SEVERAL MID-POINT AND TRAPEZOID TYPE L_p-NORM INEQUALITIES FOR ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with spectra $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \text{ins}(\gamma)$. In this paper we show among others that

$$\begin{aligned}&\left\|\int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right)\right\|\\&\leq \frac{1}{8}\frac{1}{2^{1/p}(2q+1)^{1/q}\pi}\|y-x\|^2\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}\\&\times\left[\int_\gamma\left[(|\xi|-\|x\|)^{-3p}+(|\xi|-\|y\|)^{-3p}\right]|d\xi|\right]^{1/p}\end{aligned}$$

and

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2}-\int_0^1 f((1-t)x+ty)\,dt\right\|\\&\leq \frac{1}{2^{1+1/p}\pi}[B(q+1,q+1)]^{1/q}\|y-x\|^2\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}\\&\times\left[\int_\gamma\left[(|\xi|-\|x\|)^{-3p}+(|\xi|-\|y\|)^{-3p}\right]|d\xi|\right]^{1/p}.\end{aligned}$$

Some examples for exponential function are also given.

1. Introduction

The following mid-point inequality for twice differentiable functions holds, see for instance [13] and [15]:

Theorem 1. *Assume that f is twice differentiable on (a,b) and such that*

$$\|f''\|_{(a,b),p} := \left(\int_a^b |f''(u)|^p\,dt\right)^{1/p} < \infty,\ p>1,$$

then

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{8(2q+1)^{1/q}}(b-a)^{1+1/q}\|f''\|_{(a,b),p}, \tag{1.1}$$

where $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

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The corresponding trapezoid inequality is as follows, see for instance [14] and [3]:

Theorem 2. *With the assumptions of Theorem 1,*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{1}{2}\left[B(q+1,q+1)\right]^{1/q}(b-a)^{1+1/q}\left\|f''\right\|_{(a,b),p}, \tag{1.2}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *where the Beta function is defined by the integral*

$$B(u,v):=\int_0^1 t^{u-1}(1-t)^{v-1}\,dt$$

for complex number inputs $u,\ v$ *such that* $\operatorname{Re}u>0$, $\operatorname{Re}v>0$.

In order to extend mid-point and trapezoid inequalities for functions defined on Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\|\leq\|a\|\,\|b\|$ for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a,b\in\operatorname{Inv}(\mathcal{B})$ then $ab\in\operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in\operatorname{Inv}(\mathcal{B})$;
(ii) $\{a\in\mathcal{B}:\ \|1-b\|<1\}\subset\operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B})\ni a\longmapsto a^{-1}\in\operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a:\rho(a)\to\operatorname{Inv}(\mathcal{B})$, $R_a(z):=(z-a)^{-1}$. For each $z,w\in\rho(a)$ we have the identity

$$R_a(w)-R_a(z)=(z-w)\,R_a(z)\,R_a(w).$$

We also have that

$$\sigma(a)\subset\{z\in\mathbb{C}:\ |z|\leq\|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a)=\sup\{|z|:z\in\sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda:\mathcal{B}\to\mathbb{C}$, the function $\lambda\circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;

(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i} \int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;

(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.3}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.4}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [16] and [18].

For some recent norm inequalities for functions on Banach algebras, see [8], [2] and [5]-[12].

2. Preliminary Results

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \; t \in [0,1]. \tag{2.1}$$

Lemma 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f'_{x,y}(t) = D(f)((1-t)x + ty)(y - x) \tag{2.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f'_{x,y}(0+) = D(f)(x)(y-x) \text{ and } f'_{x,y}(1-) = D(f)(y)(y-x). \tag{2.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f_{x,y}^{'}(t) &= \frac{df_{x,y}(t)}{dt} = \lim_{h\to 0}\frac{f_{x,y}(t+h)-f(t)}{h} \\ &= \lim_{h\to 0}\left[\frac{f((1-t)x+ty+h(y-x))-f((1-t)x+ty)}{h}\right] \\ &= D(f)((1-t)x+ty)(y-x), \end{aligned}$$

which proves (2.7).

The proof is similar for the lateral derivatives. □

We also have:

Lemma 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is twice differentiable on $(0,1)$ as a function of t and we have*

$$f_{x,y}''(t) = D^2(f)((1-t)x+ty)(y-x, y-x) \tag{2.4}$$

for all $t \in (0,1)$, where $D^2(f)(\cdot)(\cdot,\cdot)$ is the second Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f_{x,y}''(0+) = D^2(f)(x)(y-x, y-x), \quad f_{x,y}''(1-) = D^2(f)(y)(y-x, y-x). \tag{2.5}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{1}{h}\left[\frac{df_{x,y}(t+h)}{dt} - \frac{df_{x,y}(t)}{dt}\right] \\ &= \frac{D(f)((1-(t+h))x+(t+h)y)(y-x) - D(f)((1-t)x+ty)(y-x)}{h} \\ &= \frac{D(f)((1-t)x+ty+h(y-x))(y-x) - D(f)((1-t)x+ty)(y-x)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f_{x,y}''(t) &= \frac{d^2 f_{x,y}(t)}{dt^2} = \lim_{h\to 0}\left\{\frac{1}{h}\left[\frac{df_{x,y}(t+h)}{dt} - \frac{df_{x,y}(t)}{dt}\right]\right\} \\ &= D^2(f)((1-t)x+ty)(y-x, y-x). \end{aligned}$$

The proof is similar for the lateral derivatives. □

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{2.6}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x+hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned} & f(x+hv) - f(x) \\ &= \frac{1}{2\pi i}\left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi\right) \\ &= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} - (\xi - x)^{-1}\right] d\xi \\ &= \frac{1}{2\pi i} h \int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} v (\xi - x)^{-1}\right] d\xi, \end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x+hv) - f(x)}{h} = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x - hv)^{-1} v (\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (2.6). □

Corollary 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f'_{x,y}(t) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1}(y-x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{2.7}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f'_{x,y}(0+) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1}(y-x)(\xi - x)^{-1} d\xi, \tag{2.8}$$

and

$$f'_{x,y}(1-) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - y)^{-1}(y-x)(\xi - y)^{-1} d\xi. \tag{2.9}$$

The proof is obvious by Lemmas 1 and 2.

Lemma 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for u, $v \in \mathcal{B}$ we have*

$$\begin{aligned} D^2(f)(x)(u,v) &= \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} u (\xi - x)^{-1} d\xi \\ &\quad + \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1} u (\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \end{aligned} \tag{2.10}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$.

In particular, we have

$$D^2(f)(x)(v,v) = \frac{1}{\pi i}\int_\gamma f(\xi)\left((\xi - x)^{-1} v\right)^2 (\xi - x)^{-1} d\xi, \tag{2.11}$$

for $v \in \mathcal{B}$.

Proof. Let $u,v\in\mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x+hv)\subset \operatorname{ins}(\delta)\subset G$. Then

$$D^2(f)(x)(u,v)=\lim_{h\to 0}\frac{D(f)(x+hv)(u)-D(f)(x)(u)}{h}.$$

We have by (2.6)

$$\begin{aligned}&D(f)(x+hv)(u)-D(f)(x)(u)\\&=\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-x-hv)^{-1}u(\xi-x-hv)^{-1}d\xi\\&-\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-x)^{-1}u(\xi-x)^{-1}d\xi\\&=\frac{1}{2\pi i}\int_\gamma f(\xi)k_{\xi,h}(x,u,v)\,d\xi,\end{aligned}$$

where

$$k_{\xi,h}(x,u,v):=(\xi-x-hv)^{-1}u(\xi-x-hv)^{-1}-(\xi-x)^{-1}u(\xi-x)^{-1}\tag{2.12}$$

and $\xi\in\gamma$ while h is in a small interval around 0, $h\neq 0$.

If we multiply both sides of (2.12) by $\xi-x-hv$, then we get

$$\begin{aligned}&(\xi-x-hv)k_{\xi,h}(x,u,v)(\xi-x-hv)\\&=u-(\xi-x-hv)(\xi-x)^{-1}u(\xi-x)^{-1}(\xi-x-hv)\\&=u-\left(1-hv(\xi-x)^{-1}\right)u\left(1-h(\xi-x)^{-1}v\right)\\&=u-\left(u-hv(\xi-x)^{-1}u\right)\left(1-h(\xi-x)^{-1}v\right)\\&=u-u+hv(\xi-x)^{-1}u+hu(\xi-x)^{-1}v-h^2v(\xi-x)^{-1}u(\xi-x)^{-1}v\\&=h\left[v(\xi-x)^{-1}u+u(\xi-x)^{-1}v-hv(\xi-x)^{-1}u(\xi-x)^{-1}v\right].\end{aligned}\tag{2.13}$$

The identity (2.13) implies that

$$\begin{aligned}\frac{1}{h}k_{\xi,h}(x,u,v)&=(\xi-x-hv)^{-1}\\&\times\left[v(\xi-x)^{-1}u+u(\xi-x)^{-1}v-hv(\xi-x)^{-1}u(\xi-x)^{-1}v\right]\\&\times(\xi-x-hv)^{-1},\end{aligned}$$

which gives that

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{h}k_{\xi,h}(x,u,v)\\&=(\xi-x)^{-1}\left[v(\xi-x)^{-1}u+u(\xi-x)^{-1}v\right](\xi-x)^{-1}\\&=(\xi-x)^{-1}v(\xi-x)^{-1}u(\xi-x)^{-1}+(\xi-x)^{-1}u(\xi-x)^{-1}v(\xi-x)^{-1}\end{aligned}$$

for $\xi\in\gamma$.

Therefore

$$\begin{aligned}
&\lim_{h\to 0}\frac{D(f)(x+hv)(u)-D(f)(x)(u)}{h}\\
&=\frac{1}{2\pi i}\lim_{h\to 0}\int_\gamma f(\xi)\frac{1}{h}k_{\xi,h}(x,u,v)\,d\xi=\frac{1}{2\pi i}\int_\gamma f(\xi)\lim_{h\to 0}\frac{1}{h}k_{\xi,h}(x,u,v)\,d\xi\\
&=\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-x)^{-1}v(\xi-x)^{-1}u(\xi-x)^{-1}\,d\xi\\
&+\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-x)^{-1}u(\xi-x)^{-1}v(\xi-x)^{-1}\,d\xi
\end{aligned}$$

and the lemma is thus proved. □

Corollary 2. *Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and x, $y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)$, $\sigma(y)\subset G$ while $\gamma\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y)\subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}
&f''_{x,y}(t)\\
&=\frac{1}{\pi i}\int_\gamma f(\xi)\left((\xi-(1-t)x-ty)^{-1}(y-x)\right)^2(\xi-(1-t)x-ty)^{-1}\,d\xi
\end{aligned}\tag{2.14}$$

for all $t\in(0,1)$.

We also have the lateral derivatives

$$f''_{x,y}(0+)=\frac{1}{2\pi i}\int_\gamma f(\xi)\left((\xi-x)^{-1}(y-x)\right)^2(\xi-x)^{-1}\,d\xi,\tag{2.15}$$

and

$$f''_{x,y}(1-)=\frac{1}{2\pi i}\int_\gamma f(\xi)\left((\xi-y)^{-1}(y-x)\right)^2(\xi-y)^{-1}\,d\xi.\tag{2.16}$$

Lemma 5. *Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and x, $y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)$, $\sigma(y)\subset G$ while $\gamma\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y)\subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned}
&\left\|f''_{x,y}(t)\right\|\\
&\le\frac{1}{\pi}\|y-x\|^2\int_\gamma|f(\xi)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^3|d\xi|\\
&\le\frac{1}{\pi}\|y-x\|^2\int_\gamma|f(\xi)|\left(|\xi|-\|(1-t)x+ty\|\right)^{-3}|d\xi|\\
&\le\frac{1}{\pi}\|y-x\|^2\int_\gamma|f(\xi)|\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-3}|d\xi|\\
&\le\frac{1}{\pi}\|y-x\|^2\int_\gamma|f(\xi)|\left[(1-t)(|\xi|-\|x\|)^{-3}+t(|\xi|-\|y\|)^{-3}\right]|d\xi|
\end{aligned}\tag{2.17}$$

for all $t\in[0,1]$.

Proof. By taking the norm in (2.14) and using the integral's properties, we have

$$\begin{aligned}
&\left\| f''_{x,y}(t) \right\| \\
&\leq \frac{1}{\pi} \int_{\gamma} |f(\xi)| \\
&\times \left\| \left((\xi - (1-t)x - ty)^{-1} (y-x) \right)^{2} (\xi - (1-t)x - ty)^{-1} \right\| |d\xi| \\
&\leq \frac{1}{\pi} \|y - x\|^{2} \int_{\gamma} |f(\xi)| \left\| (\xi - (1-t)x - ty)^{-1} \right\|^{3} |d\xi| \\
&= \frac{1}{\pi} \|y - x\|^{2} \int_{\gamma} |f(\xi)| |\xi|^{-3} \left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\|^{3} |d\xi|
\end{aligned} \tag{2.18}$$

for all $t \in [0,1]$, which proves the first inequality in (2.17).

Since

$$\left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \leq (1-t)\left\| \frac{x}{\xi} \right\| + t\left\| \frac{y}{\xi} \right\| < 1 - t + t = 1$$

for $\xi \in \gamma$, hence

$$\left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} = \sum_{k=0}^{\infty} \left[(1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right]^{k}.$$

Therefore

$$\begin{aligned}
\left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\| &\leq \sum_{k=0}^{\infty} \left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\|^{k} \\
&= \left(1 - \left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \right)^{-1} \\
&= \left(\frac{|\xi|}{|\xi|} - \left\| (1-t)\frac{x}{\xi} + t\frac{y}{\xi} \right\| \right)^{-1} \\
&= |\xi| \left(|\xi| - \|(1-t)x + ty\| \right)^{-1}
\end{aligned}$$

for $\xi \in \gamma$, which implies that

$$\left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\|^{3} \leq |\xi|^{2} \left(|\xi| - \|(1-t)x + ty\| \right)^{-3}$$

for $\xi \in \gamma$.

Therefore

$$\begin{aligned}
&\int_{\gamma} |f(\xi)| |\xi|^{-3} \left\| \left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi} \right)^{-1} \right\|^{3} |d\xi| \\
&\leq \int_{\gamma} |f(\xi)| \left(|\xi| - \|(1-t)x + ty\| \right)^{-3} |d\xi|
\end{aligned}$$

and we derive the second inequality in (2.17).

By the triangle inequality we have

$$\begin{aligned}
|\xi| - \|(1-t)x + ty\| &\geq |\xi| - (1-t)\|x\| - t\|y\| \\
&= (1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) > 0
\end{aligned}$$

for $\xi \in \gamma$.

This implies that

$$\left(|\xi| - \|(1-t)x + ty\|\right)^{-1} \leq \left[(1-t)\left(|\xi| - \|x\|\right) + t\left(|\xi| - \|y\|\right)\right]^{-1}$$

and thus

$$\left(|\xi| - \|(1-t)x + ty\|\right)^{-3} \leq \left[(1-t)\left(|\xi| - \|x\|\right) + t\left(|\xi| - \|y\|\right)\right]^{-3}$$

for $\xi \in \gamma$ and $t \in [0,1]$. This proves the third inequality in (2.17).

By the convexity of the power function $(\cdot)^{-3}$ we also have

$$\begin{aligned} &\left[(1-t)\left(|\xi| - \|x\|\right) + t\left(|\xi| - \|y\|\right)\right]^{-3} \\ &\leq (1-t)\left(|\xi| - \|x\|\right)^{-3} + t\left(|\xi| - \|y\|\right)^{-3} \end{aligned}$$

for $t \in [0,1]$, which proves the fourth inequality in (2.17). □

Corollary 3. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then*

$$\begin{aligned} &\left\|f''_{x,y}(t)\right\| \\ &\leq 2R\|y-x\|^2 \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| \left\|\left(Re^{2\pi is} - (1-t)x - ty\right)^{-1}\right\|^3 ds \\ &\leq 2R\|y-x\|^2 \left(R - \|(1-t)x + ty\|\right)^{-3} \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds \\ &\leq 2R\|y-x\|^2 \left[(1-t)\left(R - \|x\|\right) + t\left(R - \|y\|\right)\right]^{-3} \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds \\ &\leq 2R\|y-x\|^2 \left[(1-t)\left(R - \|x\|\right)^{-3} + t\left(R - \|y\|\right)^{-3}\right] \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds \end{aligned} \tag{2.19}$$

for all $t \in [0,1]$.

It follows by taking γ parametrized by $\xi(s) = Re^{2\pi is}$ where $s \in [0,1]$, then $d\xi(s) = 2\pi i Re^{2\pi is} ds$, $|d\xi(s)| = 2\pi R ds$, $|\xi| = R$ and by (2.17) we get (2.19).

Remark 1. *If $\|f\|_{\gamma,\infty} := \sup_{\xi \in \gamma} |f(\xi)| < 0$, then from (2.17) we get*

$$\begin{aligned} &\left\|f''_{x,y}(t)\right\| \\ &\leq \frac{1}{\pi}\|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left\|\left(\xi - (1-t)x - ty\right)^{-1}\right\|^3 |d\xi| \\ &\leq \frac{1}{\pi}\|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left(|\xi| - \|(1-t)x + ty\|\right)^{-3} |d\xi| \\ &\leq \frac{1}{\pi}\|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left[(1-t)\left(|\xi| - \|x\|\right) + t\left(|\xi| - \|y\|\right)\right]^{-3} |d\xi| \\ &\leq \frac{1}{\pi}\|y-x\|^2 \|f\|_{\gamma,\infty} \int_\gamma \left[(1-t)\left(|\xi| - \|x\|\right)^{-3} + t\left(|\xi| - \|y\|\right)^{-3}\right] |d\xi| \end{aligned} \tag{2.20}$$

for all $t \in [0,1]$.

Also, if $\|f\|_{R,\infty} := \sup_{s\in[0,1]} \left|f\left(Re^{2\pi is}\right)\right| < \infty$*, then from (2.19) we get*

$$\begin{aligned} &\left\|f''_{x,y}(t)\right\| \\ &\leq 2R\|y-x\|^2 \|f\|_{R,\infty} \int_0^1 \left\|\left(Re^{2\pi is}-(1-t)x-ty\right)^{-1}\right\|^3 ds \\ &\leq 2R\|y-x\|^2 \left(R-\|(1-t)x+ty\|\right)^{-3} \|f\|_{R,\infty} \\ &\leq 2R\|y-x\|^2 \left[(1-t)(R-\|x\|)+t(R-\|y\|)\right]^{-3} \|f\|_{R,\infty} \\ &\leq 2R\|y-x\|^2 \left[(1-t)(R-\|x\|)^{-3}+t(R-\|y\|)^{-3}\right] \|f\|_{R,\infty} \end{aligned} \tag{2.21}$$

for all $t\in[0,1]$.

3. Main Results

We have the following mid-point type inequality:

Theorem 3. *Assume that* $f: G\to\mathbb{C}$ *is analytic on the convex domain* G *and* x, $y\in\mathcal{B}$, $x\neq y$ *with* $\sigma(x)$, $\sigma(y)\subset G$. *Then we have the norm inequality*

$$\begin{aligned} &\left\|\int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right)\right\| \\ &\leq \frac{1}{(2q+1)^{1/q}\,2^{3+1/q}}\left[\left\|f''_{x,y}\right\|_{[0,1/2],p}+\left\|f''_{x,y}\right\|_{[1/2,1],p}\right] \\ &\leq \frac{1}{8(2q+1)^{1/q}}\left\|f''_{x,y}\right\|_{[0,1],p} \end{aligned} \tag{3.1}$$

for p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the integration by parts formula twice for the Bochner integral [17], we have

$$\begin{aligned} \frac{1}{2}\int_0^{1/2} t^2 f''_{x,y}(t)\,dt &= \frac{1}{2}\left(t^2 f'_{x,y}(t)\Big|_0^{1/2} - 2\int_0^{1/2} t f'_{x,y}(t)\,dt\right) \\ &= \frac{1}{2}\left(\frac{1}{4}f'_{x,y}(1/2) - 2\int_0^{1/2} t f'_{x,y}(t)\,dt\right) \\ &= \frac{1}{8}f'_{x,y}(1/2) - \left(t f_{x,y}(t)\Big|_0^{1/2} - \int_0^{1/2} f_{x,y}(t)\,dt\right) \\ &= \frac{1}{8}f'_{x,y}(1/2) - \frac{1}{2}f_{x,y}(1/2) + \int_0^{1/2} f_{x,y}(t)\,dt \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(3.3)\qquad &\frac{1}{2}\int_{1/2}^{1}(t-1)^2 f''_{x,y}(t)\,dt\\ &=\frac{1}{2}\left((t-1)^2 f'_{x,y}(t)\Big|_{1/2}^{1}-2\int_{1/2}^{1}(t-1)\,f'_{x,y}(t)\,dt\right)\\ &=\frac{1}{2}\left(-\frac{1}{4}f'_{x,y}\left(\frac{1}{2}\right)-2\int_{1/2}^{1}(t-1)\,f'_{x,y}(t)\,dt\right)\\ &=-\frac{1}{8}f'_{x,y}(1/2)-\left((t-1)\,f_{x,y}(t)\,dt|_{1/2}^{1}-\int_{1/2}^{1}f_{x,y}(t)\,dt\right)\\ &=-\frac{1}{8}f'_{x,y}(1/2)-\frac{1}{2}f_{x,y}(1/2)+\int_{1/2}^{1}f_{x,y}(t)\,dt.\end{aligned}$$

If we add (3.2) with (3.3), then we obtain the identity of interest

$$\begin{aligned}(3.4)\qquad &\int_0^1 f((1-t)\,x+ty)\,dt-f\left(\frac{x+y}{2}\right)\\ &=\frac{1}{2}\int_0^{1/2}t^2 f''_{x,y}(t)\,dt+\frac{1}{2}\int_{1/2}^{1}(t-1)^2 f''_{x,y}(t)\,dt.\end{aligned}$$

By taking the norm in (3.4) and using the properties of the integral, we get

$$\begin{aligned}(3.5)\qquad &\left\|\int_0^1 f((1-t)\,x+ty)\,dt-f\left(\frac{x+y}{2}\right)\right\|\\ &\le\frac{1}{2}\left\|\int_0^{1/2}t^2 f''_{x,y}(t)\,dt\right\|+\frac{1}{2}\left\|\int_{1/2}^{1}(t-1)^2 f''_{x,y}(t)\,dt\right\|\\ &\le\frac{1}{2}\int_0^{1/2}t^2\left\|f''_{x,y}(t)\right\|dt+\frac{1}{2}\int_{1/2}^{1}(t-1)^2\left\|f''_{x,y}(t)\right\|dt=:M.\end{aligned}$$

By Hölder's integral inequality, we have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, that

$$\begin{aligned}\int_0^{1/2}t^2\left\|f''_{x,y}(t)\right\|dt&\le\left(\int_0^{1/2}t^{2q}dt\right)^{1/q}\left(\int_0^{1/2}\left\|f''_{x,y}(t)\right\|^p dt\right)^{1/p}\\ &=\frac{1}{(2q+1)^{1/q}\,2^{2+1/q}}\left\|f''_{x,y}\right\|_{[0,1/2],p}\end{aligned}$$

and

$$\begin{aligned}\int_{1/2}^{1}(t-1)^2\left\|f''_{x,y}(t)\right\|dt&\le\left\{\int_0^{1/2}(1-t)^{2q}\,dt\right\}^{1/q}\left(\int_{1/2}^{1}\left\|f''_{x,y}(t)\right\|^p dt\right)^{1/p}\\ &=\frac{1}{(2q+1)^{1/q}\,2^{2+1/q}}\left\|f''_{x,y}\right\|_{[1/2,1],p}\end{aligned}$$

Therefore

$$\begin{aligned} M &\leq \frac{1}{(2q+1)^{1/q}\, 2^{3+1/q}} \left\| f''_{x,y} \right\|_{[0,1/2],p} + \frac{1}{(2q+1)^{1/q}\, 2^{3+1/q}} \left\| f''_{x,y} \right\|_{[1/2,1],p} \\ &= \frac{1}{(2q+1)^{1/q}\, 2^{3+1/q}} \left[\left\| f''_{x,y} \right\|_{[0,1/2],p} + \left\| f''_{x,y} \right\|_{[1/2,1],p} \right] \end{aligned}$$

and the first inequality (3.1) is proved.

Using the concavity of the function $(\cdot)^{1/p}$, $p > 1$ we have

$$\begin{aligned} \left\| f''_{x,y} \right\|_{[0,1/2],p} + \left\| f''_{x,y} \right\|_{[1/2,1],p} &= \left(\left\| f''_{x,y} \right\|^p_{[0,1/2],p} \right)^{1/p} + \left(\left\| f''_{x,y} \right\|^p_{[1/2,1],p} \right)^{1/p} \\ &\leq 2 \left(\frac{\left\| f''_{x,y} \right\|^p_{[0,1/2],p} + \left\| f''_{x,y} \right\|^p_{[1/2,1],p}}{2} \right)^{1/p} \\ &= 2^{1-1/p} \left(\left\| f''_{x,y} \right\|^p_{[0,1/2],p} + \left\| f''_{x,y} \right\|^p_{[1/2,1],p} \right)^{1/p} \\ &= 2^{1-1/p} \left(\int_0^{1/2} \left\| f''_{x,y}(t) \right\|^p dt + \int_{1/2}^1 \left\| f''_{x,y}(t) \right\|^p dt \right)^{1/p} \\ &= 2^{1-1/p} \left\| f''_{x,y} \right\|_{[0,1],p} . \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{(2q+1)^{1/q}\, 2^{3+1/q}} \left[\left\| f''_{x,y} \right\|_{[0,1/2],p} + \left\| f''_{x,y} \right\|_{[1/2,1],p} \right] \\ &\leq \frac{2^{1-1/p}}{(2q+1)^{1/q}\, 2^{3+1/q}} \left\| f''_{x,y} \right\|_{[0,1],p} = \frac{1}{8\,(2q+1)^{1/q}} \left\| f''_{x,y} \right\|_{[0,1],p} \end{aligned}$$

and the last inequality in (3.1) is proved. □

Corollary 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\begin{aligned} (3.6)\qquad & \left\| \int_0^1 f((1-t)\,x + ty)\, dt - f\left(\frac{x+y}{2} \right) \right\| \\ &\leq \frac{1}{8\,(2q+1)^{1/q}} \frac{1}{\pi} \left\| y - x \right\|^2 \left(\int_\gamma |f(\xi)|^q \,|d\xi| \right)^{1/q} \\ &\times \left(\frac{1}{(3p-1)\,(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^{3p-1} - (|\xi| - \|y\|)^{3p-1}}{(|\xi| - \|x\|)^{3p-1}\,(|\xi| - \|y\|)^{3p-1}} \,|d\xi| \right)^{1/p} \\ &\leq \frac{1}{8} \frac{1}{2^{1/p}\,(2q+1)^{1/q}} \frac{1}{\pi} \left\| y - x \right\|^2 \left(\int_\gamma |f(\xi)|^q \,|d\xi| \right)^{1/q} \\ &\times \left[\int_\gamma \left[(|\xi| - \|x\|)^{-3p} + (|\xi| - \|y\|)^{-3p} \right] |d\xi| \right]^{1/p} . \end{aligned}$$

Proof. From (2.17) and Hölder's inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left\| f''_{x,y}(t)\right\| \leq \frac{1}{\pi}\left\| y-x\right\|^{2}\left(\int_{\gamma}\left\vert f(\xi)\right\vert^{q}\left\vert d\xi\right\vert \right)^{1/q} \times \left(\int_{\gamma}\left[(1-t)(\left\vert \xi\right\vert -\left\| x\right\|) +t(\left\vert \xi\right\vert -\left\| y\right\|)\right]^{-3p}\left\vert d\xi\right\vert \right)^{1/p}$$

for all $t\in[0,1]$.

If we take the power p, integrate and use Fubini's theorem, then we get

$$\int_{0}^{1}\left\| f''_{x,y}(t)\right\|^{p}dt \leq \left(\frac{1}{\pi}\right)^{p}\left\| y-x\right\|^{2p}\left(\int_{\gamma}\left\vert f(\xi)\right\vert^{q}\left\vert d\xi\right\vert \right)^{p/q} \times \int_{0}^{1}\left(\int_{\gamma}\left[(1-t)(\left\vert \xi\right\vert -\left\| x\right\|) +t(\left\vert \xi\right\vert -\left\| y\right\|)\right]^{-3p}\left\vert d\xi\right\vert \right) dt$$
$$=\left(\frac{1}{\pi}\right)^{p}\left\| y-x\right\|^{2p}\left(\int_{\gamma}\left\vert f(\xi)\right\vert^{p}\left\vert d\xi\right\vert \right)^{p/q} \times \int_{\gamma}\left(\int_{0}^{1}\left[(1-t)(\left\vert \xi\right\vert -\left\| x\right\|) +t(\left\vert \xi\right\vert -\left\| y\right\|)\right]^{-3p}dt\right) \left\vert d\xi\right\vert . \tag{3.7}$$

Observe that

$$\int_{\gamma}\left(\int_{0}^{1}\left[(1-t)(\left\vert \xi\right\vert -\left\| x\right\|) +t(\left\vert \xi\right\vert -\left\| y\right\|)\right]^{-3p}dt\right) \left\vert d\xi\right\vert$$
$$=\int_{\gamma}\left(\left. \frac{\left[(1-t)(\left\vert \xi\right\vert -\left\| x\right\|) +t(\left\vert \xi\right\vert -\left\| y\right\|)\right]^{-3p+1}}{(1-3p)(\left\| x\right\| -\left\| y\right\|)}\right\vert_{0}^{1}\right) \left\vert d\xi\right\vert$$
$$=\int_{\gamma}\frac{(\left\vert \xi\right\vert -\left\| y\right\|)^{-3p+1}-(\left\vert \xi\right\vert -\left\| x\right\|)^{-3p+1}}{(1-3p)(\left\| x\right\| -\left\| y\right\|)}\left\vert d\xi\right\vert$$
$$=\int_{\gamma}\frac{\frac{1}{(\left\vert \xi\right\vert -\left\| y\right\|)^{3p-1}}-\frac{1}{(\left\vert \xi\right\vert -\left\| x\right\|)^{3p-1}}}{(1-2p)(\left\| x\right\| -\left\| y\right\|)}\left\vert d\xi\right\vert$$
$$=\frac{1}{(3p-1)(\left\| y\right\| -\left\| x\right\|)}\int_{\gamma}\frac{(\left\vert \xi\right\vert -\left\| x\right\|)^{3p-1}-(\left\vert \xi\right\vert -\left\| y\right\|)^{3p-1}}{(\left\vert \xi\right\vert -\left\| x\right\|)^{3p-1}(\left\vert \xi\right\vert -\left\| y\right\|)^{3p-1}}\left\vert d\xi\right\vert .$$

Then by (3.7)

$$\int_{0}^{1}\left\| f''_{x,y}(t)\right\|^{p}dt$$
$$\leq \left(\frac{1}{\pi}\right)^{p}\left\| y-x\right\|^{2p}\left(\int_{\gamma}\left\vert f(\xi)\right\vert^{q}\left\vert d\xi\right\vert \right)^{p/q}$$
$$\times \frac{1}{(3p-1)(\left\| y\right\| -\left\| x\right\|)}\int_{\gamma}\frac{(\left\vert \xi\right\vert -\left\| x\right\|)^{3p-1}-(\left\vert \xi\right\vert -\left\| y\right\|)^{3p-1}}{(\left\vert \xi\right\vert -\left\| x\right\|)^{3p-1}(\left\vert \xi\right\vert -\left\| y\right\|)^{3p-1}}\left\vert d\xi\right\vert .$$

If we take the power $1/p$, then we get

$$\begin{aligned}&\left(\int_0^1 \left\|f''_{x,y}(t)\right\|^p dt\right)^{1/p}\\&\le \frac{1}{\pi}\|y-x\|^2\left(\int_\gamma |f(\xi)|^q |d\xi|\right)^{1/q}\\&\times\left(\frac{1}{(3p-1)(\|y\|-\|x\|)}\int_\gamma \frac{(|\xi|-\|x\|)^{3p-1}-(|\xi|-\|y\|)^{3p-1}}{(|\xi|-\|x\|)^{3p-1}(|\xi|-\|y\|)^{3p-1}}|d\xi|\right)^{1/p},\end{aligned}$$

which, by (3.1) proves the first part of (3.6).

By Hermite-Hadamard inequality for convex functions we also have

$$\begin{aligned}&\int_0^1 [(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-3p} dt\\&\le \frac{1}{2}\left[(|\xi|-\|x\|)^{-3p}+(|\xi|-\|y\|)^{-3p}\right],\end{aligned}$$

which proves the last part of (3.6). □

Remark 2. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then by (3.6) we get*

$$\begin{aligned}&\left\|\int_0^1 f((1-t)x+ty)\,dt - f\left(\frac{x+y}{2}\right)\right\| \qquad (3.8)\\&\le \frac{R}{4(2q+1)^{1/q}}\|y-x\|^2\left(\int_0^1 \left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q}\\&\times\left(\frac{1}{(3p-1)(\|y\|-\|x\|)}\frac{(R-\|x\|)^{3p-1}-(R-\|y\|)^{3p-1}}{(R-\|x\|)^{3p-1}(R-\|y\|)^{3p-1}}\right)^{1/p}\\&\le \frac{1}{4}\frac{R}{2^{1/p}(2q+1)^{1/q}}\|y-x\|^2\left(\int_0^1 \left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q}\\&\times\left[(R-\|x\|)^{-3p}+(R-\|y\|)^{-3p}\right]^{1/p}.\end{aligned}$$

We have the following trapezoid type inequality:

Theorem 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$. Then we have the norm inequality*

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt\right\| \qquad (3.9)\\&\le \frac{1}{2}[B(q+1,q+1)]^{1/q}\left\|f''_{x,y}\right\|_{[0,1],p},\end{aligned}$$

for p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, where $B(\cdot,\cdot)$ is Beta function.

Proof. Using integration by parts formula for Bochner integral, then we have

$$\begin{aligned}\frac{1}{2}\int_0^1 t(1-t)f''_{x,y}(t)\,dt &= \frac{1}{2}\left(t(1-t)f'_{x,y}(t)\Big|_0^1 - \int_0^1 (1-2t)f'_{x,y}(t)\,dt\right)\\ &= \int_0^1 \left(t-\frac{1}{2}\right)f'_{x,y}(t)\,dt\\ &= \frac{f_{x,y}(1)+f_{x,y}(0)}{2} - \int_0^1 f_{x,y}(t)\,dt,\end{aligned}$$

which produces the following identity of interest

$$\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt = \frac{1}{2}\int_0^1 t(1-t)f''_{x,y}(t)\,dt.$$

By taking the norm, we get

$$\begin{aligned}&\left\|\frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt\right\| \\ &\le \frac{1}{2}\left\|\int_0^{1/2} t(1-t)f''_{x,y}(t)\,dt\right\| + \frac{1}{2}\left\|\int_{1/2}^1 t(1-t)f''_{x,y}(t)\,dt\right\| \\ &\le \frac{1}{2}\int_0^{1/2} t(1-t)\left\|f''_{x,y}(t)\right\|dt + \frac{1}{2}\int_{1/2}^1 t(1-t)\left\|f''_{x,y}(t)\right\|dt \\ &= \frac{1}{2}\int_0^1 t(1-t)\left\|f''_{x,y}(t)\right\|dt.\end{aligned} \tag{3.10}$$

By Hölder's integral inequality, we have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, that

$$\begin{aligned}\int_0^1 t(1-t)\left\|f''_{x,y}(t)\right\|dt &\le \left(\int_0^1 [t(1-t)]^q\,dt\right)^{1/q}\left(\int_0^1 \left\|f''_{x,y}(t)\right\|^p dt\right)^{1/p}\\ &= \left(\int_0^1 t^q(1-t)^q\,dt\right)^{1/q}\left(\int_0^1 \left\|f''_{x,y}(t)\right\|^p dt\right)^{1/p}\\ &= [B(q+1,q+1)]^{1/q}\left\|f''_{x,y}\right\|_{[0,1],p}.\end{aligned} \tag{3.11}$$

By making use of (3.10) and (3.11) we derive the desired result (3.9). □

Corollary 5. *Assume that $f: G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve* *in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt \right\| \tag{3.12}$$
$$\leq \frac{1}{2}[B(q+1,q+1)]^{1/q} \frac{1}{\pi}\|y-x\|^2 \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q}$$
$$\times \left(\frac{1}{(3p-1)(\|y\|-\|x\|)} \int_\gamma \frac{(|\xi|-\|x\|)^{3p-1} - (|\xi|-\|y\|)^{3p-1}}{(|\xi|-\|x\|)^{3p-1}(|\xi|-\|y\|)^{3p-1}} |d\xi| \right)^{1/p}$$
$$\leq \frac{1}{2^{1+1/p}}[B(q+1,q+1)]^{1/q} \frac{1}{\pi}\|y-x\|^2 \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q}$$
$$\times \left[\int_\gamma \left[(|\xi|-\|x\|)^{-3p} + (|\xi|-\|y\|)^{-3p} \right] |d\xi| \right]^{1/p}.$$

Remark 3. *Assume that $f : D \subset \mathbb{C} \to \mathbb{C}$ is an analytic function on the convex domain D and $x, y \in \mathcal{B}$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then by (3.6) we get*

$$\left\| \frac{f(x)+f(y)}{2} - \int_0^1 f((1-t)x+ty)\,dt \right\| \tag{3.13}$$
$$\leq R[B(q+1,q+1)]^{1/q} \|y-x\|^2 \left(\int_0^1 \left| f\left(Re^{2\pi is}\right)\right|^q ds \right)^{1/q}$$
$$\times \left(\frac{1}{(3p-1)(\|y\|-\|x\|)} \frac{(R-\|x\|)^{3p-1} - (R-\|y\|)^{3p-1}}{(R-\|x\|)^{3p-1}(R-\|y\|)^{3p-1}} \right)^{1/p}$$
$$\leq \frac{1}{2^{1/p}} R[B(q+1,q+1)]^{1/q} \|y-x\|^2 \left(\int_0^1 \left| f\left(Re^{2\pi is}\right)\right|^q ds \right)^{1/q}$$
$$\times \left[(R-\|x\|)^{-3p} + (R-\|y\|)^{-3p} \right]^{1/p}.$$

4. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t - \nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n = 0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that x, $y \in \mathcal{B}$ and $\|x\|$, $\|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned}(4.1)\qquad & \int_0^1 \exp\left[R\cos(2\pi t)\right]dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[R\cos\theta\right]d\theta \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi}\exp\left[R\cos\theta\right]d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi}\exp\left[R\cos\theta\right]d\theta\right) \\ &= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).\end{aligned}$$

Therefore

$$\left|\exp\left(Re^{2\pi it}\right)\right|^q = \exp\left[qR\cos(2\pi t)\right]$$

and

$$\left(\int_0^1 \left|\exp\left(Re^{2\pi it}\right)\right|^q dt\right)^{1/q} = \left(\int_0^1 \exp\left[qR\cos(2\pi t)\right]dt\right)^{1/q} = I_0^{1/q}(qR) \text{ for } q \geq 1.$$

Let x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset D$ where $D(0,R)$ is an open disk centered in 0 and of radius R, then by (3.8) we get for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned}(4.2)\qquad & \left\|\int_0^1 \exp\left((1-t)x + ty\right)dt - \exp\left(\frac{x+y}{2}\right)\right\| \\ &\leq \frac{RI_0^{1/q}(qR)}{4(2q+1)^{1/q}}\|y-x\|^2 \\ &\times \left(\frac{1}{(3p-1)(\|y\|-\|x\|)}\frac{(R-\|x\|)^{3p-1} - (R-\|y\|)^{3p-1}}{(R-\|x\|)^{3p-1}(R-\|y\|)^{3p-1}}\right)^{1/p} \\ &\leq \frac{1}{4}\frac{RI_0^{1/q}(qR)}{2^{1/p}(2q+1)^{1/q}}\|y-x\|^2\left[(R-\|x\|)^{-3p} + (R-\|y\|)^{-3p}\right]^{1/p}\end{aligned}$$

and by (3.13)

$$\left\| \frac{\exp x + \exp y}{2} - \int_0^1 \exp((1-t)x + ty)\,dt \right\| \le R I_0^{1/q}(qR)\left[B(q+1,q+1)\right]^{1/q} \|y-x\|^2 \times \left(\frac{1}{(3p-1)(\|y\|-\|x\|)} \frac{(R-\|x\|)^{3p-1} - (R-\|y\|)^{3p-1}}{(R-\|x\|)^{3p-1}(R-\|y\|)^{3p-1}} \right)^{1/p} \le \frac{1}{2^{1/p}} R I_0^{1/q}(qR)\left[B(q+1,q+1)\right]^{1/q} \|y-x\|^2 \times \left[(R-\|x\|)^{-3p} + (R-\|y\|)^{-3p} \right]^{1/p}. \tag{4.3}$$

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standars, Applied Mathematics Series, **55**, 1972.
[2] M. V. Boldea, S. S. Dragomir and M. Megan, New bounds for Čebyšev functional for power series in Banach algebras via a Grüss-Lupaş type inequality. *PanAmer. Math. J.* **26** (2016), no. 3, 71–88.
[3] P. Cerone and S. S. Dragomir, Trapezoidal-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 65–134, Chapman & Hall/CRC, Boca Raton, FL, 2000.
[4] J. B. Conway, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York, 1990.
[5] S. S. Dragomir, Inequalities for power series in Banach algebras. *SUT J. Math.* **50** (2014), no. 1, 25–45.
[6] S. S. Dragomir, Inequalities of Lipschitz type for power series in Banach algebras. *Ann. Math. Sil.* **No**. 29 (2015), 61–83.
[7] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp.
[8] S. S. Dragomir, Lipschitz-type inequalities for analytic functions in Banach algebras, *Bull. Aust. Math. Soc.* **100** (2019), no. 3, 489–497.
[9] S. S. Dragomir, M. V. Boldea and M. Megan, New norm inequalities of Čebyšev type for power series in Banach algebras. *Sarajevo J. Math.* **11** (24) (2015), no. 2, 253–266.
[10] S. S. Dragomir, M. V. Boldea, C. Buşe and M. Megan, Norm inequalities of Čebyšev type for power series in Banach algebras. *J. Inequal. Appl.* **2014**, 2014:294, 19 pp.
[11] S. S. Dragomir, M. V. Boldea and M. Megan, Further bounds for Čebyšev functional for power series in Banach algebras via Grüss-Lupaş type inequalities for p-norms. *Mem. Grad. Sch. Sci. Eng. Shimane Univ. Ser. B Math.* **49** (2016), 15–34.
[12] S. S. Dragomir, M. V. Boldea and M. Megan, Inequalities for Chebyshev functional in Banach algebras. *Cubo* **19** (2017), no. 1, 53–77.
[13] S. S. Dragomir, P. Cerone and A. Sofo, Some remarks on the midpoint rule in numerical integration. *Studia Univ. Babeş-Bolyai Math.* **45** (2000), no. 1, 63–74. Preprint *RGMIA Res. Rep. Coll.* **1** (1998), No. 1, Art. 4. [Online https://rgmia.org/papers/v1n2/sever1.pdf].
[14] S. S. Dragomir, P. Cerone and A. Sofo, Some remarks on the trapezoid rule in numerical integration. *Indian J. Pure Appl. Math.* **31** (2000), no. 5, 475–494. Preprint *RGMIA Res. Rep. Coll.* **2** (1999), No. 5, Art. 1. [Online https://rgmia.org/papers/v2n5/trapezoid.pdf].
[15] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: http://ajmaa.org/RGMIA/monographs.php).
[16] R. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, 1972.
[17] J. Mikusiński, *The Bochner Integral*, Birkhäuser Verlag, 1978.
[18] W. Rudin, *Functional Analysis*, McGraw Hill, 1973.

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