MONOTONICITY AND INEQUALITIES RELATED TO THE GENERALIZED INVERSE LEMNISCATE FUNCTIONS

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Abstract. In the paper, we mainly studied the monotonicity and inequalities for some functions related to the generalized inverse lemniscate and the generalized hyperbolic inverse lemniscate functions. At first, we presented a bound estimation of the generalized inverse lemniscate functions by using Lerch Phi function. Finally, some Shafer-Fink, Wilker and Huygens type inequalities for the generalized inverse lemniscate and the generalized hyperbolic inverse lemniscate functions were obtained.

1. introduction

In [37], Takeuchi studied deeply the (p,q)-trigonometric functions depending on two parameters. The generalized trigonometric functions with two parameters are well studied in the context of nonlinear differential equations. The following (p,q)-eigenvalue problem with Dirichlét boundary condition was considered by Drábek and Manásevich [16]. Let $\phi_p(x)=|x|^{p-2}x$. For $T,\lambda>0$ and $p,q>1$

$$\begin{cases} (\phi_p(u'))'+\lambda\,\phi_q(u)=0, \quad t\in(0,T),\\ u(0)=u(T)=0. \end{cases}$$

The solution of this problem also appears in [37, Thm 2.1]. In particular, for $T=\pi_{p,q}$ the function $u(t)=\sin_{p,q}(t)$ is a solution to this problem with $\lambda=p/q(p-1)$, where

$$\pi_{p,q}=2\int_0^1(1-t^q)^{-1/p}\,dt=\frac{2}{q}B\left(1-\frac{1}{p},\frac{1}{q}\right). \tag{1.1}$$

In order to give the definition of the function $\sin_{p,q}$, first we define its inverse function $\sin_{p,q}^{-1}$, then the function itself. For $x\in[0,1]$, set

$$F_{p,q}(x)=\sin_{p,q}^{-1}=\int_0^x(1-t^q)^{-1/p}\,dt\,. \tag{1.2}$$

The function $F_{p,q}:[0,1]\to[0,\pi_{p,q}/2]$ is an increasing homeomorphism, and

$$\sin_{p,q}=F_{p,q}^{-1}$$

is defined on the the interval $[0,\pi_{p,q}/2]$. The function $\sin_{p,q}$ can be extended to $[0,\pi_{p,q}]$ by

$$\sin_{p,q}(x)=\sin_{p,q}(\pi_{p,q}-x), \quad x\in[\pi_{p,q}/2,\pi_{p,q}].$$

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By oddness, the further extension can be made to $[-\pi_{p,q}, \pi_{p,q}]$. Finally, the functions $\sin_{p,q}$ is extended to whole $\mathbb{R}$ by $2\pi_{p,q}$-periodicity, see [17]. Naturally, we also define $\cos_{p,q}$ by $\cos_{p,q}(x) = \frac{d}{dx}(\sin_{p,q}(x))$. For $p = q$, these functions reduce to the so-called p-trigonometric functions introduced by Lindqvist in his highly cited paper [23]. $\pi_{p,q}$ reduces to π_p, see [4]. In case $(p, q) = (2, 2)$, it is obvious that $\sin_{p,q}, \cos_{p,q}, \pi_{p,q}$ are reduced to the $\sin, \cos, \pi$, respectively. This is a reason why these functions and the constant are called generalized trigonometric functions (with parameter (p, q)) and the generalized π, respectively.

In present, there has been a vivid interest on the generalized trigonometric and hyperbolic functions and the generalized complete elliptic integrals defined by generalized trigonometric functions. Numerous papers have been published on the studies of generalized trigonometric functions, the generalized complete elliptic integrals and their inequalities. The reader may see references [3, 5, 6, 7, 8, 9, 13, 14, 15, 17, 18, 19, 21, 22, 28, 29, 30, 31, 32, 34, 35, 37, 38, 39, 40, 41, 44, 45].

Recently, the arc lemniscate sine function and the hyperbolic arc lemniscate sine function defined by

$$\operatorname{arcsl} = \int_0^x \left(1 - t^4\right)^{-1/2} dt\,, 0 < |x| < 1, \tag{1.3}$$

and

$$\operatorname{arcslh} = \int_0^x \left(1 + t^4\right)^{-1/2} dt\,, x \in \mathbb{R}, \tag{1.4}$$

are deeply investigated. The arc lemniscate sine function arcsl shows the arc length of the lemniscate $r^2 = \cos 2\theta$ from the origin to the point with radial position x. In fact, the arc lemniscate sine function and the hyperbolic arc lemniscate sine function are the generalized $(2, 4)$-trigonometric sine and $(2, 4)$-hyperbolic sine functions, respectively. In [25, 26], Neumann defined the arc lemniscate tangent function and the hyperbolic arc lemniscate tangent function in terms of the arc lemniscate sine function and the hyperbolic arc lemniscate sine function, respectively, as follows:

$$\operatorname{arctl} = \operatorname{arcsl}\left(\frac{x}{\left(1 + t^4\right)^{1/4}}\right), x \in \mathbb{R}, \tag{1.5}$$

and

$$\operatorname{arctlh} = \operatorname{arcslh}\left(\frac{x}{\left(1 - t^4\right)^{1/4}}\right), 0 < |x| < 1. \tag{1.6}$$

In [10], Chen established several Wilker and Huygens type inequalities by using power series expansions of the lemniscate functions. Next, Chen [11] obtained some new Wilker and Huygens type inequalities for the lemniscate functions. Deng and Chen[12] established some Shafer-Fink type inequalities for Gauss lemniscate functions. Very recently, Wei, He and Wang[36] discussed the monotonicity and inequalities for some functions involving the arc lemniscate and the hyperbolic arc lemniscate functions. In particular, they established sharp Shafer-Fink type inequalities for the arc lemniscate and the hyperbolic arc lemniscate functions. In [24], Mahmoud and Agarwal showed some new bounds which be better than the bounds of a conjecture about the function arctl posed by Sun and Chen [27]. In particular, they presented new bounds for arcsl in terms of the Lerch zeta function. Very quickly, Alzer and Richard improved the elegant result by the monotone form of l'Hôpital rule.

In [42, 43], Takeuchi established the double-angle formulas related to the inverse functions of $\sin_{2,6}^{-1}$ and $\sinh_{2,6}^{-1}$ defined by

$$\sin_{2,6}^{-1}(x) = \int_0^x \left(1-t^6\right)^{-1/2} dt\,, 0<|x|<1, \tag{1.7}$$

and

$$\sinh_{2,6}^{-1} = \int_0^x \left(1+t^6\right)^{-1/2} dt\,, x\in\mathbb{R}. \tag{1.8}$$

This function in (1.7) appeared in Ramanujan's Notebooks[2] and is regarded as a generalized version of the lemniscate function. In this paper, we mainly studied bound estimation of the generalized inverse lemniscate functions by using Lerch Phi function. Next, some Shafer-Fink, Wilker and Huygens type inequalities for the generalized inverse lemniscate and the generalized hyperbolic inverse lemniscate functions were proved.

2. Bound of the generalized inverse Lemniscate function

Lemma 2.1. [1, Theorem 1.1] *Let $-\infty<a<b<+\infty$, and let $f,g:[a,b]\to\mathbb{R}$ be continuous functions that are differentiable on (a,b) with $f(a)=g(a)=0$ or $f(b)=g(b)=0$. Assume that $g'(x)\neq 0$ for each $x\in(a,b)$. If f'/g' is increasing (decreasing) on (a,b), then so is f/g.*

Lemma 2.2. [20, p.119] *Let $-1\le\lambda\le\frac{1}{4}$. Then*

$$\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n+\lambda)}}. \tag{2.1}$$

If $\lambda\ge\frac{n+1}{4n+3}$, the inequality (2.1) is reversed.

Theorem 2.1. *For all $x\in(0,1)$, we have*

$$\alpha x\Phi(x^6,3/2,1/6) < \sin_{2,6}^{-1}(x) < \beta x\Phi(x^6,3/2,1/6) \tag{2.2}$$

with the best possible constants $\alpha=\frac{1}{6\sqrt{6}}$ and $\beta=\frac{B(1/6,1/2)}{6\zeta(3/2,1/6)}$ where

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

$$\Phi(z,s,\alpha)=\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s}, \alpha\neq 0,-1\cdots, |z|<1,$$

and

$$\zeta\left(s,\alpha\right)=\Phi(1,s,\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s}$$

are classical beta function, Lerch Phi function and Hurwitz zeta function, respectively.

Proof. Let $F(x)=\frac{F_{11}(x)}{F_{12}(x)}=\frac{\sin_{2,6}^{-1}(x)}{x\Phi(x^6,3/2,1/6)}$, where $F_{11}(x)=\sin_{2,6}^{-1}(x)$ and $F_{12}(x)=x\Phi(x^6,3/2,1/6)$. Then $F_{11}(0^+)=F_{12}(0^+)=0$. Simple computation yields $F_{11}'(x)=\frac{1}{\sqrt{1-x^6}}$ and $F_{12}'(x)=6\sqrt{6}\sum\limits_{n=0}^{\infty}\frac{x^{6n}}{\sqrt{6n+1}}$. So, we obtain

$$\frac{F_{11}'(x)}{F_{12}'(x)}=\frac{1}{6\sqrt{6}h\left(x^6\right)}$$

where

$$h(t)=\sqrt{1-t}\sum_{n=0}^{\infty}\frac{t^n}{\sqrt{6n+1}}.$$

By differentiation, we get

$$\begin{aligned}2\sqrt{1-t}h'(t)&=2(1-t)\sum_{n=1}^{\infty}\frac{nt^{n-1}}{\sqrt{6n+1}}-\sum_{n=0}^{\infty}\frac{t^n}{\sqrt{6n+1}}\\&=\sum_{n=0}^{\infty}\left(\frac{2n+2}{\sqrt{6n+7}}-\frac{2n+1}{\sqrt{6n+1}}\right)t^n.\end{aligned}$$

Since

$$\frac{2n+2}{\sqrt{6n+7}}-\frac{2n+1}{\sqrt{6n+1}}<0\Leftrightarrow 2n+3>0,$$

we have $h'(t)<0$. This implies that $\frac{F_{11}'(x)}{F_{12}'(x)}$ is strictly increasing on $(0,1)$, so that the Lemma 2.1 reveals that $F(x)$ is strictly increasing on $(0,1)$. It follows that $F(0)<F(x)<F(1)$. The limit values

$$\begin{aligned}F(0^+)&=\frac{1}{6\sqrt{6}}\lim_{x\to0}\frac{\sin_{2,6}^{-1}(x)}{x}=\frac{1}{6\sqrt{6}},\\F(1^-)&=\frac{\sin_{2,6}^{-1}(1)}{\Phi(1,3/2,1/6)}=\frac{B(1/6,1/2)}{6\zeta(3/2,1/6)}.\end{aligned}$$

is clear by using l'Hôspital rule. □

Remark 2.1. By using Lemma 2.2, we may give a new proof of left side of inequality (2.2). Using the identities

$$\frac{1}{\sqrt{1-x^6}}=1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^{6n} \tag{2.3}$$

and

$$\frac{d}{dz}\Phi(z^k,s,\alpha)=kz^{k-1}\Phi'(z^k,s,\alpha),$$

we get

$$\begin{aligned}H'(x)&=\frac{\sqrt{6}}{2}+\sum_{n=1}^{\infty}\frac{1}{2\sqrt{1/6+n}}x^{6n}-\frac{\sqrt{6}}{2}\frac{1}{\sqrt{1-x^6}}\\&=\frac{\sqrt{6}}{2}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{6n+1}}-\frac{(2n-1)!!}{(2n)!!}\right)x^{6n}\end{aligned}$$

where

$$H(x)=\frac{x}{12}\Phi(x^6,3/2,1/6)-\frac{\sqrt{6}}{2}\sin_{2,6}^{-1}(x).$$

Considering to $\frac{(n+1)\pi}{4n+3}<1$, we easily obtain

$$\frac{1}{\sqrt{6n+1}}<\frac{1}{\sqrt{\pi n+\pi(n+1)/(4n+3)}}<\frac{(2n-1)!!}{(2n)!!}$$

by using Lemma 2.2. This implies that $H'(x)<0$ and the function $H(x)$ is strictly decreasing on $(0,1)$. Then $H(x)<H(0)=0$. The proof is completed.

It is worth noting that we can obtain an upper bound of $\sin_{2,6}^{-1}(x)$ by using Lemma 2.2.

Theorem 2.2. *For all $x \in (0,1)$, we have*

$$\sin_{2,6}^{-1}(x) < \frac{1}{6} x \Phi(x^6, 3/2, 1/6). \tag{2.4}$$

Proof. Completely similar to Remark 2.1, we get

$$\begin{aligned} G'(x) &= \tfrac{1}{6}\Phi(x^6,3/2,1/6) + x^6\Phi'(x^6,3/2,1/6) - \tfrac{1}{\sqrt{1-x^6}} \\ &= \sqrt{6} - 1 + \sum_{n=1}^{\infty}\left(\tfrac{1}{\sqrt{1/6+n}} - \tfrac{(2n-1)!!}{(2n)!!}\right)x^{6n} \end{aligned}$$

where

$$G(x) = \frac{x}{6}\Phi(x^6,3/2,1/6) - \sin_{2,6}^{-1}(x).$$

Lemma 2.2 may easily yield

$$\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(1/6+n)}} < \frac{1}{\sqrt{1/6+n}}.$$

This implies that $G'(x) > 0$ and the function $G(x)$ is strictly increasing on $(0,1)$. Then $G(x) > G(0) = 0$. This completes the proof. □

3. Shafer-Fink type inequalities

Lemma 3.1. *(i) The function $h_1(x) = \frac{\sin_{2,6}^{-1}(x)}{x}$ is strictly increasing on $(0,1)$ with range $(1, \pi_{2,6}/2)$ where $\pi_{2,6} = \frac{1}{3}B\left(\frac{1}{2},\frac{1}{6}\right) = \frac{\sqrt{\pi}}{3}\frac{\Gamma(1/6)}{\Gamma(2/3)}$;*

(ii) The function $h_2(x) = \frac{\sinh_{2,6}^{-1}(x)}{x}$ is strictly decreasing on $(0,\infty)$ with range $(0,1)$.

Proof. By using Lemma 2.1, we easily complete the proof. Here, we omit the details for the sake of simplicity. □

Lemma 3.2. *(i) The function $g_1(x) = \frac{x - \sqrt[6]{1-x^6}\sin_{2,6}^{-1}(x)}{\sin_{2,6}^{-1}(x) - x}$ is strictly increasing on $(0,1)$ with range $\left(\frac{4}{3}, \frac{2}{\pi_{2,6}-2}\right)$;*

(ii) The function $g_2(x) = \frac{\sqrt[6]{1+x^6}\sinh_{2,6}^{-1}(x) - x}{x - \sinh_{2,6}^{-1}(x)}$ is strictly decreasing on $(0,\infty)$ with range $\left(\frac{\pi_{3/2,6}}{2} - 1, \frac{4}{3}\right)$ where $\pi_{3/2,6} = \frac{1}{3}B\left(\frac{1}{3},\frac{1}{6}\right) = \frac{\Gamma(1/3)\Gamma(1/6)}{3\sqrt{\pi}}$.

Proof. (i) Let $g_1(x) = \frac{g_{11}(x)}{g_{12}(x)}$ where $g_{11}(x) = x - \sqrt[6]{1-x^6}\sin_{2,6}^{-1}(x)$ and $g_{12}(x) = \sin_{2,6}^{-1}(x) - x$. Then $g_{11}(0^+) = g_{12}(0^+) = 0$. Direct computation results in

$$\frac{g_{11}'(x)}{g_{12}'(x)} = \frac{1 + x^5(1-x^6)^{-5/6}\sin_{2,6}^{-1}(x) - (1-x^6)^{-1/3}}{(1-x^6)^{-1/2} - 1}$$

with $g_{11}'(0^+) = g_{12}'(0^+) = 0$. Computing again, we get

$$\frac{g_{11}''(x)}{g_{12}''(x)} = \frac{5}{3}\frac{\sin_{2,6}^{-1}(x)}{x}\frac{1}{(1-x^6)^{1/3}} - \frac{1}{3}(1-x^6)^{1/6}$$

which is strictly increasing by Lemma 3.1. Hence $g_1(x)$ is strictly increasing by Lemma 2.1. The limiting values

$$g_1(0^+) = \lim_{x\to 0}\frac{g_{11}''(x)}{g_{12}''(x)} = \frac{4}{3}$$

and

$$g_1(1^-) = \frac{1}{\sin_{2,6}^{-1}(1) - 1} = \frac{2}{\pi_{2,6} - 2}$$

are obvious.

(ii) Let us write $g_2(x) = \frac{g_{21}(x)}{g_{22}(x)}$ where $g_{21}(x) = \sqrt[6]{1+x^6}\sinh_{2,6}^{-1}(x) - x$ and $g_{22}(x) = x - \sinh_{2,6}^{-1}(x)$ with $g_{21}(0^+) = g_{22}(0^+) = 0$.

Direct computation results in

$$\frac{g'_{21}(x)}{g'_{22}(x)} = \frac{x^5(1+x^6)^{-5/6}\sinh_{2,6}^{-1}(x) + (1+x^6)^{-1/3} - 1}{1 - (1+x^6)^{-1/2}}$$

and

$$\frac{g''_{21}(x)}{g''_{22}(x)} = \frac{5}{3}\frac{\sinh_{2,6}^{-1}(x)}{x}\frac{1}{(1+x^6)^{1/3}} - \frac{1}{3}(1+x^6)^{1/6}$$

with $g'_{21}(0^+) = g'_{22}(0^+) = 0$ which is strictly decreasing by Lemma 3.1. Hence $g_2(x)$ is strictly decreasing by Lemma 2.1. The limiting values read as follows

$$g_2(0^+) = \lim_{x\to 0}\frac{g''_{21}(x)}{g''_{22}(x)} = \frac{4}{3}$$

and

$$\begin{aligned} g_2(\infty) &= \sinh_{2,6}^{-1}(\infty) - 1 = \int_0^\infty (1+t^6)^{-1/2}\,dt \, - 1 \\ &= \int_0^1 (1-s^6)^{-2/3}\,ds \, - 1 = \tfrac{\pi_{3/2,6}}{2} - 1, \end{aligned}$$

where we apply the substitution $1 + t^6 = \frac{1}{1-s^6}$. The proof is complete. □

Theorem 3.1. *The following inequalities hold true:*

(i) $\frac{\pi_{2,6}}{2+(\pi_{2,6}-2)\sqrt[6]{1-x^6}} < \frac{\sin_{2,6}^{-1}(x)}{x} < \frac{7}{4+3\sqrt[6]{1-x^6}}, 0 < |x| < 1;$

(ii) $\frac{\pi_{3/2,6}}{(\pi_{3/2,6}-2)+2\sqrt[6]{1+x^6}} < \frac{\sinh_{2,6}^{-1}(x)}{x} < \frac{7}{4+3\sqrt[6]{1+x^6}}, |x| > 0.$

Proof. Applying Lemma (3.2), we easily complete the proof of Theorem 3.1. □

4. Wilker and Huygens type inequalities

It is known that the function $\frac{1}{\sqrt{1+x^6}}$ can be expressed in series form as follows:

$$\frac{1}{\sqrt{1+x^6}} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}x^{6n}. \tag{4.1}$$

Applying expansions of the power series (2.3) and (4.1), we easily obtain expansions of the power series of the functions $\sin_{2,6}^{-1}(x)$ and $\sinh_{2,6}^{-1}(x)$ as follows:

$$\sin_{2,6}^{-1}(x) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{6n+1}\frac{x^{6n+1}}{n!}, \tag{4.2}$$

$$\sinh_{2,6}^{-1}(x) = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+1/2)}{6n+1}\frac{x^{6n+1}}{n!}. \tag{4.3}$$

It is worth noting that the functions $\sin_{2,6}^{-1}(x)$ and $\sinh_{2,6}^{-1}(x)$ can be expressed in Gaussian hypergeometric function as follows:

$$\sin_{2,6}^{-1}(x) = xF\left(\frac{1}{2}, \frac{1}{6}; \frac{7}{6}; x^6\right),$$

$$\sinh_{2,6}^{-1}(x) = xF\left(\frac{1}{2}, \frac{1}{6}; \frac{7}{6}; -x^6\right).$$

Further results can refer to the reference [4]. Here, we give a new expansion of series for $\sin_{2,6}^{-1}(x)$.

Lemma 4.1. [46, Lemma 2.1] *Suppose that the function G given by $G(x) = \frac{g(x)}{(1-\alpha x^\eta)^\xi}$ satisfies $g, G \in L^1[0,1]$ where $0 < \alpha \leq 1, \eta > 0, \lambda < \frac{1}{2}, \xi \in \mathbb{R}, x \in [0,1]$, and*

$$b_j = b_j(\alpha, \eta, x) = \alpha^j \int_0^x t^{j\eta} g(t) dt,$$

it follows that

$$\int_0^x \frac{g(x)}{(1-\alpha x^\eta)^\xi} dx = \sum_{n=0}^\infty \frac{(\xi)_n}{n!(1-\lambda)^{n+\xi}} \sum_{j=0}^n \binom{n}{j} (-\lambda)^{n-j} b_j(\alpha, \eta, x). \tag{4.4}$$

Putting $\alpha = 1, \eta = 6, \xi = \frac{1}{2}, g(x) = 1$ in the Lemma 4.1, we have

$$\sin_{2,6}^{-1}(x) = \sum_{n=0}^\infty \frac{(1/2)_n}{n!(1-\lambda)^{n+\frac{1}{2}}} \sum_{j=0}^n \binom{n}{j} (-\lambda)^{n-j} \frac{x^{6j+1}}{6j+1}. \tag{4.5}$$

If $\lambda = 0$ in formula (4.5), we can get formula (4.2) by using formula $(1/2)_n = \frac{\Gamma(n+1/2)}{\sqrt{\pi}}$.

As an analogy to arc lemniscate functions, the generalized inverse lemniscate tangent $\tan_{2,6}^{-1}$ and the generalized hyperbolic inverse lemniscate tangent $\tanh_{2,6}^{-1}$ have been defined as follows:

$$\begin{aligned} \tan_{2,6}^{-1}(x) &= \sin_{2,6}^{-1}\left(\tfrac{x}{\sqrt[6]{1+x^6}}\right), \\ \tanh_{2,6}^{-1}(x) &= \sinh_{2,6}^{-1}\left(\tfrac{x}{\sqrt[6]{1-x^6}}\right). \end{aligned} \tag{4.6}$$

Lemma 4.2. *Let $p \geq 0$ be an integer. Then for $0 < x < 1$*

$$\sum_{k=0}^{2p-1} (-1)^k u_k(x) \leq \tan_{2,6}^{-1}(x) \leq \sum_{k=0}^{2p} (-1)^k u_k(x) \tag{4.7}$$

where

$$u_k(x) = \frac{\Gamma(k+2/3)}{\Gamma(2/3)\, k!} \frac{x^{6k+1}}{6k+1}.$$

Proof. Elementary calculations reveal that for $0 < x < 1$,

$$\begin{aligned} &\tfrac{d}{dx}\left(\tan_{2,6}^{-1}(x)\right) = \tfrac{d}{dx} \int_0^{\frac{x}{\sqrt[6]{1+x^6}}} \tfrac{1}{\sqrt{1-x^6}} dx = \left(1+x^6\right)^{-\frac{2}{3}} \\ &= \sum_{n=0}^\infty \binom{-\frac{2}{3}}{n} x^{6n} = \sum_{n=0}^\infty (-1)^n \tfrac{\Gamma(n+2/3)}{\Gamma(2/3) n!} x^{6n}. \end{aligned}$$

So, we get

$$\tan_{2,6}^{-1}(x)=\sum_{n=0}^{\infty}(-1)^n u_n(x)=\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma\left(n+2/3\right)}{\Gamma\left(2/3\right)n!}\frac{x^{6n+1}}{6n+1}.$$

Using the property of alternating series, we easily complete the proof. □

Lemma 4.3. *Let $p\geq 0$ be an integer. Then for $0<x<1$*

$$\sum_{k=0}^{2p-1}(-1)^k v_k(x)\leq \sinh_{2,6}^{-1}(x)\leq \sum_{k=0}^{2p}(-1)^k v_k(x) \tag{4.8}$$

where

$$v_k(x)=\frac{\Gamma\left(k+1/2\right)}{\sqrt{\pi}k!}\frac{x^{6k+1}}{6k+1}.$$

Proof. Using formula (4.3) and the property of alternating series, we easily complete the proof. □

Lemma 4.4. *For $0<x<1$, we have*

$$\tanh_{2,6}^{-1}(x)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+2/3\right)}{\Gamma\left(2/3\right)n!}\frac{x^{6n+1}}{6n+1}. \tag{4.9}$$

Proof. Direct calculation yields

$$\begin{aligned}&\tfrac{d}{dx}\left(\tanh_{2,6}^{-1}(x)\right)=\tfrac{d}{dx}\int_0^{\frac{x}{\sqrt[6]{1-x^6}}}\tfrac{1}{\sqrt{1+x^6}}dx\\&=\left(1-x^6\right)^{-\frac{2}{3}}=\sum_{n=0}^{\infty}\tfrac{\Gamma(n+2/3)}{\Gamma(2/3)n!}x^{6n}\end{aligned}$$

by using (4.6). Completely similar to Lemma 4.2, we complete the proof. □

Theorem 4.1. *For $0<|x|<1$, we have*

$$\left(\frac{x}{\sin_{2,6}^{-1}(x)}\right)^2+\frac{x}{\tan_{2,6}^{-1}(x)}<2. \tag{4.10}$$

Proof. For $0<|x|<1$, we get

$$\left(\frac{\sin_{2,6}^{-1}(x)}{x}\right)^2=\left(1+\frac{x^6}{14}+\frac{3x^{12}}{104}+\cdots\right)^2>1+\frac{x^6}{7}+\frac{40x^{12}}{637}$$

and

$$1-\frac{2x^6}{21}<\frac{\tan_{2,6}^{-1}(x)}{x}<1$$

by using formulas (4.2) and (4.7). So, we find for $0<|x|<1$

$$\begin{aligned}&\left(\tfrac{x}{\sin_{2,6}^{-1}(x)}\right)^2+\tfrac{x}{\tan_{2,6}^{-1}(x)}-2<\tfrac{1}{1+\frac{x^6}{7}+\frac{40x^{12}}{637}}+\tfrac{1}{1-\frac{2x^6}{21}}-2\\&=\tfrac{x^6\left(-\frac{1}{21}-\frac{3332x^6}{93639}+\frac{160x^{12}}{13377}\right)}{\left(1+\frac{x^6}{7}+\frac{40x^{12}}{637}\right)\left(1-\frac{2x^6}{21}\right)}<0.\end{aligned}$$

□

Theorem 4.2. *For $0 < |x| < 1$, we have*

$$\left(\frac{\sin_{2,6}^{-1}(x)}{x}\right)^2 + \frac{\tan_{2,6}^{-1}(x)}{x} > 2 \tag{4.11}$$

and

$$\frac{2\sin_{2,6}^{-1}(x)}{x} + \frac{\tan_{2,6}^{-1}(x)}{x} > 3. \tag{4.12}$$

Proof. Inequality (4.10) can be rewritten as

$$\frac{2}{\frac{1}{\left(\frac{\sin_{2,6}^{-1}(x)}{x}\right)^2} + \frac{1}{\frac{\tan_{2,6}^{-1}(x)}{x}}} > 1.$$

By using the arithmetic-geometric-harmonic mean inequality, we get

$$\begin{aligned}\frac{\left(\frac{\sin_{2,6}^{-1}(x)}{x}\right)^2 + \frac{\tan_{2,6}^{-1}(x)}{x}}{2} &\geq \sqrt{\left(\frac{\sin_{2,6}^{-1}(x)}{x}\right)^2 \frac{\tan_{2,6}^{-1}(x)}{x}} \\ &\geq \frac{2}{\frac{1}{\left(\frac{\sin_{2,6}^{-1}(x)}{x}\right)^2} + \frac{1}{\frac{\tan_{2,6}^{-1}(x)}{x}}} > 1\end{aligned}$$

and

$$\frac{2\sin_{2,6}^{-1}(x)}{x} + \frac{\tan_{2,6}^{-1}(x)}{x} \geq 3\sqrt[3]{\left(\frac{\sin_{2,6}^{-1}(x)}{x}\right)^2 \frac{\tan_{2,6}^{-1}(x)}{x}} > 3$$

for $0 < |x| < 1$. □

Theorem 4.3. *For $0 < |x| < 1$, we have*

$$\frac{x}{\sinh_{2,6}^{-1}(x)} + \left(\frac{x}{\tanh_{2,6}^{-1}(x)}\right)^2 < 2. \tag{4.13}$$

Proof. For $0 < |x| < 1$, we get

$$\left(\frac{\tanh_{2,6}^{-1}(x)}{x}\right)^2 = \left(1 + \frac{2x^6}{21} + \frac{20x^{12}}{234} + \cdots\right)^2 > 1 + \frac{4x^6}{21} \tag{4.14}$$

and

$$1 - \frac{x^6}{14} < \frac{\sinh_{2,6}^{-1}(x)}{x} < 1 - \frac{x^6}{14} + \frac{3x^{12}}{104} \tag{4.15}$$

by using formulas (4.8) and (4.9). So, we find for $0 < |x| < 1$,

$$\begin{aligned}&\frac{x}{\sinh_{2,6}^{-1}(x)} + \left(\frac{x}{\tanh_{2,6}^{-1}(x)}\right)^2 - 2 \\ &< \frac{1}{1 - \frac{x^6}{14}} + \frac{1}{1 + \frac{4x^6}{21}} - 2 \\ &= \frac{-x^6\left(\frac{5}{42} - \frac{4x^6}{147}\right)}{\left(1 - \frac{x^6}{14}\right)\left(1 + \frac{4x^6}{21}\right)} < 0.\end{aligned}$$

□

Theorem 4.4. *For $0 < |x| < 1$, we have*

$$\frac{\sinh_{2,6}^{-1}(x)}{x} + \left(\frac{\tanh_{2,6}^{-1}(x)}{x}\right)^2 > 2 \tag{4.16}$$

and

$$\frac{\sinh_{2,6}^{-1}(x)}{x} + \frac{2\tanh_{2,6}^{-1}(x)}{x} > 3. \tag{4.17}$$

Proof. Inequality (4.13) can be rewritten as

$$\frac{2}{\frac{1}{\left(\frac{\sin_{2,6}^{-1}(x)}{x}\right)} + \frac{1}{\left(\frac{\tan_{2,6}^{-1}(x)}{x}\right)^2}} > 1.$$

By using the arithmetic-geometric-harmonic mean inequality again, we get

$$\frac{\sinh_{2,6}^{-1}(x)}{x} + \left(\frac{\tanh_{2,6}^{-1}(x)}{x}\right)^2 \geq \frac{2}{\frac{1}{\left(\frac{\sinh_{2,6}^{-1}(x)}{x}\right)} + \frac{1}{\left(\frac{\tanh_{2,6}^{-1}(x)}{x}\right)^2}} > 2$$

and

$$\frac{\sinh_{2,6}^{-1}(x)}{x} + \frac{2\tanh_{2,6}^{-1}(x)}{x} \geq 3\sqrt[3]{\frac{\sinh_{2,6}^{-1}(x)}{x}\left(\frac{\tanh_{2,6}^{-1}(x)}{x}\right)^2} > 3.$$

□

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References

[1] G. D. Anderson, M. Vuorinen and X.-H. Zhang: *Topics in special functions III.* Analytic Number Theory, Approximation Theory, and Special Functions, 2014 `http://arXiv:1305.0938v1[math.CA]` 3

[2] B.C. Berndt: *Ramanujan's notebooks.* Part IV, Springer-Verlag, New York, 1994. 3

[3] Á. Baricz, B. A. Bhayo, T. K. Pogány: *Functional inequalities for generalized inverse trigonometric and hyperbolic functions.* J. Math. Anal. Appl., **417** (2014), 244-259. `http://arxiv.org/abs/1401.4863` 2

[4] Á. Baricz, B. A. Bhayo, M. Vuorinen: *Turán type inequalities for generalized inverse trigonometric functions.* Filomat, **29**, No. 2 (2015), 303-313, `http://arxiv.org/abs/1209.1696`. 2, 7

[5] Á. Baricz, B. A. Bhayo, R. Klén: *Convexity properties of generalized trigonometric and hyperbolic functions.* Aequat. Math., **89** (2015), 473-484. `http://arxiv.org/abs/1301.0699` 2

[6] B. A. Bhayo, M. Vuorinen: *On generalized trigonometric functions with two parameters.* J. Approx. Theory, **164** (2012), 1415-1426. `http://arxiv.org/abs/1112.0483` 2

[7] B.A. Bhayo, M. Vuorinen: *Power mean inequalities generalized trigonometric functions.* Math. Vesnik, **67**, No. 1 (2015), 17-25. `http://arxiv.org/abs/1209.0983` 2

[8] B. A. Bhayo and L. Yin: *Logarithmic mean inequality for generalized trigonometric and hyperbolic functions.* Acta. Univ. Sapientiae Math., **6**, No. 2 (2014), 135-145. `http://arxiv.org/abs/1404.6732` 2

[9] B. A. Bhayo and L. Yin: *On the conjecture of generalized trigonometric and hyperbolic functions.* Math. Pannon., Vol. 24, No. 2 (2013), 1-8. http://arxiv.org/abs/1402.7331 2
[10] C.-P. Chen: *Wilker and Huygens type inequalities for the lemniscate functions.* J. Math. Inequal., **6**, No. 4 (2012), 673-684. 2
[11] C.-P. Chen: *Wilker and Huygens type inequalities for the lemniscate functions* Π. Math. Inequal. Appl., **16**, No. 2 (2013), 577-586. 2
[12] J.-E. Deng and C.-P. Chen: *Sharp Shafer-Fink type inequalities for Gauss lemniscate functions.* J. Inequal. Appl., (2014), 35. 2
[13] Y.-M. Chu, M.-K. Wang, S.-L. Qiu and Y.-P. Jiang: *Bounds for complete elliptic integrals of the second kind with applications.* Comput. Math. Appl., 2012, **63**(7), 1177-1184. 2
[14] Y.-M. Chu, Y.-F. Qiu and M.-K. Wang: *Hölder mean inequalities for the complete elliptic integrals.* Integral Transforms Spec. Funct., 2012, **23**(7), 521-527. 2
[15] Y.-M. Chu, M.-K. Wang, Y.-P. Jiang and S.-L. Qiu: *Concavity of the complete elliptic integrals of the second kind with respect to Hölder means.* J. Math. Anal. Appl., 2012, **395**(2), 637-642. 2
[16] P. Drábek, R. Manásevich: *On the closed solution to some p-Laplacian nonhomogeneous eigenvalue problems.* Diff. and Int. Eqns., **12** (1999), 723-740. 1
[17] D. E. Edmunds, P. Gurka, J. Lang: *Properties of generalized trigonometric functions.* J. Approx. Theory, **164** (2012), 47-56. 2
[18] W. -D. Jiang, M. -K. Wang, Y. -M. Chu, Y. -P. Jiang, F. Qi: *Convexity of the generalized sine function and the generalized hyperbolic sine function.* J. Approx. Theory, **174** (2013), 1-9. 2
[19] T. Kamiya and S. Takeuchi: *Complete (p, q)-elliptic integrals with application to a family of means.* J. Classical Anal.,, **10**, No. 1:15-25 http://arxiv.org/abs/1507.01383 2
[20] J. -C. Kuang: *Applied inequalities(Second edition).* Shan Dong Science and Technology Press. Jinan, 2002. 3
[21] R. Klén, M. Vuorinen, X.-H. Zhang: *Inequalities for the generalized trigonometric and hyperbolic functions.* J. Math. Anal. Appl., **409** (2014), 521-529. http://arxiv.org/abs/1210.6749 2
[22] J. Lang and D. E. Edmunds: *Eigenvlues, Embeddings and generalized trigonometric functions.* Lecture Notes in Mathematics, 2016, Springer, 2011. 2
[23] P. Lindqvist: *Some remarkable sine and cosine functions.* Ricerche di Math., **XLIV** (1995), 269-290. 2
[24] M. Mahmoud and R. P. Agarwal: *On some bounds of Gauss arc lemniscate sine and tangent functions.* Ricerche di Math., **8** No:5 (2017), 46-58. 2
[25] E. Neuman: *On Gauss lemniscate functions and lemniscatic mean.* Math. Pannon., **18** (2007), 77-94. 2
[26] E. Neuman: *On Gauss lemniscate functions and lemniscatic mean* Π. Math. Pannon., **23** (2012), 65-73. 2
[27] J.-L. Sun, C.-P. Chen: *Shafer-type inequalities for inverse trigonometric functions and Gauss lemniscate functions.* J. Inequal. Appl., **2016** (2016), 212. 2
[28] M.-K. Wang, Z.-Y. He and Y.-M. Chu: *Sharp Power Mean Inequalities for the Generalized Elliptic Integral of the First Kind.* Comput. Methods Funct. Theory, 2020, 20(1), 111-124. 2
[29] M.-K. Wang, H.-H. Chu and Y.-M. Chu: *Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals.* J. Math. Anal. Appl., 2019, 480(2), Article ID 123388, 9 pages 2
[30] M.-K. Wang, Y.-M. Chu and W. Zhang: *Precise estimates for the solution of Ramanujan's generalized modular equation.* Ramanujan J., 2019,49(3), 653-668. 2
[31] M.-K. Wang, Y.-M. Chu and W. Zhang: *Monotonicity and inequalities involving zero-balanced hypergeometric function.* Math. Inequal. Appl.,2019, 22(2), 601-617. 2
[32] S.-L. Qiu, X.-Y. Ma and Y.-M. Chu: *Sharp Landen transformation inequalities for hypergeometric functions with applications.* J. Math. Anal. Appl., 2019, 474(2), 1306-1337. 2
[33] M.-K. Wang, W. Zhang and Y.-M. Chu: *Monotonicity, convexity and inequalities involving the generalized elliptic integrals.* Acta Math. Sci., 2019, 39B(5), 1440-1450 2
[34] M.-K. Wang, Y.-M. Li and Y.-M. Chu: *Inequalities and infinite product formula for Ramanujan generalized modular equation function.* Ramanujan J., 2018, 46(1), 189-200. 2

[35] G.-D. Wang, X.-H. Zhang and Y.-M. Chu: *A power mean inequality involving the complete elliptic integrals.* Rocky Mountain J. Math., 2014,44(5), 1661-1667. 2
[36] M. -J. Wei, Y. He and G. -D. Wang: *Shafer-Fink type inequalities for arc lemniscate functions.* RACSAM, **114** (2020), 53. 2
[37] S. Takeuchi: *Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian.* J. Math. Anal. Appl., **385** (2012), 24-35. 1, 2
[38] S. Takeuchi: *The complete p-elliptic integrals and a computation formula of π_p for $p = 4$.* Ramanujan J., 2018, 46 (2) :309-321 http://arxiv.org/abs/1503.02394 2
[39] S. Takeuchi: *A new form of the generalized complete elliptic integrals.* Kodai J. Math., **39**, No. 1 (2016), 202-226. http://arxiv.org/abs/1411.4778 2
[40] S. Takeuchi: *Multiple-angle formulas of generalized trigonometric functions with two parameters.* http://arxiv.org/abs/1603.06709 2
[41] S. Takeuchi: *Legendre-type relations for generalized complete elliptic integrals.* Journal of Classical Analysis, **9**, No. 1 (2016), 35-42. http://arxiv.org/abs/1606.05115 2
[42] S. Takeuchi: *Some double-angle formulas related to a generalized lemniscate function.* http://arxiv.org/abs/1912.11483 3
[43] S. Takeuchi: *Applications of a duality between generalized trigonometric and hyperbolic functions.* http://arxiv.org/abs/2011.06834 3
[44] L. Yin and L. G. Huang: *Inequalities for generalized trigonometric and hyperbolic functions with two parameters.* J. Nonlinear Sci. Appl., **8**, No. 4 (2015), 315-323. 2
[45] L. Yin, L. -G. Huang, Y. -L. Wang and X. -L. Lin: *A survey for generalized trigonometric and hyperbolic functions.* J. Math. Inequal., **13**(3), 2019, 833-854. 2
[46] L. Yin, W. -Y. Cui and X. -K. Dou: *Series representations for some special functions.* Proceeding of the Jangjeon Mathematical Society, **19**(1), 2016, 101-105. 7

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