NORM INEQUALITIES FOR JENSEN'S GAP OF ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x \in \mathcal{L}(\mathcal{B}; E, \mathcal{A}, \mu)$, the set of Bochner-integrable functions on a measurable space $(E, \mathcal{A}, \mu)$ endowed with a countably-additive scalar measure μ on a σ-algebra $\mathcal{A}$ of subsets of E and with values in the Banach algebra $\mathcal{B}$. If the spectrum $\sigma(x(t)) \subset G$ for all $t \in E$ and $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x(t)) \subset \operatorname{ins}(\gamma)$ for all $t \in E$, then, in this paper, we show among others that

$$\left\| \int_E (f \circ x)(u) \, d\mu(u) - f\left(\int_E x(u) \, d\mu(u) \right) \right\|$$
$$\leq \frac{1}{2\pi} \int_E \|\bar{x}_E - x(v)\| \left(\int_\gamma \frac{|f(\xi)|}{(|\xi| - \|\bar{x}_E\|)(|\xi| - \|x(v)\|)} |d\xi| \right) d\mu(v),$$

where

$$\bar{x}_E := \int_E x(u) \, d\mu(u).$$

Some examples for exponential function in Banach algebras are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega w(x) |f(x)| \, d\mu(x) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x) \, d\mu(x)$.

Assume that $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on (m, M), $f : \Omega \to [m, M]$ and $w \geq 0$ μ-a.e. on Ω with $\int_\Omega w d\mu = 1$ and such that $\Phi \circ f$, $f \in L_w(\Omega, \mu)$. We define the *integral Jensen's gap* by

$$J(\Phi, f, w, \mu, \Omega) := \int_\Omega w(\Phi \circ f) \, d\mu - \Phi\left(\int_\Omega w f d\mu \right).$$

We have the following string of upper bounds, see [7], [8] and [9]:

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$$
\begin{aligned}
(1.1)\qquad 0 &\leq J(\Phi, f, w, \mu, \Omega) \\
&\leq \int_{\Omega} w(\Phi' \circ f) f d\mu - \int_{\Omega} w(\Phi' \circ f) d\mu \int_{\Omega} wf d\mu \\
&\leq \frac{1}{2}[\Phi'(M) - \Phi'(m)] \int_{\Omega} w \left| f - \int_{\Omega} wf d\mu \right| d\mu \\
&\leq \frac{1}{2}[\Phi'(M) - \Phi'(m)] \left[\int_{\Omega} wf^2 d\mu - \left(\int_{\Omega} wf d\mu \right)^2 \right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}[\Phi'(M) - \Phi'(m)](M - m),
\end{aligned}
$$

provided that $\Phi' \circ f,\ (\Phi' \circ f) f \in L_w(\Omega, \mu)$.

For a recent survey presenting different upper bounds for the integral Jensen's gap, see the survey paper [12] and the references therein.

In order to obtain upper bounds for the norm of the integral Jensen's gap for analytic functions with values in Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \rightarrow [0, \infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \leq \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\mathrm{Inv}(\mathcal{B})$. If $a, b \in \mathrm{Inv}(\mathcal{B})$ then $ab \in \mathrm{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \mathrm{Inv}(\mathcal{B})$;
(ii) $\{b \in \mathcal{B}: \ \|1 - b\| < 1\} \subset \mathrm{Inv}(\mathcal{B})$;
(iii) $\mathrm{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\mathrm{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \mathrm{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : \ z - a \in \mathrm{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \mathrm{Inv}(\mathcal{B})$, $R_a(z) := (z - a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z - w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;

(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;

(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;

(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;

(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_\delta f(\xi)(\xi - a)^{-1} d\xi, \tag{1.2}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [6, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.3}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [20] and [27].

For some recent norm inequalities for functions on Banach algebras, see [13], [3] and [10]-[17].

2. Some Facts on Bochner Integral

Let $\mathcal{F}(B; E, \mathcal{A}, \mu)$ be the linear space of functions $x(v)$, $v \in E$, with values in a real or complex Banach space B, given on a measurable space $(E, \mathcal{A}, \mu)$ endowed with a countably-additive scalar measure μ on a σ-algebra $\mathcal{A}$ of subsets of E.

A function $x_0 \in \mathcal{F}$ is called *simple* if can be defined as, see [28]

$$x_0(v) := \begin{cases} x_i \in B, & v \in A_i \in \mathcal{A},\ \mu(A_i) < \infty, i \in \{1, ..., n\} \\ & A_k \cap A_j = \emptyset, k \neq j, k, j \in \{1, ..., n\}, \\ & \\ 0, & v \in E \setminus \cup_{i=1}^n A_i,\ n \in \mathbb{N}. \end{cases}$$

A function $x \in \mathcal{F}$ is called *strongly measurable* if there exists a sequence $\{x_n\}$ of simple functions with $\|x_n - x\| \to 0$ almost-everywhere with respect to the measure μ on E. As a consequence of this, the scalar function $\|x\|$ is $\mathcal{A}$-measurable.

For the simple function $x_0 \in \mathcal{F}$ as above we define the integral by

$$\int_E x_0(v)\, d\mu(v) := \sum_{i=1}^n x_i \mu(A_i).$$

A function $x \in \mathcal{F}$ is said to be *Bochner integrable* if it is strongly measurable and if for some approximating sequence $\{x_n\}$ of simple functions we have

$$\lim_{n\to\infty}\int_E \|x(v) - x_n(v)\|\, d\mu(v) = 0.$$

The *Bochner integral* of such a function over a set $A \in \mathcal{A}$ is defined as

$$\int_A x(v)\, d\mu(v) = \lim_{n\to\infty}\int_E \chi_A(v)\, x_n(v)\, d\mu(v),$$

where χ_A is the *characteristic function* of A, and the limit is understood in the sense of strong convergence in the Banach space E. This limit exists, and is independent of the choice of the approximation sequence of simple functions.

It is well-known that, for a strongly-measurable function to be Bochner integrable it is necessary and sufficient for the norm of this function to be integrable, i.e.

$$\int_A \|x(v)\|\, d\mu(v) < \infty.$$

The set of Bochner-integrable functions forms a vector subspace $\mathcal{L}(B; E, \mathcal{A}, \mu)$ of $\mathcal{F}(B; E, \mathcal{A}, \mu)$, and the Bochner integral is a linear operator on this subspace.

Some fundamental properties of Bochner integrals are as follows [28], see also [2], [21], [22], [23] and [29]:

(1) For any $x \in \mathcal{L}(B; E, \mathcal{A}, \mu)$ we have the norm inequality

$$\left\| \int_A x(v)\, d\mu(v) \right\| \le \int_A \|x(v)\|\, d\mu(v).$$

(2) Bochner integral is a countably-additive μ-absolutely continuous set-function on the σ-algebra $\mathcal{A}$, i.e.

$$\int_{\cup_{i=1}^{\infty} A_i} x(v)\, d\mu(v) = \sum_{i=1}^{\infty} \int_{A_i} x(v)\, d\mu(v)$$

if $A_i \in \mathcal{A}$, $\mu(A_i) < \infty, i \in \mathbb{N}$, $A_k \cap A_j = \emptyset, k \neq j, k, j \in \mathbb{N}$, and

$$\left\| \int_A x(v)\, d\mu(v) \right\| \to 0 \text{ if } \mu(A) \to 0,$$

uniformly over $A \in \mathcal{A}$.

(3) If $x_n \in F, x_n \to x$ almost-everywhere with respect to the measure μ on $A \in \mathcal{A}$, if $\|x_n\| \le f$ almost-everywhere with respect to μ on A, and if $\int_A f(v)\, d\mu(v) < \infty$, then $x \in \mathcal{L}(B; E, \mathcal{A}, \mu)$ and

$$\int_A x_n(v)\, d\mu(v) \to \int_A x(v)\, d\mu(v).$$

(4) The space is complete with respect to the norm

$$\|x\| := \int_A \|x(v)\|\, d\mu(v).$$

(5) If T is a closed linear operator from a Banach space X into a Banach space Y and if $x \in \mathcal{L}(X; E, \mathcal{A}, \mu)$ and $Tx \in \mathcal{L}(Y; E, \mathcal{A}, \mu)$, then

$$\int_A Tx(v)\, d\mu(v) = T\left(\int_A x(v)\, d\mu(v) \right).$$

If T is bounded, the condition $Tx \in \mathcal{L}(Y; E, \mathcal{A}, \mu)$ is automatically satisfied.

3. Preliminary Results

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \ t \in [0,1]. \tag{3.1}$$

Lemma 1. *Assume that $f: G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$. The function $f_{x,y}: [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f_{x,y}^{'}(t) = D(f)((1-t)x + ty)(y-x) \tag{3.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.2).

We also have the lateral derivatives

$$f_{x,y}^{'}(0+) = D(f)(x)(y-x) \text{ and } f_{x,y}^{'}(1-) = D(f)(y)(y-x). \tag{3.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}
&\frac{f_{x,y}(t+h) - f(t)}{h} \\
&= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\
&= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
f_{x,y}^{'}(t) = \frac{df_{x,y}(t)}{dt} &= \lim_{h\to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\
&= \lim_{h\to 0}\left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}\right] \\
&= D(f)((1-t)x + ty)(y-x),
\end{aligned}$$

which proves (3.5).

The proof is similar for the lateral derivatives. □

Lemma 2. *Assume that $f: G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{3.4}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x + hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned}
&f(x+hv) - f(x) \\
&= \frac{1}{2\pi i}\left(\int_{\gamma} f(\xi)(\xi - x - hv)^{-1} d\xi - \int_{\gamma} f(\xi)(\xi - x)^{-1} d\xi\right) \\
&= \frac{1}{2\pi i}\int_{\gamma} f(\xi)\left[(\xi - x - hv)^{-1} - (\xi - x)^{-1}\right] d\xi \\
&= \frac{1}{2\pi i} h \int_{\gamma} f(\xi)\left[(\xi - x - hv)^{-1} v (\xi - x)^{-1}\right] d\xi,
\end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x+hv)-f(x)}{h} = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x - hv)^{-1} v(\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (3.4). □

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f'_{x,y}(t) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1}(y-x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{3.5}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f'_{x,y}(0+) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1}(y-x)(\xi - x)^{-1} d\xi, \tag{3.6}$$

and

$$f'_{x,y}(1-) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - y)^{-1}(y-x)(\xi - y)^{-1} d\xi. \tag{3.7}$$

The proof is obvious by Lemmas 1 and 2.

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x \in \mathcal{L}(\mathcal{B}; E, \mathcal{A}, \mu)$ with $\sigma(x(u)) \subset G$ for all $u \in E$.

In the following we assume that $\int_E d\mu(u) = 1$. We define the *Jensen's gap* by

$$J(f, x; E, \mathcal{A}, \mu) := \int_E (f \circ x)(u)\, d\mu(u) - f\left(\int_E x(u)\, d\mu(u)\right). \tag{3.8}$$

We have the following representation for the Jensen's gap:

Theorem 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x \in \mathcal{L}(\mathcal{B}; E, \mathcal{A}, \mu)$ with $\sigma(x(t)) \subset G$ for all $t \in E$. If $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x(t)) \subset \operatorname{ins}(\gamma)$ for all $t \in E$, then*

$$\begin{aligned} J(f, x; E, \mathcal{A}, \mu) = \frac{1}{2\pi i}\int_\gamma f(\xi)\Bigg(\int_0^1 \Bigg[\int_E (\xi - (1-t)x(v) - t\bar{x}_E)^{-1} \\ \times (\bar{x}_E - x(v))(\xi - (1-t)x(v) - t\bar{x}_E)^{-1} d\mu(v)\Bigg] dt\Bigg) d\xi, \end{aligned} \tag{3.9}$$

where

$$\bar{x}_E := \int_E x(u)\, d\mu(u).$$

Proof. We have for x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ that

$$f(y) - f(x) = f_{x,y}(1) - f_{x,y}(0) = \int_0^1 f'_{x,y}(t)\, dt.$$

By Lemma 3 and Fubini's theorem, we have

$$\begin{aligned}
&\int_0^1 f_{x,y}^{'}(t)\,dt \\
&= \frac{1}{2\pi i}\int_0^1 \left(\int_\gamma f(\xi)(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\,d\xi\right)dt \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi)\left(\int_0^1 (\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\,dt\right)d\xi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&f(y)-f(x) \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi) \\
&\times\left(\int_0^1 (\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\,dt\right)d\xi.
\end{aligned} \tag{3.10}$$

From (3.10) we get

$$\begin{aligned}
&f(x(v))-f\left(\int_E x(u)\,d\mu(u)\right) \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi)\left(\int_0^1 \left(\xi-(1-t)x(v)-t\int_E x(u)\,d\mu(u)\right)^{-1}\right. \\
&\times\left(\int_E x(u)\,d\mu(u)-x(v)\right) \\
&\times\left.\left(\xi-(1-t)x(v)-t\int_E x(u)\,d\mu(u)\right)^{-1}\right)dt\,d\xi.
\end{aligned} \tag{3.11}$$

By taking the Bochner integral in (3.11) over the variable $v\in E$, use the fact that $\int_E d\mu(v)=1$, then we get by Fubini's theorem

$$\begin{aligned}
&\int_E f(x(v))\,d\mu(v)-f\left(\int_E x(u)\,d\mu(u)\right) \\
&= \frac{1}{2\pi i}\int_E\left[\int_\gamma f(\xi)\left(\int_0^1\left(\xi-(1-t)x(v)-t\int_E x(u)\,d\mu(u)\right)^{-1}\right.\right. \\
&\times\left(\int_E x(u)\,d\mu(u)-x(v)\right) \\
&\times\left.\left.\left(\xi-(1-t)x(v)-t\int_E x(u)\,d\mu(u)\right)^{-1}dt\right)d\xi\right]d\mu(v) \\
&= \frac{1}{2\pi i}\int_\gamma f(\xi)\left(\int_0^1\left[\int_E\left(\xi-(1-t)x(v)-t\int_E x(u)\,d\mu(u)\right)^{-1}\right.\right. \\
&\times\left(\int_E x(u)\,d\mu(u)-x(v)\right) \\
&\times\left.\left.\left(\xi-(1-t)x(v)-t\int_E x(u)\,d\mu(u)\right)^{-1}d\mu(v)\right]dt\right)d\xi,
\end{aligned}$$

which proves the desired identity (3.9). □

Remark 1. *Assume that* $f : G \subset \mathbb{C} \to \mathbb{C}$ *is an analytic function on the convex domain* G *and* $x \in \mathcal{L}(\mathcal{B}; E, \mathcal{A}, \mu)$ *with* $\sigma(x(t)) \subset D(0,R) \subset G$ *for all* $t \in E$, *where* $D(0,R)$ *is an open disk centered in* 0 *and of radius* R. *By taking* γ *parametrized by* $\xi(t) = Re^{2\pi is}$ *where* $s \in [0,1]$, *then* $d\xi(s) = 2\pi i Re^{2\pi is} ds$, $|d\xi(s)| = 2\pi R ds$, $|\xi| = R$ *and by (3.9) we obtain the representation*

$$J(f, x; E, \mathcal{A}, \mu) = R\int_0^1 e^{2\pi is} f\left(Re^{2\pi is}\right)\left(\int_0^1 \left[\int_E \left(Re^{2\pi is} - (1-t)x(v) - t\bar{x}_E\right)^{-1} \times (\bar{x}_E - x(v))\left(Re^{2\pi is} - (1-t)x(v) - t\bar{x}_E\right)^{-1} d\mu(v)\right] dt\right) ds. \tag{3.12}$$

4. General Norm Inequalities

We have the following norm inequalities:

Theorem 2. *Assume that* $f : G \to \mathbb{C}$ *is analytic on the convex domain* G *and* $x \in \mathcal{L}(\mathcal{B}; E, \mathcal{A}, \mu)$ *with* $\sigma(x(t)) \subset G$ *for all* $t \in E$. *If* $\gamma \subset G$ *is taken to be close rectifiable curve in* G *and such that* $\sigma(x(t)) \subset \operatorname{ins}(\gamma)$ *for all* $t \in E$, *then*

$$\begin{aligned} &\|J(f, x; E, \mathcal{A}, \mu)\| \\ &\le \frac{1}{2\pi}\int_E \|\bar{x}_E - x(v)\| \\ &\quad\times \left[\int_\gamma |f(\xi)|\left(\int_0^1 \left\|(\xi - (1-t)x(v) - t\bar{x}_E)^{-1}\right\|^2 dt\right)|d\xi|\right] d\mu(v) \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\le \frac{1}{2\pi}\int_E \|\bar{x}_E - x(v)\| \\ &\quad\times \left[\int_\gamma |f(\xi)|\left(\int_0^1 (|\xi| - \|(1-t)x(v) + t\bar{x}_E\|)^{-2} dt\right)|d\xi|\right] d\mu(v) \\ &\le \frac{1}{2\pi}\int_E \|\bar{x}_E - x(v)\| \\ &\quad\times \left(\int_\gamma \frac{|f(\xi)|}{(|\xi| - \|\bar{x}_E\|)(|\xi| - \|x(v)\|)}|d\xi|\right) d\mu(v). \end{aligned}$$

Proof. By taking the norm and using the properties of the integral, we get

$$\begin{aligned} &\|J(f, x; E, \mathcal{A}, \mu)\| \\ &\le \frac{1}{2\pi}\int_\gamma |f(\xi)| \left\|\int_0^1 \left[\int_E (\xi - (1-t)x(v) - t\bar{x}_E)^{-1} \right.\right. \\ &\quad\left.\left. \times (\bar{x}_E - x(v))(\xi - (1-t)x(v) - t\bar{x}_E)^{-1} d\mu(v)\right] dt\right\| |d\xi| \end{aligned}$$

$$\leq \frac{1}{2\pi}\int_{\gamma}|f(\xi)|\int_0^1\left[\int_E\left\|(\xi-(1-t)x(v)-t\bar{x}_E)^{-1}\right.\right.$$
$$\left.\left.\times\ (\bar{x}_E-x(v))(\xi-(1-t)x(v)-t\bar{x}_E)^{-1}\right\|d\mu(v)\right]dt\Big)|d\xi|$$
$$\leq \frac{1}{2\pi}\int_{\gamma}|f(\xi)|$$
$$\times\left(\int_0^1\left(\int_E\|\bar{x}_E-x(v)\|\left\|(\xi-(1-t)x(v)-t\bar{x}_E)^{-1}\right\|^2 d\mu(v)\right)dt\right)|d\xi|$$
$$=\frac{1}{2\pi}\int_E\|\bar{x}_E-x(v)\|$$
$$\times\left[\int_{\gamma}|f(\xi)|\left(\int_0^1\left\|(\xi-(1-t)x(v)-t\bar{x}_E)^{-1}\right\|^2 dt\right)|d\xi|\right]d\mu(v),$$

which proves the first inequality in (4.1).

Let $v\in E$. We have

$$B(f,x(v),\bar{x}_E)$$
$$:=\frac{1}{2\pi}\int_{\gamma}|f(\xi)|\int_0^1\left(\left\|(\xi-(1-t)x(v)-t\bar{x}_E)^{-1}\right\|^2 dt\right)|d\xi|$$
$$=\frac{1}{2\pi}\int_{\gamma}|f(\xi)||\xi|^{-2}\left(\int_0^1\left\|\left(1-(1-t)\frac{x(v)}{\xi}-t\frac{\bar{x}_E}{\xi}\right)^{-1}\right\|^2 dt\right)|d\xi|.$$

Since $\sigma(\bar{x}_E)\subset \operatorname{ins}(\gamma)$,

$$\left\|(1-t)\frac{x(v)}{\xi}+t\frac{\bar{x}_E}{\xi}\right\|\leq(1-t)\left\|\frac{x(v)}{\xi}\right\|+t\left\|\frac{\bar{x}_E}{\xi}\right\|<1-t+t=1$$

for $\xi\in\gamma$, hence

$$\left(1-(1-t)\frac{x(v)}{\xi}-t\frac{\bar{x}_E}{\xi}\right)^{-1}=\sum_{k=0}^{\infty}\left[(1-t)\frac{x(v)}{\xi}+t\frac{\bar{x}_E}{\xi}\right]^k.$$

Therefore

$$\begin{aligned}\left\|\left(1-(1-t)\frac{x(v)}{\xi}-t\frac{\bar{x}_E}{\xi}\right)^{-1}\right\|&\leq\sum_{k=0}^{\infty}\left\|(1-t)\frac{x(v)}{\xi}+t\frac{\bar{x}_E}{\xi}\right\|^k\\&=\left(1-\left\|(1-t)\frac{x(v)}{\xi}+t\frac{\bar{x}_E}{\xi}\right\|\right)^{-1}\\&=\left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x(v)}{\xi}+t\frac{\bar{x}_E}{\xi}\right\|\right)^{-1}\\&=|\xi|\left(|\xi|-\|(1-t)x(v)+t\bar{x}_E\|\right)^{-1}\end{aligned}$$

for $\xi\in\gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{x(v)}{\xi}-t\frac{\bar{x}_E}{\xi}\right)^{-1}\right\|^2\leq|\xi|^2\left(|\xi|-\|(1-t)x(v)+t\bar{x}_E\|\right)^{-2}$$

for $\xi\in\gamma$.

Therefore

$$\int_\gamma |f(\xi)| |\xi|^{-2} \left(\int_0^1 \left\| \left(1 - (1-t) \frac{x(v)}{\xi} - t \frac{\bar{x}_E}{\xi} \right)^{-1} \right\|^2 dt \right) |d\xi|$$
$$\leq \int_\gamma |f(\xi)| \left(\int_0^1 (|\xi| - \|(1-t) x(v) + t\bar{x}_E\|)^{-2} dt \right) |d\xi|,$$

which proves the second inequality in (4.1).

By the triangle inequality we have

$$|\xi| - \|(1-t) x(v) + t\bar{x}_E\| \geq |\xi| - (1-t) \|x(v)\| - t \|\bar{x}_E\|$$
$$= (1-t) (|\xi| - \|x(v)\|) + t (|\xi| - \|\bar{x}_E\|) > 0$$

for $\xi \in \gamma$.

This implies that

$$(|\xi| - \|(1-t) x(v) + t\bar{x}_E\|)^{-1} \leq [(1-t) (|\xi| - \|x(v)\|) + t (|\xi| - \|\bar{x}_E\|)]^{-1}$$

and thus

$$(|\xi| - \|(1-t) x(v) + t\bar{x}_E\|)^{-2} \leq [(1-t) (|\xi| - \|x(v)\|) + t (|\xi| - \|\bar{x}_E\|)]^{-2}$$

for $\xi \in \gamma$ and $t \in [0,1]$.

Taking the integral over $t \in [0,1]$, we get

$$\int_0^1 (|\xi| - \|(1-t) x(v) + t\bar{x}_E\|)^{-2} dt$$
$$\leq \int_0^1 [(1-t) (|\xi| - \|x(v)\|) + t (|\xi| - \|\bar{x}_E\|)]^{-2} dt$$
$$= -\frac{1}{\|x(v)\| - \|\bar{x}_E\|} \int_0^1 \frac{d}{dt} [(1-t) (|\xi| - \|x(v)\|) + t (|\xi| - \|\bar{x}_E\|)]^{-1} dt$$
$$= -\frac{1}{\|x(v)\| - \|\bar{x}_E\|} [(1-t) (|\xi| - \|x(v)\|) + t (|\xi| - \|\bar{x}_E\|)]^{-1} \Big|_0^1$$
$$= \frac{1}{\|\bar{x}_E\| - \|x(v)\|} \left[(|\xi| - \|\bar{x}_E\|)^{-1} - (|\xi| - \|x(v)\|)^{-1} \right]$$
$$= \frac{1}{\|\bar{x}_E\| - \|x(v)\|} \frac{\|\bar{x}_E\| - \|x(v)\|}{(|\xi| - \|\bar{x}_E\|)(|\xi| - \|x(v)\|)}$$
$$= \frac{1}{(|\xi| - \|\bar{x}_E\|)(|\xi| - \|x(v)\|)},$$

for $\|\bar{x}_E\| \neq \|x(v)\|$.

If $\|\bar{x}_E\| = \|x(v)\|$, then we have

$$\int_0^1 (|\xi| - \|(1-t) x(v) + t\bar{x}_E\|)^{-2} dt$$
$$\leq \int_0^1 [(1-t) (|\xi| - \|x(v)\|) + t (|\xi| - \|x(v)\|)]^{-2} dt = (|\xi| - \|x(v)\|)^{-2},$$

which also gives the bound for $\|\bar{x}_E\| = \|x(v)\|$.

Therefore, in all cases

$$B(f, x(v), \bar{x}_E) \leq \frac{1}{2\pi} \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|\bar{x}_E\|)(|\xi| - \|x(v)\|)} |d\xi|$$

and the last part of (4.1) is also proved. ☐

Remark 2. *Assume that* $f : G \subset \mathbb{C} \to \mathbb{C}$ *is an analytic function on the convex domain* G *and* $x \in \mathcal{L}(\mathcal{B}; E, \mathcal{A}, \mu)$ *with* $\sigma(x(t)) \subset D(0,R) \subset G$ *for all* $t \in E$, *where* $D(0,R)$ *is an open disk centered in* 0 *and of radius* R. *By taking* γ *parametrized by* $\xi(t) = Re^{2\pi i s}$ *where* $s \in [0,1]$, *then* $d\xi(s) = 2\pi i R e^{2\pi i s} ds$, $|d\xi(s)| = 2\pi R ds$, $|\xi| = R$ *and by (4.1) we get*

$$
\begin{aligned}
&\|J(f,x;E,\mathcal{A},\mu)\| \\
&\le R\int_E \|\bar{x}_E - x(v)\| \left[\int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| \right.\\
&\left.\times\left(\int_0^1 \left\|\left(Re^{2\pi i s} - (1-t)x(v) - t\bar{x}_E\right)^{-1}\right\|^2 dt\right) ds\right] d\mu(v) \\
&\le R\int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| ds \\
&\times \int_E \|\bar{x}_E - x(v)\| \left(\int_0^1 \left(R - \|(1-t)x(v) + t\bar{x}_E\|\right)^{-2} dt\right) d\mu(v) \\
&\le \frac{R}{R - \|\bar{x}_E\|}\int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| ds \int_E \frac{\|\bar{x}_E - x(v)\|}{R - \|x(v)\|} d\mu(v).
\end{aligned} \tag{4.2}
$$

Corollary 1. *With the assumptions of Theorem 2, then for* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$
\|J(f,x;E,\mathcal{A},\mu)\| \tag{4.3}
$$

$$
\le \frac{1}{2\pi}\begin{cases}
\int_E \|\bar{x}_E - x(v)\| d\mu(v) \\
\times \operatorname{esssup}_{v\in E}\left(\int_\gamma \frac{|f(\xi)|}{(|\xi| - \|\bar{x}_E\|)(|\xi| - \|x(v)\|)} |d\xi|\right), \\[2ex]
\left(\int_E \|\bar{x}_E - x(v)\|^p\right)^{1/p} \\
\times \left[\int_E \left(\int_\gamma \frac{|f(\xi)|}{(|\xi| - \|\bar{x}_E\|)(|\xi| - \|x(v)\|)} |d\xi|\right)^q d\mu(v)\right]^{1/q}, \\[2ex]
\operatorname{esssup}_{v\in E}\|\bar{x}_E - x(v)\| \\
\times \int_E \left(\int_\gamma \frac{|f(\xi)|}{(|\xi| - \|\bar{x}_E\|)(|\xi| - \|x(v)\|)} |d\xi|\right) d\mu(v),
\end{cases}
$$

$$
\le \frac{1}{2\pi}\begin{cases}
\int_E \|\bar{x}_E - x(v)\| d\mu(v) \\
\times \int_\gamma \frac{|f(\xi)|}{\left(|\xi| - \operatorname{esssup}_{v\in E}\|x(v)\|\right)(|\xi| - \|\bar{x}_E\|)} |d\xi|, \\[2ex]
\ell^{1/p}(\gamma)\left(\int_E \|\bar{x}_E - x(v)\|^p d\mu(v)\right)^{1/p} \\
\times \left[\int_\gamma \left(\int_E \frac{1}{(|\xi| - \|x(v)\|)^q} d\mu(v)\right) \frac{|f(\xi)|^q}{(|\xi| - \|\bar{x}_E\|)^q} |d\xi|\right]^{1/q}, \\[2ex]
\operatorname{esssup}_{v\in E}\|\bar{x}_E - x(v)\| \\
\times \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|\bar{x}_E\|)} \left(\int_E \frac{1}{|\xi| - \|x(v)\|} d\mu(v)\right) |d\xi|,
\end{cases}
$$

where $\ell(\gamma) := \int_\gamma |d\xi|$.

Proof. By Hölder's inequality we have for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$
\int_E \|\bar{x}_E - x(v)\| \left(\int_\gamma \frac{|f(\xi)|}{(|\xi| - \|\bar{x}_E\|)(|\xi| - \|x(v)\|)} |d\xi|\right) d\mu(v)
$$

$$\leq \begin{cases} \operatorname{esssup}_{v\in E}\left(\int_{\gamma}\frac{|f(\xi)|}{(|\xi|-\|\bar{x}_E\|)(|\xi|-\|x(v)\|)}\,|d\xi|\right)\int_E\|\bar{x}_E-x(v)\|\,d\mu(v) \\ \left(\int_E\|\bar{x}_E-x(v)\|^p\,d\mu(v)\right)^{1/p}\left[\int_E\left(\int_{\gamma}\frac{|f(\xi)|}{(|\xi|-\|\bar{x}_E\|)(|\xi|-\|x(v)\|)}\,|d\xi|\right)^q d\mu(v)\right]^{1/q} \\ \operatorname{esssup}_{v\in E}\|\bar{x}_E-x(v)\|\int_E\left(\int_{\gamma}\frac{|f(\xi)|}{(|\xi|-\|\bar{x}_E\|)(|\xi|-\|x(v)\|)}\,|d\xi|\right)d\mu(v). \end{cases}$$

Now, observe that

$$\begin{aligned} &\operatorname*{esssup}_{v\in E}\left(\int_{\gamma}\frac{|f(\xi)|}{(|\xi|-\|\bar{x}_E\|)(|\xi|-\|x(v)\|)}\,|d\xi|\right) \\ &\leq \int_{\gamma}\operatorname*{esssup}_{v\in E}\left(\frac{1}{(|\xi|-\|x(v)\|)}\right)\frac{|f(\xi)|}{(|\xi|-\|\bar{x}_E\|)}\,|d\xi| \\ &= \int_{\gamma}\frac{|f(\xi)|}{(|\xi|-\operatorname{esssup}_{v\in E}\|x(v)\|)(|\xi|-\|\bar{x}_E\|)}\,|d\xi|, \end{aligned}$$

which proves the first branch of the second inequality in (4.3).

By Jensen's inequality for power $q>1$,

$$\begin{aligned} &\left(\int_{\gamma}\frac{|f(\xi)|}{(|\xi|-\|\bar{x}_E\|)(|\xi|-\|x(v)\|)}\,|d\xi|\right)^q \\ &\leq \ell^{q-1}(\gamma)\int_{\gamma}\frac{|f(\xi)|^q}{(|\xi|-\|\bar{x}_E\|)^q(|\xi|-\|x(v)\|)^q}\,|d\xi|, \end{aligned}$$

where $\ell(\gamma):=\int_{\gamma}|d\xi|$.

Then

$$\begin{aligned} &\left[\int_E\left(\int_{\gamma}\frac{|f(\xi)|}{(|\xi|-\|\bar{x}_E\|)(|\xi|-\|x(v)\|)}\,|d\xi|\right)^q d\mu(v)\right]^{1/q} \\ &\leq \left[\ell^{q-1}(\gamma)\int_E\left(\int_{\gamma}\frac{|f(\xi)|^q}{(|\xi|-\|\bar{x}_E\|)^q(|\xi|-\|x(v)\|)^q}\,|d\xi|\right)d\mu(v)\right]^{1/q} \\ &= \ell^{1-1/q}(\gamma)\left[\int_{\gamma}\left(\int_E\frac{1}{(|\xi|-\|x(v)\|)^q}d\mu(v)\right)\frac{|f(\xi)|^q}{(|\xi|-\|\bar{x}_E\|)^q}\,|d\xi|\right]^{1/q}, \end{aligned}$$

which proves the second branch of the second inequality in (4.3).

The last part is obvious. □

Remark 3. *Assume that $f:G\subset\mathbb{C}\to\mathbb{C}$ is an analytic function on the convex domain G and $x\in\mathcal{L}(\mathcal{B};E,\mathcal{A},\mu)$ with $\sigma(x(t))\subset D(0,R)\subset G$ for all $t\in E$. Then* *by (4.3),*

$$\| J(f,x;E,\mathcal{A},\mu) \| \le \frac{R}{R-\|\bar{x}_E\|} \times \begin{cases} \int_0^1 \left| f\left(Re^{2\pi is}\right)\right| ds \int_E \|\bar{x}_E - x(v)\| \, d\mu(v) \, \frac{1}{R-\operatorname{esssup}_{v\in E}\|x(v)\|}, \\ \ell^{1/p}(\gamma)\left(\int_0^1 \left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q} \left(\int_E \|\bar{x}_E - x(v)\|^p d\mu(v)\right)^{1/p} \\ \times \left(\int_E \frac{1}{(R-\|x(v)\|)^q} d\mu(v)\right)^{1/q}, \\ \int_0^1 \left| f\left(Re^{2\pi is}\right)\right| ds \operatorname{esssup}_{v\in E} \|\bar{x}_E - x(v)\| \int_E \frac{1}{R-\|x(v)\|} d\mu(v), \end{cases} \tag{4.4}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

5. Discrete Inequalities

Let $f: G \to \mathbb{C}$ be analytic on the convex domain G and $x_k \in \mathcal{B}$ with $\sigma(x_k) \subset G$ for all $k \in I$, a finite subset of natural numbers. Suppose that $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x_k) \subset \operatorname{ins}(\gamma)$ for all $k \in I$. Assume that $(p_k)_{k\in I}$ is a probability distribution, namely $p_k \ge 0$ for all $k \in I$ with $\sum_{k\in I} p_k = 1$. We define $\bar{x}_{p,I} := \sum_{k\in I} p_k x_k$. We have $\sigma(\bar{x}_{p,I}) \subset \operatorname{ins}(\gamma)$ and can define the *discrete Jensen's gap*

$$J(f,x;I,p) := \sum_{k\in I} p_k f(x_k) - f\left(\sum_{k\in I} p_k x_k\right). \tag{5.1}$$

From (3.8) we derive the representation

$$J(f,x;I,p) = \frac{1}{2\pi i}\int_\gamma f(\xi)\left(\int_0^1 \left[\sum_{k\in I} p_k \left(\xi-(1-t)x_k - t\bar{x}_{p,I}\right)^{-1} \times (\bar{x}_{p,I}-x_k)\left(\xi-(1-t)x_k - t\bar{x}_{p,I}\right)^{-1}\right] dt\right) d\xi, \tag{5.2}$$

while from (4.1) we have the inequalities

$$\|J(f,x;I,p)\| \le \sum_{k\in I} p_k \|\bar{x}_{p,I}-x_k\| \left(\int_\gamma \frac{|f(\xi)|}{(|\xi|-\|\bar{x}_{p,I}\|)(|\xi|-\|x_k\|)} |d\xi|\right). \tag{5.3}$$

We also have the bounds

$$\|J(f,x;I,p)\| \le \frac{1}{2\pi} \begin{cases} \sum_{k\in I} p_k \|\bar{x}_{p,I}-x_k\| \int_\gamma \frac{|f(\xi)|}{(|\xi|-\max_{k\in I}\|x_k\|)(|\xi|-\|\bar{x}_{p,I}\|)} |d\xi|, \\ \ell^{1/p}(\gamma)\left(\sum_{k\in I} p_k \|\bar{x}_{p,I}-x_k\|^p\right)^{1/p} \\ \times \left[\int_\gamma \left(\sum_{k\in I} \frac{p_k}{(|\xi|-\|x_k\|)^q}\right) \frac{|f(\xi)|^q}{(|\xi|-\|\bar{x}_{p,I}\|)^q} |d\xi|\right]^{1/q}, \\ \max_{k\in I}\|\bar{x}_{p,I}-x_k\| \int_\gamma \left(\sum_{k\in I}\frac{p_k}{|\xi|-\|x_k\|}\right)\frac{|f(\xi)|}{(|\xi|-\|\bar{x}_{p,I}\|)} |d\xi|. \end{cases} \tag{5.4}$$

If $\sigma(x_k) \subset D(0,R) \subset G$ for all $k \in I$, then by (5.3) we have

$$\left\| \sum_{k\in I} p_k f(x_k) - f\left(\sum_{k\in I} p_k x_k\right) \right\| \leq \frac{R}{R - \|\bar{x}_E\|} \int_0^1 \left| f\left(Re^{2\pi i s}\right)\right| ds \sum_{k\in I} \frac{p_k \|\bar{x}_E - x_k\|}{R - \|x_k\|}. \tag{5.5}$$

6. Hermite-Hadamard Type Inequalities

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G. We consider $E = [0,1]$, x, $y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma) \subset G$ and $x(s) = (1-s)x + sy$, $s \in [0,1]$. Then $x(s) \subset \operatorname{ins}(\gamma) \subset G$, $s \in [0,1]$ and

$$\bar{x}_E = \int_0^1 x(s)\, ds = \int_0^1 [(1-s)x + sy]\, ds = \frac{x+y}{2}.$$

The Hermite-Hadamard gap is defined by

$$J(f, x; [0,1]) := \int_0^1 f((1-s)x + sy)\, ds - f\left(\frac{x+y}{2}\right). \tag{6.1}$$

From (3.9) we obtain the representation

$$\begin{aligned} &J(f, x; [0,1]) \\ &= \frac{1}{2\pi i} \int_\gamma f(\xi) \left(\int_0^1 \left[\int_0^1 \left(\xi - (1-t)[(1-s)x + sy] - t\frac{x+y}{2}\right)^{-1} \right.\right. \\ &\times \left(\frac{x+y}{2} - [(1-s)x + sy]\right) \\ &\left.\left. \left(\xi - (1-t)[(1-s)x + sy] - t\frac{x+y}{2}\right)^{-1} ds \right] dt \right) d\xi \\ &= \frac{1}{2\pi i} \int_\gamma f(\xi) \\ &\times \left(\int_0^1 \left[\int_0^1 \left(\frac{1}{2} - s\right) \left(\xi - \left(1 - s - \frac{1}{2}t + ts\right)x - \left(s + \frac{1}{2}t - ts\right)y\right)^{-1} \right.\right. \\ &\times \left.\left. (y-x) \left(\xi - \left(1 - s - \frac{1}{2}t + ts\right)x - \left(s + \frac{1}{2}t - ts\right)y\right)^{-1} ds \right] dt \right) d\xi. \end{aligned} \tag{6.2}$$

From (4.1) we have the inequality

$$\begin{aligned} &\left\| \int_0^1 f((1-s)x + sy)\, ds - f\left(\frac{x+y}{2}\right) \right\| \\ &\leq \frac{1}{2\pi} \|y - x\| \\ &\times \int_0^1 \left|\frac{1}{2} - s\right| \left(\int_\gamma \frac{|f(\xi)|}{\left(|\xi| - \left\|\frac{x+y}{2}\right\|\right)\left(|\xi| - \|(1-s)x + sy\|\right)} |d\xi| \right) ds. \end{aligned} \tag{6.3}$$

If $x,y\in\mathcal{B}$ with $\sigma(x),\sigma(y)\subset D(0,R)\subset G$, then by (6.3) we derive the simpler inequality

$$\left\|\int_0^1 f((1-s)x+sy)\,ds-f\left(\frac{x+y}{2}\right)\right\| \le \frac{R\|y-x\|}{R-\left\|\frac{x+y}{2}\right\|}\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt\int_0^1\frac{\left|\frac{1}{2}-s\right|}{R-\|(1-s)x+sy\|}ds. \tag{6.4}$$

It is easy to see that $g(s):=(R-\|(1-s)x+sy\|)^{-1}$ is convex on $[0,1]$ and $h(s):=\left|\frac{1}{2}-s\right|$ is symmetric on $[0,1]$, then by *Féjer's inequality* [18, p. 2]

$$\int_0^1 g(s)h(s)\,ds\le\frac{g(0)+g(1)}{2}\int_0^1 h(s)\,ds,$$

we have

$$\int_0^1\frac{\left|\frac{1}{2}-s\right|}{R-\|(1-s)x+sy\|}ds\le\frac{1}{8}\left((R-\|x\|)^{-1}+(R-\|y\|)^{-1}\right) = \frac{1}{4}\frac{R-\frac{\|x\|+\|y\|}{2}}{(R-\|x\|)(R-\|y\|)}$$

and by (6.4) we get the simpler upper bound

$$\left\|\int_0^1 f((1-s)x+sy)\,ds-f\left(\frac{x+y}{2}\right)\right\| \le\frac{1}{4}R\frac{\left(R-\frac{\|x\|+\|y\|}{2}\right)\|y-x\|}{(R-\|x\|)(R-\|y\|)\left(R-\left\|\frac{x+y}{2}\right\|\right)}\int_0^1\left|f\left(Re^{2\pi it}\right)\right|dt. \tag{6.5}$$

7. Some Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z)=\left(\frac{1}{2}z\right)^\nu\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z)=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned}\int_0^1 \exp\left[R\cos(2\pi t)\right]dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[R\cos\theta\right]d\theta \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi}\exp\left[R\cos\theta\right]d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi}\exp\left[R\cos\theta\right]d\theta\right) \\ &= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).\end{aligned} \tag{7.1}$$

Therefore

$$\left|\exp\left(Re^{2\pi it}\right)\right|^q = \exp\left[qR\cos(2\pi t)\right]$$

and

$$\left(\int_0^1 \left|\exp\left(Re^{2\pi it}\right)\right|^q dt\right)^{1/q} = \left(\int_0^1 \exp\left[qR\cos(2\pi t)\right]dt\right)^{1/q} = I_0^{1/q}(qR) \text{ for } q \geq 1.$$

Assume that $x \in \mathcal{L}(\mathcal{B}; E, \mathcal{A}, \mu)$ with $\sigma(x(t)) \subset D(0, R) \subset G$ for all $t \in E$. If we use the inequality (4.2) for the exponential function, then we get

$$\begin{aligned}&\left\|\int_E \exp\left[x(u)\right]d\mu(u) - \exp\left(\int_E x(u)\,d\mu(u)\right)\right\| \\ &\leq R\int_E \|\bar{x}_E - x(v)\|\left[\int_0^1 \exp\left[R\cos(2\pi t)\right]\right. \\ &\times\left.\left(\int_0^1 \left\|\left(Re^{2\pi is} - (1-t)\,x(v) - t\bar{x}_E\right)^{-1}\right\|^2 dt\right)ds\right]d\mu(v) \\ &\leq RI_0(R)\int_E \|\bar{x}_E - x(v)\|\left(\int_0^1 \left(R - \|(1-t)\,x(v) + t\bar{x}_E\|\right)^{-2}dt\right)d\mu(v) \\ &\leq \frac{RI_0(R)}{R - \|\bar{x}_E\|}\int_E \frac{\|\bar{x}_E - x(v)\|}{R - \|x(v)\|}d\mu(v).\end{aligned} \tag{7.2}$$

If we use (4.4) for the exponential function, then we also obtain

$$\begin{aligned}&\left\|\int_E \exp\left[x(u)\right]d\mu(u) - \exp\left(\int_E x(u)\,d\mu(u)\right)\right\| \\ &\leq \frac{R}{R - \|\bar{x}_E\|}\times\begin{cases} I_0(R)\int_E \|\bar{x}_E - x(v)\|\,d\mu(v)\,\frac{1}{R - \operatorname{esssup}_{v\in E}\|x(v)\|}, \\ \\ \ell^{1/p}(\gamma)\,I_0^{1/q}(qR)\left(\int_E \|\bar{x}_E - x(v)\|^p\,d\mu(v)\right)^{1/p} \\ \times\left(\int_E \frac{1}{(R - \|x(v)\|)^q}d\mu(v)\right)^{1/q}, \\ \\ I_0(R)\operatorname{esssup}_{v\in E}\|\bar{x}_E - x(v)\|\int_E \frac{1}{R - \|x(v)\|}d\mu(v).\end{cases}\end{aligned} \tag{7.3}$$

If $\sigma(x_k) \subset D(0,R)$ for all $k \in I$, then by (5.5) we get the discrete inequality

$$\left\| \sum_{k\in I} p_k \exp x_k - \exp\left(\sum_{k\in I} p_k x_k\right)\right\| \le \frac{RI_0(R)}{R-\|\bar{x}_E\|}\sum_{k\in I}\frac{p_k\|\bar{x}_E - x_k\|}{R-\|x_k\|}, \tag{7.4}$$

where $p_k \ge 0$ for all $k \in I$ with $\sum_{k\in I} p_k = 1$.

Finally, if $x,\ y \in \mathcal{B}$ with $\sigma(x)$, $\sigma(y) \subset D(0,R)$, then we have the Hermite-Hadamard type inequality

$$\begin{aligned} &\left\| \int_0^1 \exp((1-s)x + sy)\,ds - \exp\left(\frac{x+y}{2}\right)\right\| \\ &\le \frac{1}{4} RI_0(R)\frac{\left(R - \frac{\|x\|+\|y\|}{2}\right)\|y-x\|}{(R-\|x\|)(R-\|y\|)\left(R-\left\|\frac{x+y}{2}\right\|\right)}. \end{aligned} \tag{7.5}$$

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