
&= \frac{1}{2\pi i} \lim_{h \to 0} \int_\gamma f(\xi) \frac{1}{h} k_{\xi,h}(x,u,v)\, d\xi = \frac{1}{2\pi i} \int_\gamma f(\xi) \lim_{h \to 0} \frac{1}{h} k_{\xi,h}(x,u,v)\, d\xi \\
&= \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} u (\xi - x)^{-1} d\xi \\
&+ \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x)^{-1} u (\xi - x)^{-1} v (\xi - x)^{-1} d\xi
\end{aligned}$$

and the lemma is thus proved. □

Corollary 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}
&f''_{x,y}(t) \\
&= \frac{1}{\pi i} \int_\gamma f(\xi) \left((\xi - (1-t)x - ty)^{-1} (y - x)\right)^2 (\xi - (1-t)x - ty)^{-1} d\xi
\end{aligned} \tag{2.14}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f''_{x,y}(0+) = \frac{1}{2\pi i}\int_\gamma f(\xi)\left((\xi - x)^{-1}(y-x)\right)^2 (\xi - x)^{-1} d\xi, \tag{2.15}$$

and

$$f''_{x,y}(1-) = \frac{1}{2\pi i}\int_\gamma f(\xi)\left((\xi - y)^{-1}(y-x)\right)^2 (\xi - y)^{-1} d\xi. \tag{2.16}$$

Lemma 5. *Assume that* $f : G \to \mathbb{C}$ *is analytic on the convex domain* G *and* x, $y \in \mathcal{B}$, $x \neq y$ *with* $\sigma(x)$, $\sigma(y) \subset G$ *while* $\gamma \subset G$ *is taken to be close rectifiable curve in* G *and such that* $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. *Then*

$$\begin{aligned}
&\left\|f''_{x,y}(t)\right\| \\
&\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left\|(\xi - (1-t)x - ty)^{-1}\right\|^3 |d\xi| \\
&\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(|\xi| - \|(1-t)x + ty\|\right)^{-3} |d\xi| \\
&\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)\right]^{-3} |d\xi| \\
&\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left[(1-t)(|\xi| - \|x\|)^{-3} + t(|\xi| - \|y\|)^{-3}\right] |d\xi|
\end{aligned} \tag{2.17}$$

for all $t \in [0,1]$.

Proof. By taking the norm in (2.14) and using the integral's properties, we have

$$\begin{aligned}
&\left\|f''_{x,y}(t)\right\| \\
&\le \frac{1}{\pi}\int_\gamma |f(\xi)| \\
&\times \left\|\left((\xi - (1-t)x - ty)^{-1}(y-x)\right)^2 (\xi - (1-t)x - ty)^{-1}\right\| |d\xi| \\
&\le \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left\|(\xi - (1-t)x - ty)^{-1}\right\|^3 |d\xi| \\
&= \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| |\xi|^{-3} \left\|\left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi}\right)^{-1}\right\|^3 |d\xi|
\end{aligned} \tag{2.18}$$

for all $t \in [0,1]$, which proves the first inequality in (2.17).

Since

$$\left\|(1-t)\frac{x}{\xi} + t\frac{y}{\xi}\right\| \le (1-t)\left\|\frac{x}{\xi}\right\| + t\left\|\frac{y}{\xi}\right\| < 1 - t + t = 1$$

for $\xi \in \gamma$, hence

$$\left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi}\right)^{-1} = \sum_{k=0}^{\infty}\left[(1-t)\frac{x}{\xi} + t\frac{y}{\xi}\right]^k.$$

Therefore

$$\begin{aligned}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\| &\le \sum_{k=0}^{\infty}\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|^{k}\\ &=\left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=\left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=|\xi|\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\end{aligned}$$

for $\xi\in\gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{3}\le|\xi|^{2}\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}$$

for $\xi\in\gamma$.

Therefore

$$\begin{aligned}&\int_{\gamma}|f(\xi)|\,|\xi|^{-3}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{3}|d\xi|\\ &\le\int_{\gamma}|f(\xi)|\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}|d\xi|\end{aligned}$$

and we derive the second inequality in (2.17).

By the triangle inequality we have

$$\begin{aligned}|\xi|-\|(1-t)x+ty\| &\ge|\xi|-(1-t)\|x\|-t\|y\|\\ &=(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)>0\end{aligned}$$

for $\xi\in\gamma$.

This implies that

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\le\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-1}$$

and thus

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}\le\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-3}$$

for $\xi\in\gamma$ and $t\in[0,1]$. This proves the third inequality in (2.17).

By the convexity of the power function $(\cdot)^{-3}$ we also have

$$\begin{aligned}&\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-3}\\ &\le(1-t)(|\xi|-\|x\|)^{-3}+t(|\xi|-\|y\|)^{-3}\end{aligned}$$

for $t\in[0,1]$, which proves the fourth inequality in (2.17). □

Corollary 3. *Assume that $f:D\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the convex domain D and x, $y\in\mathcal{B}$ with $\sigma(x)$, $\sigma(y)\subset D(0,R)\subset D$ where $D(0,R)$ is an open* *disk centered in* 0 *and of radius* R, *then*

$$\begin{aligned}
&\left\|f''_{x,y}(t)\right\| \\
&\le 2R\|y-x\|^2\int_0^1\left|f\left(Re^{2\pi is}\right)\right|\left\|\left(Re^{2\pi is}-(1-t)x-ty\right)^{-1}\right\|^3 ds \\
&\le 2R\|y-x\|^2\left(R-\|(1-t)x+ty\|\right)^{-3}\int_0^1\left|f\left(Re^{2\pi is}\right)\right| ds \\
&\le 2R\|y-x\|^2\left[(1-t)(R-\|x\|)+t(R-\|y\|)\right]^{-3}\int_0^1\left|f\left(Re^{2\pi is}\right)\right| ds \\
&\le 2R\|y-x\|^2\left[(1-t)(R-\|x\|)^{-3}+t(R-\|y\|)^{-3}\right]\int_0^1\left|f\left(Re^{2\pi is}\right)\right| ds
\end{aligned} \tag{2.19}$$

for all $t\in[0,1]$.

It follows by taking γ parametrized by $\xi(s)=Re^{2\pi is}$ where $s\in[0,1]$, then $d\xi(s)=2\pi iRe^{2\pi is}ds$, $|d\xi(s)|=2\pi Rds$, $|\xi|=R$ and by (2.17) we get (2.19).

Remark 1. *If* $\|f\|_{\gamma,\infty}:=\sup_{\xi\in\gamma}|f(\xi)|<0$, *then from (2.17) we get*

$$\begin{aligned}
&\left\|f''_{x,y}(t)\right\| \\
&\le \frac{1}{\pi}\|y-x\|^2\|f\|_{\gamma,\infty}\int_\gamma\left\|\left(\xi-(1-t)x-ty\right)^{-1}\right\|^3|d\xi| \\
&\le \frac{1}{\pi}\|y-x\|^2\|f\|_{\gamma,\infty}\int_\gamma\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}|d\xi| \\
&\le \frac{1}{\pi}\|y-x\|^2\|f\|_{\gamma,\infty}\int_\gamma\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-3}|d\xi| \\
&\le \frac{1}{\pi}\|y-x\|^2\|f\|_{\gamma,\infty}\int_\gamma\left[(1-t)(|\xi|-\|x\|)^{-3}+t(|\xi|-\|y\|)^{-3}\right]|d\xi|
\end{aligned} \tag{2.20}$$

for all $t\in[0,1]$.

Also, if $\|f\|_{R,\infty}:=\sup_{s\in[0,1]}\left|f\left(Re^{2\pi is}\right)\right|<\infty$, *then from (2.19) we get*

$$\begin{aligned}
&\left\|f''_{x,y}(t)\right\| \\
&\le 2R\|y-x\|^2\|f\|_{R,\infty}\int_0^1\left\|\left(Re^{2\pi is}-(1-t)x-ty\right)^{-1}\right\|^3 ds \\
&\le 2R\|y-x\|^2\left(R-\|(1-t)x+ty\|\right)^{-3}\|f\|_{R,\infty} \\
&\le 2R\|y-x\|^2\left[(1-t)(R-\|x\|)+t(R-\|y\|)\right]^{-3}\|f\|_{R,\infty} \\
&\le 2R\|y-x\|^2\left[(1-t)(R-\|x\|)^{-3}+t(R-\|y\|)^{-3}\right]\|f\|_{R,\infty}
\end{aligned} \tag{2.21}$$

for all $t\in[0,1]$.

3. Bounds for the 1-Norm

We have the following upper bound for the 1-norm of $f''_{x,y}$:

Theorem 3. *Assume that* $f:G\to\mathbb{C}$ *is analytic on the convex domain* G *and* x, $y\in\mathcal{B}$, $x\neq y$ *with* $\sigma(x)$, $\sigma(y)\subset G$ *while* $\gamma\subset G$ *is taken to be close rectifiable curve* *in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\left\|f''_{x,y}\right\|_{[0,1],1} := \int_0^1 \left\|f''_{x,y}(t)\right\| dt \tag{3.1}$$
$$\leq \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 \left\|(\xi-(1-t)x-ty)^{-1}\right\|^3 dt\right)|d\xi|$$
$$\leq \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 (|\xi|-\|(1-t)x+ty\|)^{-3} dt\right)|d\xi|$$
$$\leq \frac{1}{\pi}\|y-x\|^2 \int_\gamma \frac{|f(\xi)|\left(|\xi|-\frac{\|x\|+\|y\|}{2}\right)}{(|\xi|-\|y\|)^2(|\xi|-\|x\|)^2}|d\xi|$$
$$\leq \frac{1}{2\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left[(|\xi|-\|x\|)^{-3}+(|\xi|-\|y\|)^{-3}\right]|d\xi|.$$

Proof. By taking the integral over t in $[0,1]$ and using Fubini's theorem, we get

$$\int_0^1 \left\|f''_{x,y}(t)\right\| dt \tag{3.2}$$
$$\leq \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 \left\|(\xi-(1-t)x-ty)^{-1}\right\|^3 dt\right)|d\xi|$$
$$\leq \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 (|\xi|-\|(1-t)x+ty\|)^{-3} dt\right)|d\xi|$$

$$\leq \frac{1}{\pi}\|y-x\|^2$$
$$\times \int_\gamma |f(\xi)| \left(\int_0^1 [(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-3} dt\right)|d\xi|$$
$$\leq \frac{1}{\pi}\|y-x\|^2$$
$$\times \int_\gamma |f(\xi)| \left(\int_0^1 \left[(1-t)(|\xi|-\|x\|)^{-3}+t(|\xi|-\|y\|)^{-3}\right] dt\right)|d\xi|.$$

Observe that

$$\int_0^1 [(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-3} dt$$
$$= -\frac{1}{2(\|x\|-\|y\|)}\int_0^1 \frac{d}{dt}[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2} dt$$
$$= -\frac{1}{2(\|x\|-\|y\|)}[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2}\Big|_0^1$$

$$
\begin{aligned}
&= \frac{1}{2\left(\|y\|-\|x\|\right)}\left[\left(|\xi|-\|y\|\right)^{-2}-\left(|\xi|-\|x\|\right)^{-2}\right] \\
&= \frac{1}{2\left(\|y\|-\|x\|\right)}\frac{\left(|\xi|-\|x\|\right)^{2}-\left(|\xi|-\|y\|\right)^{2}}{\left(|\xi|-\|y\|\right)^{2}\left(|\xi|-\|x\|\right)^{2}} \\
&= \frac{1}{2\left(\|y\|-\|x\|\right)}\frac{\left(\|y\|-\|x\|\right)\left(2\left|\xi\right|-\|x\|-\|y\|\right)}{\left(|\xi|-\|y\|\right)^{2}\left(|\xi|-\|x\|\right)^{2}} \\
&= \frac{|\xi|-\frac{\|x\|+\|y\|}{2}}{\left(|\xi|-\|y\|\right)^{2}\left(|\xi|-\|x\|\right)^{2}}
\end{aligned}
$$

for $\|y\| \neq \|x\|$, which, by (3.2), proves the third inequality of (3.1).

If $\|y\| = \|x\|$, then we have

$$
\begin{aligned}
&\int_0^1 \left(|\xi|-\|(1-t)\,x+ty\|\right)^{-3} dt \\
&\leq \int_0^1 \left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|x\|\right)\right]^{-3} dt = \left(|\xi|-\|x\|\right)^{-3},
\end{aligned}
$$

which also gives the bound for $\|y\| = \|x\|$.

Finally, since

$$
\begin{aligned}
&\int_0^1 \left[(1-t)\left(|\xi|-\|x\|\right)^{-3}+t\left(|\xi|-\|y\|\right)^{-3}\right] dt \\
&= \frac{1}{2}\left[\left(|\xi|-\|x\|\right)^{-3}+\left(|\xi|-\|y\|\right)^{-3}\right],
\end{aligned}
$$

hence the last part of (3.1) is also proved. □

Corollary 4. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$
\begin{aligned}
&\left\|f''_{x,y}\right\|_{[0,1],1} \\
&\leq 2R\left\|y-x\right\|^{2}\int_0^1 \left|f\left(Re^{2\pi it}\right)\right|\left(\int_0^1 \left\|\left(Re^{2\pi is}-(1-s)\,x-sy\right)^{-1}\right\|^{3} ds\right) dt \\
&\leq 2R\left\|y-x\right\|^{2}\left(\int_0^1 \left(R-\|(1-s)\,x+sy\|\right)^{-3} ds\right)\int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt \\
&\leq \frac{2R\left\|y-x\right\|^{2}\left(R-\frac{\|x\|+\|y\|}{2}\right)}{\left(R-\|y\|\right)^{2}\left(R-\|x\|\right)^{2}}\int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt \\
&\leq R\left\|y-x\right\|^{2}\left[\left(R-\|x\|\right)^{-3}+\left(R-\|y\|\right)^{-3}\right]\int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt.
\end{aligned}
\tag{3.3}
$$

From a different perspective, we also have:

Theorem 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve* *in* G *and such that* $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. *Then*

$$\begin{aligned}
&\left\|f''_{x,y}\right\|_{[0,1],1} \\
&\leq \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 \left\|(\xi-(1-t)x-ty)^{-1}\right\|^3 dt\right)|d\xi| \\
&\leq \frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 (|\xi|-\|(1-t)x+ty\|)^{-3} dt\right)|d\xi| \\
&\leq \frac{1}{2\pi}\|y-x\|^2 \left\{\frac{1}{2}\int_\gamma |f(\xi)| \left[(|\xi|-\|y\|)^{-3}+(|\xi|-\|x\|)^{-3}\right]|d\xi| \right. \\
&\left. + \int_\gamma |f(\xi)| \left(|\xi|-\left\|\frac{x+y}{2}\right\|\right)^{-3}|d\xi|\right\} \\
&\leq \frac{1}{2\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left[(|\xi|-\|y\|)^{-3}+(|\xi|-\|x\|)^{-3}\right]|d\xi|.
\end{aligned} \tag{3.4}$$

Proof. Let $\xi \in \gamma$. For $\alpha, \beta \geq 0$ with $\alpha+\beta=1$ and $t_1, t_2 \in [0,1]$, then for $g_\xi(t) := (|\xi|-\|(1-t)x+ty\|)^{-3}$, $t \in [0,1]$ we get

$$\begin{aligned}
&g_\xi(\alpha t_1+\beta t_2) \\
&= (|\xi|-\|(1-(\alpha t_1+\beta t_2))x+(\alpha t_1+\beta t_2)y\|)^{-3} \\
&= (|\xi|-\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\|)^{-3}.
\end{aligned}$$

By the properties of the norm, we have

$$\begin{aligned}
&\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\| \\
&\leq \alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|,
\end{aligned}$$

which gives that

$$\begin{aligned}
&|\xi|-\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\| \\
&\geq |\xi|-\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\| > 0
\end{aligned}$$

for $\xi \in \gamma$.

This implies that

$$\begin{aligned}
&(|\xi|-\|\alpha[(1-t_1)x+t_1y]+\beta[(1-t_2)x+t_2y]\|)^{-1} \\
&\leq (|\xi|-\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|)^{-1}
\end{aligned}$$

giving that

$$\begin{aligned}
&g_\xi(\alpha t_1+\beta t_2) \\
&\leq (|\xi|-\alpha\|(1-t_1)x+t_1y\|+\beta\|(1-t_2)x+t_2y\|)^{-3} \\
&= (\alpha[|\xi|-\|(1-t_1)x+t_1y\|]+\beta[|\xi|-\|(1-t_2)x+t_2y\|])^{-3}.
\end{aligned}$$

By using the convexity of the function $(\cdot)^{-3}$ we have

$$\begin{aligned}
&(\alpha[|\xi|-\|(1-t_1)x+t_1y\|]+\beta[|\xi|-\|(1-t_2)x+t_2y\|])^{-3} \\
&\leq \alpha[|\xi|-\|(1-t_1)x+t_1y\|]^{-3}+\beta[|\xi|-\|(1-t_2)x+t_2y\|]^{-3} \\
&= \alpha g_\xi(t_1)+\beta g_\xi(t_2).
\end{aligned}$$

Therefore

$$g_\xi(\alpha t_1 + \beta t_2) \le \alpha g_\xi(t_1) + \beta g_\xi(t_2),$$

which proves the convexity of g_ξ on $[0,1]$.

By using the Hermite-Hadamard type inequality, see for instance [15, p. 11], for g_ξ on $[0,1]$ we get

$$\int_0^1 g_\xi(t)\,dt \le \frac{1}{2}\left\{\frac{1}{2}\left[g_\xi(1) + g_\xi(0)\right] + g_\xi\left(\frac{1}{2}\right)\right\} \le \frac{1}{2}\left[g_\xi(1) + g_\xi(0)\right]$$

namely

$$\begin{aligned}&\int_0^1 \left(|\xi| - \|(1-t)x + ty\|\right)^{-3} dt \\ &\le \frac{1}{2}\left\{\frac{1}{2}\left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right] + \left(|\xi| - \left\|\frac{x+y}{2}\right\|\right)^{-3}\right\} \\ &\le \frac{1}{2}\left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right]\end{aligned}$$

for $\xi \in \gamma$.

From (3.1) we obtain

$$\begin{aligned}&\frac{1}{\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 \left(|\xi| - \|(1-t)x + ty\|\right)^{-3} dt\right)|d\xi| \\ &\le \frac{1}{2\pi}\|y-x\|^2 \left\{\frac{1}{2}\int_\gamma |f(\xi)| \left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right]|d\xi| \right. \\ &\left. + \int_\gamma |f(\xi)| \left(|\xi| - \left\|\frac{x+y}{2}\right\|\right)^{-3}|d\xi|\right\} \\ &\le \frac{1}{2\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left[\left(|\xi| - \|y\|\right)^{-3} + \left(|\xi| - \|x\|\right)^{-3}\right]|d\xi|,\end{aligned}$$

which proves the desired result (3.4). □

Corollary 5. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$\begin{aligned}&\left\|f''_{x,y}\right\|_{[0,1],1} \\ &\le 2R\|y-x\|^2 \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| \left(\int_0^1 \left\|\left(Re^{2\pi is} - (1-s)x - sy\right)^{-1}\right\|^3 ds\right) dt \\ &\le 2R\|y-x\|^2 \left(\int_0^1 \left(R - \|(1-s)x + sy\|\right)^{-3} ds\right) \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt \\ &\le R\|y-x\|^2 \left\{\frac{1}{2}\left[\left(R - \|y\|\right)^{-3} + \left(R - \|x\|\right)^{-3}\right] \right. \\ &\left. + \left(R - \left\|\frac{x+y}{2}\right\|\right)^{-3}\right\} \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt \\ &\le R\|y-x\|^2 \left[\left(R - \|x\|\right)^{-3} + \left(R - \|y\|\right)^{-3}\right] \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt.\end{aligned} \tag{3.5}$$

Remark 2. *We observe that, by Corollaries 2.6 and 2.7 we have the following upper bounds*

$$B_1(f,x,y,R) := \frac{2R\|y-x\|^2\left(R-\frac{\|x\|+\|y\|}{2}\right)}{(R-\|y\|)^2(R-\|x\|)^2}\int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt$$

and

$$B_2(f,x,y,R) := R\|y-x\|^2\left\{\frac{1}{2}\left[(R-\|y\|)^{-3}+(R-\|x\|)^{-3}\right] + \left(R-\left\|\frac{x+y}{2}\right\|\right)^{-3}\right\}\int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt$$

for the 1*-norm of the second derivative* $\left\|f''_{x,y}\right\|_{[0,1],1}$, *where* x, $y\in\mathcal{B}$ *with* $\sigma(x)$, $\sigma(y)\subset D(0,R)\subset D$.

If we consider the scalar functions g, $h:(-1,1)\times(-1,1)\to\mathbb{R}$ *defined by*

$$g(x,y) := \frac{2\left(1-\frac{|x|+|y|}{2}\right)}{(1-|y|)^2(1-|x|)^2}$$

and

$$h(x,y) := \frac{1}{2}\left[(1-|y|)^{-3}+(1-|x|)^{-3}\right] + \left(1-\left|\frac{x+y}{2}\right|\right)^{-3},$$

then a 3-dimensional plot of the difference $g(x,y)-h(x,y)$ *in the box* $(-1,1)\times(-1,1)$ *shows that it takes both positive and negative values, meaning that some time the bound* $B_1(f,x,y,R)$ *is better and other times is worse than* $B_2(f,x,y,R)$.

4. Main Results

We have the following mid-point type inequality:

Theorem 5. *Assume that* $f:G\to\mathbb{C}$ *is analytic on the convex domain* G *and* x, $y\in\mathcal{B}$, $x\neq y$ *with* $\sigma(x)$, $\sigma(y)\subset G$. *Then we have the norm inequality*

$$\left\|\int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right)\right\| \leq \frac{1}{8}\left\|f''_{x,y}\right\|_{[0,1],1}. \tag{4.1}$$

Proof. Using the integration by parts formula twice for the Bochner integral [17], we have

$$\begin{aligned}\frac{1}{2}\int_0^{1/2} t^2 f''_{x,y}(t)\,dt &= \frac{1}{2}\left(t^2 f'_{x,y}(t)\Big|_0^{1/2} - 2\int_0^{1/2} t f'_{x,y}(t)\,dt\right) \\ &= \frac{1}{2}\left(\frac{1}{4}f'_{x,y}(1/2) - 2\int_0^{1/2} t f'_{x,y}(t)\,dt\right) \\ &= \frac{1}{8}f'_{x,y}(1/2) - \left(t f_{x,y}(t)\Big|_0^{1/2} - \int_0^{1/2} f_{x,y}(t)\,dt\right) \\ &= \frac{1}{8}f'_{x,y}(1/2) - \frac{1}{2}f_{x,y}(1/2) + \int_0^{1/2} f_{x,y}(t)\,dt\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
(4.3)\qquad & \frac{1}{2}\int_{1/2}^{1}(t-1)^2 f''_{x,y}(t)\,dt \\
&= \frac{1}{2}\left((t-1)^2 f'_{x,y}(t)\Big|_{1/2}^{1} - 2\int_{1/2}^{1}(t-1) f'_{x,y}(t)\,dt\right) \\
&= \frac{1}{2}\left(-\frac{1}{4}f'_{x,y}\left(\frac{1}{2}\right) - 2\int_{1/2}^{1}(t-1) f'_{x,y}(t)\,dt\right) \\
&= -\frac{1}{8}f'_{x,y}(1/2) - \left((t-1) f_{x,y}(t)\,dt\Big|_{1/2}^{1} - \int_{1/2}^{1} f_{x,y}(t)\,dt\right) \\
&= -\frac{1}{8}f'_{x,y}(1/2) - \frac{1}{2}f_{x,y}(1/2) + \int_{1/2}^{1} f_{x,y}(t)\,dt.
\end{aligned}$$

If we add (4.2) with (4.3), then we obtain the identity of interest

$$\begin{aligned}
(4.4)\qquad & \int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right) \\
&= \frac{1}{2}\int_0^{1/2} t^2 f''_{x,y}(t)\,dt + \frac{1}{2}\int_{1/2}^{1}(t-1)^2 f''_{x,y}(t)\,dt.
\end{aligned}$$

By taking the norm in (4.4) and using the properties of the integral, we get

$$\begin{aligned}
(4.5)\qquad & \left\|\int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right)\right\| \\
&\le \frac{1}{2}\left\|\int_0^{1/2} t^2 f''_{x,y}(t)\,dt\right\| + \frac{1}{2}\left\|\int_{1/2}^{1}(t-1)^2 f''_{x,y}(t)\,dt\right\| \\
&\le \frac{1}{2}\int_0^{1/2} t^2 \left\|f''_{x,y}(t)\right\|dt + \frac{1}{2}\int_{1/2}^{1}(t-1)^2 \left\|f''_{x,y}(t)\right\|dt =: M.
\end{aligned}$$

Observe that

$$\int_0^{1/2} t^2 \left\|f''_{x,y}(t)\right\|dt \le \sup_{t\in[0,1/2]}\left\{t^2\right\}\int_0^{1/2}\left\|f''_{x,y}(t)\right\|dt = \frac{1}{4}\left\|f''_{x,y}\right\|_{[0,1/2],1}$$

and

$$\int_{1/2}^{1}(t-1)^2 \left\|f''_{x,y}(t)\right\|dt \le \sup_{t\in[1/2,1]}\left\{(1-t)^2\right\}\int_{1/2}^{1}\left\|f''_{x,y}(t)\right\|dt = \frac{1}{4}\left\|f''_{x,y}\right\|_{[1/2,1],1}.$$

Therefore

$$M \le \frac{1}{8}\left\|f''_{x,y}\right\|_{[0,1/2],1} + \frac{1}{8}\left\|f''_{x,y}\right\|_{[1/2,1],1} = \frac{1}{8}\left\|f''_{x,y}\right\|_{[0,1],1}$$

and the inequality (4.1) is proved. □

Corollary 6. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve* *in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}
&\left\| \int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right) \right\| \\
&\leq \frac{1}{8\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 \left\| (\xi-(1-t)x-ty)^{-1} \right\|^3 dt \right) |d\xi| \\
&\leq \frac{1}{8\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 (|\xi| - \|(1-t)x+ty\|)^{-3}\,dt \right) |d\xi| \\
&\leq \frac{1}{8\pi}\|y-x\|^2 \begin{cases} \int_\gamma \frac{1}{(|\xi|-\|y\|)^2(|\xi|-\|x\|)^2}\left(|\xi| - \frac{\|x\|+\|y\|}{2}\right)|f(\xi)|\,|d\xi|, \\ \left\{\frac{1}{2}\int_\gamma |f(\xi)| \left[(|\xi|-\|y\|)^{-3} + (|\xi|-\|x\|)^{-3}\right]|d\xi| \right. \\ \left. + \int_\gamma |f(\xi)| \left(|\xi| - \left\|\frac{x+y}{2}\right\|\right)^{-3}|d\xi|\right\} \end{cases} \\
&\leq \frac{1}{16\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left[(|\xi|-\|x\|)^{-3} + (|\xi|-\|y\|)^{-3}\right]|d\xi|.
\end{aligned} \tag{4.6}$$

The case when the spectra $\sigma(x), \sigma(y)$ are included in a disk is as follows:

Corollary 7. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$\begin{aligned}
&\left\| \int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right) \right\| \\
&\leq \frac{1}{4}R\|y-x\|^2 \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| \left(\int_0^1 \left\| \left(Re^{2\pi is}-(1-s)x-sy\right)^{-1} \right\|^3 ds \right) dt \\
&\leq \frac{1}{4}R\|y-x\|^2 \left(\int_0^1 (R - \|(1-s)x+sy\|)^{-3}\,ds \right) \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt \\
&\leq \frac{1}{4}R\|y-x\|^2 \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt \\
&\times \begin{cases} \frac{1}{(R-\|y\|)^2(R-\|x\|)^2}\left(R - \frac{\|x\|+\|y\|}{2}\right), \\ \left\{\frac{1}{2}\left[(R-\|y\|)^{-3} + (R-\|x\|)^{-3}\right] + \left(R - \left\|\frac{x+y}{2}\right\|\right)^{-3}\right\} \end{cases} \\
&\leq \frac{1}{8}R\|y-x\|^2 \left[(R-\|x\|)^{-3} + (R-\|y\|)^{-3}\right] \int_0^1 \left|f\left(Re^{2\pi it}\right)\right| dt.
\end{aligned} \tag{4.7}$$

We have the following trapezoid type inequality:

Theorem 6. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. Then we have the norm inequality*

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt \right\| \leq \frac{1}{8}\left\|f''_{x,y}\right\|_{[0,1],1}. \tag{4.8}$$

Proof. Using integration by parts formula for Bochner integral, then we have

$$\begin{aligned}\frac{1}{2}\int_0^1 t(1-t) f''_{x,y}(t)\,dt &= \frac{1}{2}\left(t(1-t) f'_{x,y}(t)\Big|_0^1 - \int_0^1 (1-2t) f'_{x,y}(t)\,dt \right)\\ &= \int_0^1 \left(t-\frac{1}{2}\right) f'_{x,y}(t)\,dt\\ &= \frac{f_{x,y}(1)+f_{x,y}(0)}{2} - \int_0^1 f_{x,y}(t)\,dt,\end{aligned}$$

which produces the following identity of interest

$$\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt = \frac{1}{2}\int_0^1 t(1-t) f''_{x,y}(t)\,dt.$$

By taking the norm, we get

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt\right\|\\ &\le \frac{1}{2}\left\|\int_0^{1/2} t(1-t) f''_{x,y}(t)\,dt\right\| + \frac{1}{2}\left\|\int_{1/2}^1 t(1-t) f''_{x,y}(t)\,dt\right\|\\ &\le \frac{1}{2}\int_0^{1/2} t(1-t)\left\|f''_{x,y}(t)\right\|dt + \frac{1}{2}\int_{1/2}^1 t(1-t)\left\|f''_{x,y}(t)\right\|dt\\ &\le \frac{1}{2}\sup_{t\in[0,1/2]}\{t(1-t)\}\left\|f''_{x,y}\right\|_{[0,1/2],1} + \frac{1}{2}\sup_{t\in[1/2,1]}\{t(1-t)\}\left\|f''_{x,y}\right\|_{[1/2,1],1}\\ &= \frac{1}{8}\left\|f''_{x,y}\right\|_{[0,1/2],1} + \frac{1}{8}\left\|f''_{x,y}\right\|_{[1/2,1],1} = \frac{1}{8}\left\|f''_{x,y}\right\|_{[0,1],1},\end{aligned}$$

which proves (4.8). □

Corollary 8. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt\right\| \qquad (4.9)\\ &\le \frac{1}{8\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 \left\|(\xi-(1-t)x-ty)^{-1}\right\|^3 dt\right)|d\xi|\\ &\le \frac{1}{8\pi}\|y-x\|^2 \int_\gamma |f(\xi)| \left(\int_0^1 (|\xi| - \|(1-t)x+ty\|)^{-3} dt\right)|d\xi|\\ &\le \frac{1}{8\pi}\|y-x\|^2 \begin{cases} \int_\gamma \frac{1}{(|\xi|-\|y\|)^2(|\xi|-\|x\|)^2}\left(|\xi| - \frac{\|x\|+\|y\|}{2}\right)|f(\xi)|\,|d\xi|,\\ \left\{\frac{1}{2}\int_\gamma |f(\xi)|\left[(|\xi|-\|y\|)^{-3} + (|\xi|-\|x\|)^{-3}\right]|d\xi| \right.\\ \left. + \int_\gamma |f(\xi)|\left(|\xi| - \left\|\frac{x+y}{2}\right\|\right)^{-3}|d\xi|\right\}\end{cases}\\ &\le \frac{1}{16\pi}\|y-x\|^2 \int_\gamma |f(\xi)|\left[(|\xi|-\|x\|)^{-3} + (|\xi|-\|y\|)^{-3}\right]|d\xi|.\end{aligned}$$

The case when the spectra $\sigma(x)$, $\sigma(y)$ are included in a disk is as follows:

Corollary 9. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0, R) \subset D$ where $D(0, R)$ is an open disk centered in 0 and of radius R, then*

$$\begin{aligned}
&\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt\right\| \\
&\le \frac{1}{4}R\|y-x\|^2 \\
&\times \int_0^1 \left|f\left(Re^{2\pi it}\right)\right|\left(\int_0^1 \left\|\left(Re^{2\pi is}-(1-s)x-sy\right)^{-1}\right\|^3 ds\right)dt \\
&\le \frac{1}{4}R\|y-x\|^2\left(\int_0^1 (R-\|(1-s)x+sy\|)^{-3}\,ds\right)\int_0^1 \left|f\left(Re^{2\pi it}\right)\right|dt \\
&\le \frac{1}{4}R\|y-x\|^2\int_0^1 \left|f\left(Re^{2\pi it}\right)\right|dt \\
&\times \begin{cases} \frac{1}{(R-\|y\|)^2(R-\|x\|)^2}\left(R-\frac{\|x\|+\|y\|}{2}\right), \\ \left\{\frac{1}{2}\left[(R-\|y\|)^{-3}+(R-\|x\|)^{-3}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-3}\right\} \end{cases} \\
&\le \frac{1}{8}R\|y-x\|^2\left[(R-\|x\|)^{-3}+(R-\|y\|)^{-3}\right]\int_0^1 \left|f\left(Re^{2\pi it}\right)\right|dt.
\end{aligned} \tag{4.10}$$

5. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z)=\left(\frac{1}{2}z\right)^\nu\sum_{k=0}^\infty\frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n = 0$ we have $I_0(z)$ given by

$$I_0(z)=\sum_{k=0}^\infty\frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n = 0$ we have

$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right|=\left|\exp\left[R\left(\cos(2\pi t)+i\sin(2\pi t)\right)\right]\right|=\exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi} d\theta$ and

$$\begin{aligned}
&\int_0^1 \exp\left[R\cos(2\pi t)\right] dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \exp\left[R\cos\theta\right] d\theta \\
&= \frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi} \exp\left[R\cos\theta\right] d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} \exp\left[R\cos\theta\right] d\theta\right) \\
&= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).
\end{aligned} \tag{5.1}$$

Assume that $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R. By (4.7) and (4.10) for the exponential function we get

$$\begin{aligned}
&\left\| \int_0^1 \exp\left((1-t)x + ty\right) dt - \exp\left(\frac{x+y}{2}\right) \right\| \\
&\le \frac{1}{4} R I_0(R) \|y-x\|^2 \\
&\times \begin{cases} \frac{1}{(R-\|y\|)^2 (R-\|x\|)^2}\left(R - \frac{\|x\|+\|y\|}{2}\right), \\ \left\{\frac{1}{2}\left[(R-\|y\|)^{-3} + (R-\|x\|)^{-3}\right] + \left(R - \left\|\frac{x+y}{2}\right\|\right)^{-3}\right\} \end{cases} \\
&\le \frac{1}{8} R I_0(R) \|y-x\|^2 \left[(R-\|x\|)^{-3} + (R-\|y\|)^{-3}\right]
\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}
&\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f\left((1-t)x + ty\right) dt \right\| \\
&\le \frac{1}{4} R I_0(R) \|y-x\|^2 \\
&\times \begin{cases} \frac{1}{(R-\|y\|)^2 (R-\|x\|)^2}\left(R - \frac{\|x\|+\|y\|}{2}\right), \\ \left\{\frac{1}{2}\left[(R-\|y\|)^{-3} + (R-\|x\|)^{-3}\right] + \left(R - \left\|\frac{x+y}{2}\right\|\right)^{-3}\right\} \end{cases} \\
&\le \frac{1}{8} R I_0(R) \|y-x\|^2 \left[(R-\|x\|)^{-3} + (R-\|y\|)^{-3}\right].
\end{aligned} \tag{5.3}$$

Let f be an analytic functions on the open disk $D(0,\rho)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $(|\lambda| < \rho)$. If $\nu(a) < \rho$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. We define the associated function

$$\lambda \mapsto f_A(\lambda) : D(0,\rho) \to \mathbb{C}, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \tag{5.4}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \lambda\in D(0,1); \tag{5.5}$$
$$g_A(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \lambda\in\mathbb{C};$$

$$h_A(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \lambda\in\mathbb{C};$$
$$l_A(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n \qquad \lambda\in\mathbb{C}, \tag{5.6}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda\in D(0,1);$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda\in D(0,1);$$
$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda\in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n,\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1);$$

where Γ is *Gamma function.*

Lemma 6. *Let f be an analytic functions on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. We have*

$$|f(\lambda)| \le f_A(|\lambda|) \text{ for } \lambda \in D(0,\rho). \tag{5.7}$$

Proof. For $\lambda \in D(0,\rho)$ we have

$$\begin{aligned} |f(\lambda)| &= \left|\sum_{j=0}^{\infty} \alpha_j \lambda^j\right| = \lim_{n\to\infty}\left|\sum_{j=0}^{n} \alpha_j \lambda^j\right| \le \lim_{n\to\infty}\sum_{j=0}^{n}\left|\alpha_j \lambda^j\right| \\ &= \lim_{n\to\infty}\sum_{j=0}^{n}|\alpha_j|\,|\lambda|^j = \sum_{j=0}^{\infty}|\alpha_j|\,|\lambda|^j = f_A(|\lambda|), \end{aligned}$$

which proves the statement. □

Proposition 1. *Let f be an analytic function on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. If $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$ and $0 < R < \rho$, then*

$$\begin{aligned} &\left\|\int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right)\right\| \\ &\le \frac{1}{4} R f_A(R)\,\|y-x\|^2 \\ &\times \begin{cases} \frac{1}{(R-\|y\|)^2(R-\|x\|)^2}\left(R-\frac{\|x\|+\|y\|}{2}\right), \\ \left\{\frac{1}{2}\left[(R-\|y\|)^{-3}+(R-\|x\|)^{-3}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-3}\right\} \end{cases} \\ &\le \frac{1}{8} R f_A(R)\,\|y-x\|^2\left[(R-\|x\|)^{-3}+(R-\|y\|)^{-3}\right] \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &\left\|\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt\right\| \\ &\le \frac{1}{4} R\,\|y-x\|^2 f_A(R) \\ &\times \begin{cases} \frac{1}{(R-\|y\|)^2(R-\|x\|)^2}\left(R-\frac{\|x\|+\|y\|}{2}\right), \\ \left\{\frac{1}{2}\left[(R-\|y\|)^{-3}+(R-\|x\|)^{-3}\right]+\left(R-\left\|\frac{x+y}{2}\right\|\right)^{-3}\right\} \end{cases} \\ &\le \frac{1}{8} R f_A(R)\,\|y-x\|^2\left[(R-\|x\|)^{-3}+(R-\|y\|)^{-3}\right]. \end{aligned} \tag{5.9}$$

The proof follows by (4.7), (4.10) and Lemma 6. As examples, one can consider the functions f and f_A listed above.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa