On fractional inequalities using generalized proportional Hadamard fractional integral operators

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Abstract

In this paper, we use generalized proportional Hadamard fractional integral operators to establish some new fractional integral inequalities for extend Chebhychev functional. In addition, we investigate Pólya-Szegö-type fractional integral inequalities by employing generalized proportional Hadamard fractional integral operators.

Keywords : Pólya-Szegö-type inequality, extend Chebhychev functional, fractional integral inequality, generalised proportional Hadamard fractional integral operator.
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1 Introduction

Fractional calculus is the study of generalization of traditional calculus into non-integer differential and integral order. Fractional derivative and integrals have become more popular and helpful due to it's various application in field of science and technology. Fractional integral inequalities play accomplish a vital role in obtaining uniqueness of solution of fractional ordinary differential equations, fractional partial differential equations and fractional boundary value problems.
In [7], the Chebyshev for two integrable functions f and g on $[a, b]$ is defined as

$$T(u,v) = \frac{1}{b-a}\int_a^b u(x)v(x)dx - \frac{1}{b-a}\left(\int_a^b u(x)dx\right)\frac{1}{b-a}\left(\int_a^b v(x)dx\right). \tag{1.1}$$

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Many application and several inequalities related to Chebyshev functional are found in [2, 6, 20, 21]. Let us consider the extended Chebyshev functional [17]

$$\begin{aligned} T(u,v,p,q) = & \int_a^b q(x)dx \int_a^b p(x)u(x)v(x)dx + \int_a^b p(x)dx \int_a^b q(x)u(x)v(x)dx \\ & - \left(\int_a^b p(x)u(x)dx\right)\left(\int_a^b q(x)v(x)dx\right) \\ & - \left(\int_a^b q(x)u(x)dx\right)\left(\int_a^b p(x)v(x)dx\right), \end{aligned} \tag{1.2}$$

where f and g are two integrable functions on $[a,b]$and p and q are positive integrable functions on $[a,b]$. If f an g are synchronous on $[a,b]$, then $T(u,v,p,q) \geq 0$.

Recently, many mathematicians have been work with slightly different fractional integral formulas, for example, Riemann-Liouville, Hadamard, Saigo, generalized Katugamapola, Erd*é*lyi-Kober, Riemann-Liouville k-fractional, Hadamard k-fractional, (k,s)-Riemann-Liouville and k-generalized (in terms hypergeometric function) fractional integral operators, see [1, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 23, 26, 27, 29, 30, 36, 37, 38, 46, 48, 50]. In [17], Dahmani investigated fractional integral inequalities for extended Chebyshev functional by employing Riemann-Liouville fractional integral. V. L. Chinchane et al. [13, 15] proposed fractional inequalities for extended Chebyshev functional via Hadamard fractional integral operators and generalized K-fractional integral (in terms of hypergeometric function). A. Anber and et al. [3] presented some fractional integral inequalities which is similar to Minkowski fractional integral inequality using Riemann-Liouville fractional integral. In [35], S. K. Panchal et al. investigated weighted fractional integral inequalities using generalized Katugampola fractional integral operator. In [4] M. Andric et al. proposed the reverse fractional Minkowski integral inequality using extended Mittag-Leffler function with the corresponding fractional integral operator is proved, as well as several related Minkowski-type inequalities. E. Deniz et al. [22] proved some new P*ó*lya-Szeg*ö* inequalities via conformable fractional integral operator and use them to prove some new fractional Chebyshev type inequalities concerning the integral of the product of two functions and the product of two integrals. In [44] G. Rahman et al. established certain new P*ó*lya-Szeg*ö*-type tempered fractional integral inequalities by considering the generalized tempered fractional integral. S. Rashid et al. [45] some new important inequalities of P*ó*lya-Szeg*ö* and Chebyshev types by use of the generalized k-fractional integral.

G. Rahman et al. [39, 40, 41] investigated Minkowski inequality and some other fractional inequalities for convex functions by employing fractional proportional integral operators. Atangana and Baleanu proposed a new fractional derivative operator with the non-local and non-singular kernel, see [5]. In [25], F. Jarad et al. proposed the fractional conformable integral and derivative operators. In [24, 42, 43], F. Jarad et al. and G. Rahman presented concepts of non-local fractional proportional and generalized Hadamard proportional integrals involving exponential functions in their kernels. In [32, 33, 34, 49], authors investigated various integral inequalities by employing conformable and generalized conformable fractional integrals. M. Caputo and M. Fabrizio [8] introduced new fractional derivative and integral without singular kernel. Later on, Lasada and Niteto proposed certain properties of fraction derivative without a singular kernel, see [28].
Motivated from [3, 14, 17, 40, 41, 42, 43], our purpose in this paper is to propose use generalized Hadamard proportional integrals extended Chebyshev functional and Pólya-Szegö integral inequalities. The paper has been organized as follows, in Section 2, we recall basic definitions, remarks and lemma related to generalized Hadamard proportional integrals. In Section 3, we obtain fractional integral inequalities for extended Chebyshev functional using generalized Hadamard proportional integrals, in Section 4, we present some Pólya-Szegö integral inequalities using generalized Hadamard proportional integrals. In section 5, we give the concluding remarks.

2 Preliminary

Here, we present some important definition, remarks and lemma of generalised proportional Hadamard fractional integral operator which will be used throughout this paper.

Definition 2.1 *The left and right sided generalized proportional fractional integrals are respectively defined by*

$$({}_a\mathfrak{J}^{\alpha,\beta}z)(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_a^x e^{[\frac{\beta-1}{\beta}(x-t)]}(x-t)^{\alpha-1}z(t)dt, a < x \tag{2.1}$$

and

$$(\mathfrak{J}_b^{\alpha,\beta}z)(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_x^b e^{[\frac{\beta-1}{\beta}(t-x)]}(t-x)^{\alpha-1}z(t)dt, x < b, \tag{2.2}$$

where the proportionality index $\beta \in (0,1]$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$.

Remark 2.1 *If we consider $\beta = 1$ in(2.1) and (2.2), then we get the well known left and right Riemann-Liouville integrals which are respectively defined by*

$$({}_a\mathfrak{J}^{\alpha} z)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} z(t)dt, a < x \tag{2.3}$$

and

$$(\mathfrak{J}_b^{\alpha} z)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} z(t)dt, x < b, \tag{2.4}$$

where $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$.

Recently, Rahman et al.[43] proposed the following generalized Hadamard proportional fractional integrals.

Definition 2.2 *The left sided generalized Hadamard proportional fractional integral of order $\alpha > 0$ and proportional index $\beta \in (0, 1]$ is defined by*

$$({}_a\mathcal{H}^{\alpha,\beta} z)(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_a^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln t)\right]} (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, a < x. \tag{2.5}$$

Definition 2.3 *The right sided generalized Hadamard proportional fractional integral of order $\alpha > 0$ and proportional index $\beta \in (0, 1]$ is defined by*

$$(\mathcal{H}_b^{\alpha,\beta} z)(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_x^b e^{\left[\frac{\beta-1}{\beta}(\ln t - \ln x)\right]} (\ln t - \ln x)^{\alpha-1} \frac{z(t)}{t} dt, x < b. \tag{2.6}$$

Definition 2.4 *The one sided generalized Hadamard proportional fractional integral of order $\alpha > 0$ and proportional index $\beta \in (0, 1]$ is defined by*

$$(\mathcal{H}_{1,x}^{\alpha,\beta} z)(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln t)\right]} (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, t > 1, \tag{2.7}$$

where $\Gamma(\alpha)$ is the classical well known gamma function.

Remark 2.2 *If we consider $\beta = 1$, then (2.5)-(2.7) will led to the following well known Hadamard fractional integrals*

$$({}_a\mathcal{H}^{\alpha} z)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, a < x, \tag{2.8}$$

$$(\mathcal{H}_b^{\alpha} z)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln t - \ln x)^{\alpha-1} \frac{z(t)}{t} dt, x < b, \tag{2.9}$$

and

$$(\mathcal{H}_{1,x}^{\alpha} z)(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, x > 1. \tag{2.10}$$

One can easily prove the following results:

Lemma 2.1

$$(\mathcal{H}^{\alpha,\beta}_{1,x} e^{[\frac{\beta-1}{\beta}(\ln x)]}(\ln x)^{\lambda-1})(x) = \frac{\Gamma(\lambda)}{\beta^{\alpha}\Gamma(\alpha+\lambda)} e^{[\frac{\beta-1}{\beta}(\ln x)]}(\ln x)^{\alpha+\lambda-1}, \qquad (2.11)$$

and the semigroup property

$$(\mathcal{H}^{\alpha,\beta}_{1,x})(\mathcal{H}^{\lambda,\beta}_{1,x})z(x) = (\mathcal{H}^{\alpha+\lambda,\beta}_{1,x})z(x). \qquad (2.12)$$

Remark 2.3 *If $\beta = 1$, then* (2.11) *will reduce to the result of [47] as defined by*

$$(\mathcal{H}^{\alpha}_{1,x}(\ln x)^{\lambda-1})(x) = \frac{\Gamma(\lambda)}{\Gamma(\alpha+\lambda)}(\ln x)^{\alpha+\lambda-1}. \qquad (2.13)$$

3 Fractional Integral Inequalities

In this section, we establish fractional integral inequality involving generalized proportional Hadamard fractional integral operators. We now prove the following lemma.

Lemma 3.1 *Let f and g be two integrable and synchronous functions on $[1, \infty)$. and $u, v : [1, \infty) \to [1, \infty)$. then for all $x > 1$, $\alpha > 0$, $\beta \in (0, 1]$ $\alpha \in \mathbb{C}$ and $\mathbb{R}(\alpha) > 0$, we have*

$$\begin{aligned} &\mathcal{H}^{\alpha,\beta}_{1,x}[u(x)]\mathcal{H}^{\alpha,\beta}_{1,x}[vfg(x)] + \mathcal{H}^{\alpha,\beta}_{1,x}[v(x)]\mathcal{H}^{\alpha,\beta}_{1,x}[ufg(x)] \geq \\ &\mathcal{H}^{\alpha,\beta}_{1,x}[uf(x)]\mathcal{H}^{\alpha,\beta}_{1,x}[vg(x)] + \mathcal{H}^{\alpha,\beta}_{1,x}[vf(x)]\mathcal{H}^{\alpha,\beta}_{1,x}[ug(x)]. \end{aligned} \qquad (3.1)$$

Proof: Since f and g are synchronous functions on $[1, \infty)$ for all $\tau \geq 0$, $\sigma \geq 0$, we have

$$\Big(f(\tau) - f(\sigma)\Big)\Big(g(\tau) - g(\sigma)\Big) \geq 0. \qquad (3.2)$$

From (3.2),

$$f(\tau)g(\tau) + f(\sigma)g(\sigma) \geq f(\tau)g(\sigma) + f(\sigma)g(\tau). \qquad (3.3)$$

Consider

$$\psi(x, \tau) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)\tau} e^{[\frac{\beta-1}{\beta}(lnx - ln\tau)]}(lnx - ln\tau)^{\alpha-1}. \qquad (3.4)$$

Clearly, we can say that the function $\psi(x, \tau)u(\tau)$ remain positive because for all $\tau \in (1, x)$, $(x > 1), \alpha, \beta > 0$. Multiplying both side of (3.3) by $\psi(x, \tau)$, then integrating resulting identity with respect to τ from 1 to x, we get

$$
\begin{aligned}
&\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(\ln x-\ln\tau)]}(\ln x-\ln\tau)^{\alpha-1}u(\tau)f(\tau)g(\tau)\frac{d\tau}{\tau}\\
&+\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(\ln x-\ln\tau)]}(\ln x-\ln\tau)^{\alpha-1}u(\tau)f(\sigma)g(\sigma)\frac{d\tau}{\tau}\\
&\geq\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(\ln x-\ln\tau)]}(\ln x-\ln\tau)^{\alpha-1}u(\tau)f(\tau)g(\sigma)\frac{d\tau}{\tau}\\
&+\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(\ln x-\ln\tau)]}(\ln x-\ln\tau)^{\alpha-1}u(\tau)f(\sigma)g(\tau)\frac{d\tau}{\tau},
\end{aligned}
\tag{3.5}
$$

consequently,

$$
\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)]+f(\sigma)g(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\\
&\geq g(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)]+f(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)].
\end{aligned}
\tag{3.6}
$$

Multiplying both side of (3.6) by $\psi(x,\sigma)v(\sigma)$ remain positive because for all $\sigma\in(1,x)$, $(x>1),\alpha,\beta>0$. Then integrating resulting identity with respect to σ from 1 to x, we get

$$
\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)]\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(lnx-ln\sigma)]}(lnx-ln\sigma)^{\alpha-1}v(\sigma)\frac{d\sigma}{\sigma}\\
&+\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(lnx-ln\sigma)]}(lnx-ln\sigma)^{\alpha-1}v(\sigma)f(\sigma)g(\sigma)\frac{d\sigma}{\sigma}\\
&\geq\mathcal{H}_{1,x}^{\alpha,\beta}[uf(t)]\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(lnx-ln\sigma)]}(lnx-ln\sigma)^{\alpha-1}v(\sigma)g(\sigma)\frac{d\sigma}{\sigma}\\
&+\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(lnx-ln\sigma)]}(lnx-ln\sigma)^{\alpha-1}v(\sigma)f(\sigma)\frac{d\sigma}{\sigma}.
\end{aligned}
\tag{3.7}
$$

This completes the proof of inequality (3.1).
Now, we give our main result.

Theorem 3.2 *Let f and g be two integrable and synchronous functions on $[1,\infty)$, and $r,p,q:[1,\infty)\to[1,\infty)$, then for all $x>1$ $\alpha>0$, $\beta\in(0,1]$ $\alpha\in\mathbb{C}$, $\Re(\alpha)>0$, we have*

$$
\begin{aligned}
&2\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]\right]+\\
&2\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]\right]+\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right]+\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right].
\end{aligned}
\tag{3.8}
$$

Proof: To prove theorem, put $u = p,\ v = q$, and using lemma 3.1, we get

$$\begin{aligned}&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)].\end{aligned} \tag{3.9}$$

Now, multiplying both side by (3.9) $\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]$, we have

$$\begin{aligned}&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]\right] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]\right],\end{aligned} \tag{3.10}$$

again, put $u = r, v = q$, and using lemma 3.1, we get

$$\begin{aligned}&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)],\end{aligned} \tag{3.11}$$

multiplying both side of (3.11) by $\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]$, we have

$$\begin{aligned}&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\right] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right],\end{aligned} \tag{3.12}$$

with the same arguments as in equation (3.10) and (3.12), we can write

$$\begin{aligned}&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg](x)\right] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right],\end{aligned} \tag{3.13}$$

adding the inequalities (3.10), (3.12) and (3.13), we get required inequality (3.8).

Lemma 3.3 *Let f and g be two integrable and synchronous functions on* $[0, \infty)$*, and* $x, y : [0, \infty[\to [0, \infty)$*, then for all* $x > 1$ $\alpha, \phi > 0$*,* $\beta, \varphi \in (0, 1]$ α*,* $\phi \in \mathbb{C}$*,* $\Re(\alpha) > 0$ *and* $\Re(\phi) > 0$*, we have*

$$\begin{aligned}&\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[vfg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[v(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[vg(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[vf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)].\end{aligned} \tag{3.14}$$

Proof:- Multiplying both sides of (3.6) by $\frac{1}{\varphi^{\phi}\Gamma(\phi)\sigma}e^{[\frac{\varphi-1}{\varphi}(\ln x - \ln\sigma)]}(\ln x - \ln\sigma)^{\phi-1}, \sigma \in (1, x), x > 1, \phi, \varphi > 0$ which (in view of the argument mentioned above in proof of lemma 3.1) remain positive. Then integrating resulting identity with respect to σ from 1 to x, we have

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)]\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}v(\sigma)\frac{d\sigma}{\sigma}\\
&+\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}v(\sigma)f(\sigma)g(\sigma)\frac{d\sigma}{\sigma}\\
&\geq \mathcal{H}_{1,x}^{\alpha,\beta}[uf(t)]\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}v(\sigma)g(\sigma)\frac{d\sigma}{\sigma}\\
&+\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}v(\sigma)f(\sigma)\frac{d\sigma}{\sigma}.
\end{aligned}\tag{3.15}$$

This completes the proof of inequality (3.14).

Theorem 3.4 *Let f and g be two integrable and synchronous functions on $[1,\infty)$, and $r,p,q:[1,\infty)\to[1,\infty)$, then for all $\alpha,\phi>0$, $\beta,\phi\in(0,1]$ $\alpha,\phi\in\mathbb{C}$, $\Re(\alpha)>0$ and $\Re(\phi)>0$, we have*

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)][\,\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[pfg(t)]+2\,\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg](\mathcal{H}_{1,x}^{\phi,\varphi})\\
&+\,\mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]]\\
&+\left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[p(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\right]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]\right]+\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right]+\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[pg(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[pf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right].
\end{aligned}\tag{3.16}$$

Proof: To prove theorem, we put $u=p$, $v=q$ and using lemma 3.3 we get

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)].
\end{aligned}\tag{3.17}$$

Now, multiplying both side by (3.17) $\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]$, we have

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]\right]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]\right],
\end{aligned}\tag{3.18}$$

putting $u=r,\ v=q$, and using lemma 3.3, we get

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)]+\,\mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)],
\end{aligned}\tag{3.19}$$

multiplying both side by (3.19) $\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]$, we have

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)]+\mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\right]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)]+\mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right].
\end{aligned}\tag{3.20}$$

With the same argument as in equation (3.18) and (3.20), we obtain

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[pfg(x)]+\mathcal{H}_{1,x}^{\phi,\varphi}[p(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\right]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[pg(x)]+\mathcal{H}_{1,x}^{\phi,\varphi}[pf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right].
\end{aligned}\tag{3.21}$$

Adding the inequalities (3.18), (3.20) and (3.21), we get the inequality (3.16).

Remark 3.1 *If f, g, r, p and q satisfy the following conditions,*

1. *The function f and g is asynchronous on $[1,\infty)$.*
2. *The function r,p,q are negative on $[1,\infty)$.*
3. *Two of the function r,p,q are positive and the third is negative on $[1,\infty)$.*

then the inequalities (3.8) *and* (3.16) *are reversed.*

4 Fractional P*ó*lya-Szeg*ö*-type inequalities

In this section, we proposed fractional Pólya-Szegö-type by considering generalized proportional Hadamard fractional integral operators.

Theorem 4.1 *Let p and q be two integrable functions on $[1,\infty)$. Assume that there exist four positive integrable functions u_1,u_2, v_1 and v_2 on $[1,\infty)$ such that*

$$0<u_1(\tau)\leq p(\tau)\leq u_2(\tau),\ 0<v_1(\tau)\leq q(\tau)\leq v_2(\tau),\ (\tau\in(1,x)).$$

then for all $x>1$, $\alpha>0$, $\beta\in(0,1]$ $\alpha\in\mathbb{C}$ and $\Re(\alpha)>0$, the following inequality holds

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[v_1v_2p^2](x)\mathcal{H}_{1,x}^{\alpha,\beta}[u_1u_2q^2](x)}{\left(\mathcal{H}_{1,x}^{\alpha,\beta}[(v_1u_1+v_2u_2)pq](x)\right)^2}\leq\frac{1}{4}.\tag{4.1}$$

Proof:- Using given condition in Theorem, we find

$$\left(\frac{u_2(\tau)}{v_1(\tau)}-\frac{p(\tau)}{q(\tau)}\right)\geq 0. \tag{4.2}$$

Also we have,

$$\left(\frac{p(\tau)}{q(\tau)}-\frac{u_1(\tau)}{v_2(\tau)}\right)\geq 0, \tag{4.3}$$

multiplying (4.2) and (4.3), we have

$$\left(\frac{u_2(\tau)}{v_1(\tau)}-\frac{p(\tau)}{q(\tau)}\right)\left(\frac{p(\tau)}{q(\tau)}-\frac{u_1(\tau)}{v_2(\tau)}\right)\geq 0, \tag{4.4}$$

which implies that

$$\left(\frac{u_2(\tau)}{v_1(\tau)}-\frac{p(\tau)}{q(\tau)}\right)\frac{x(\tau)}{q(\tau)}-\left(\frac{u_2(\tau)}{v_1(\tau)}-\frac{p(\tau)}{q(\tau)}\right)\frac{u_1(\tau)}{v_2(\tau)}\geq 0, \tag{4.5}$$

so

$$\left(\frac{u_2(\tau)}{v_1(\tau)}+\frac{u_1(\tau)}{v_2(\tau)}\right)\frac{p(\tau)}{q(\tau)}\geq\frac{p^2(\tau)}{q^2(\tau)}+\frac{u_1(\tau)u_2(\tau)}{v_1(\tau)v_2(\tau)}, \tag{4.6}$$

and

$$\left[u_1(\tau)v_1(\tau)+u_2(\tau)v_2(\tau)\right]p(\tau)q(\tau)\geq v_1(\tau)v_2(\tau)p^2(\tau)+u_1(\tau)u_2(\tau)q^2(\tau). \tag{4.7}$$

Now, multiplying both sides of (4.7) by $\psi(x,\tau)$, ($\tau\in(1,x)$, $x>1$), where $\psi(x,\tau)$ is defined by (3.4). Then integrating resulting identity with respect to τ from 1 to x, we have

$$\begin{aligned}
&\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x-\ln\tau)\right]}(\ln x-\ln\tau)^{\alpha-1}\left[u_1(\tau)v_1(\tau)+u_2(\tau)v_2(\tau)\right]p(\tau)q(\tau)\frac{d\tau}{\tau}\\
&\geq\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x-\ln\tau)\right]}(\ln x-\ln\tau)^{\alpha-1}[v_1(\tau)v_2(\tau)p^2(\tau)]\frac{d\tau}{\tau}\\
&+\frac{1}{\beta^\alpha\Gamma(\alpha)}\int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x-\ln\tau)\right]}(\ln x-\ln\tau)^{\alpha-1}[u_1(\tau)u_2(\tau)q^2(\tau)]\frac{d\tau}{\tau},
\end{aligned} \tag{4.8}$$

which implies that

$$\mathcal{H}_{1,x}^{\alpha,\beta}\left[u_1v_1+u_2v_2\right]pq(x)\geq\mathcal{H}_{1,x}^{\alpha,\beta}[v_1v_2p^2](x)+\mathcal{H}_{1,x}^{\alpha,\beta}[u_1u_2q^2](x). \tag{4.9}$$

By using the Arithmetic-geometric mean inequality $a+b\geq 2\sqrt{ab}$, where $a,b\in\mathbb{R}^+$, we have

$$\mathcal{H}_{1,x}^{\alpha,\beta}\left[u_1v_1+u_2v_2\right]pq(x)\geq 2\sqrt{\mathcal{H}_{1,x}^{\alpha,\beta}[v_1v_2p^2](x)\,\mathcal{H}_{1,x}^{\alpha,\beta}[u_2u_1q^2](x)}, \tag{4.10}$$

which leads to

$$\left(\mathcal{H}_{1,x}^{\alpha,\beta}\left[u_1v_1+u_2v_2\right]pq(x)\right)^2 \geq 4\left(\mathcal{H}_{1,x}^{\alpha,\beta}[v_1v_2p^2](x)\,\mathcal{H}_{1,x}^{\alpha,\beta}[u_1u_2q^2](x)\right). \quad (4.11)$$

It follows that

$$\mathcal{H}_{1,x}^{\alpha,\beta}[v_1v_2p^2](x)\,\mathcal{H}_{1,x}^{\alpha,\beta}[u_1u_2q^2](x) \leq \frac{1}{4}\left(\mathcal{H}_{1,x}^{\alpha,\beta}\left[u_1v_1+u_1u_2\right]pq(x)\right)^2, \quad (4.12)$$

which gives the required inequality (4.1).

Theorem 4.2 *Let p and q be two integrable functions on $[1,\infty]$. Assume that there exist four positive integrable functions u_1,u_2, v_1 and v_2 on $[1,\infty]$ such that*

$$0 < u_1(\tau) \leq p(\tau) \leq u_2(\tau),\; 0 < v_1(\sigma) \leq q(\sigma) \leq v_2(\sigma),\; (\tau,\sigma \in (1,x), x > 1).$$

Then for $x > 1$ and $\alpha,\phi > 0$, $\beta,\varphi \in (0,1]$ $\alpha,\phi \in \mathbb{C}$ and $\Re(\alpha),\Re(\varphi) > 0$, the following inequality holds

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}[u_1u_2](x)\mathcal{H}_{1,x}^{\phi,\varphi}[v_1v_2](x)\mathcal{H}_{1,x}^{\alpha,\beta}[p^2](x)\mathcal{H}_{1,x}^{\phi,\varphi}[q^2](x)}{\left(\mathcal{H}_{1,x}^{\alpha,\beta}[u_1p](x)\mathcal{H}_{1,x}^{\phi,\varphi}[v_1q](x)+\mathcal{H}_{1,x}^{\alpha,\beta}[u_2p](x)\mathcal{H}_{1,x}^{\alpha,\beta}[v_2q](x)\right)^2} \leq \frac{1}{4}. \quad (4.13)$$

Proof:- To prove (4.13), since $\tau,\sigma \in (1,x)$ and $x > 1$, we have

$$\left(\frac{u_2(\tau)}{v_1(\sigma)}-\frac{p(\tau)}{q(\rho)}\right) \geq 0, \quad (4.14)$$

also we have,

$$\left(\frac{p(\tau)}{q(\sigma)}-\frac{u_1(\tau)}{v_2(\sigma)}\right) \geq 0. \quad (4.15)$$

Multiplying (4.13) and (4.14), we have

$$\left(\frac{u_2(\tau)}{v_1(\sigma)}-\frac{p(\tau)}{q(\sigma)}\right)\left(\frac{p(\tau)}{q(\sigma)}-\frac{u_1(\tau)}{v_2(\sigma)}\right) \geq 0, \quad (4.16)$$

$$\left(\frac{u_2(\tau)}{v_1(\sigma)}-\frac{p(\tau)}{q(\sigma)}\right)\frac{p(\tau)}{q(\sigma)}-\left(\frac{u_2(\tau)}{v_1(\sigma)}-\frac{p(\tau)}{q(\sigma)}\right)\frac{u_1(\tau)}{v_2(\sigma)} \geq 0, \quad (4.17)$$

it follows that

$$\left(\frac{u_2(\tau)}{v_1(\sigma)}+\frac{u_1(\tau)}{v_2(\sigma)}\right)\frac{p(\tau)}{q(\sigma)} \geq \frac{p^2(\tau)}{q^2(\sigma)}+\frac{u_1(\tau)u_2(\tau)}{v_1(\sigma)v_2(\sigma)}. \quad (4.18)$$

Multiplying both side of equation (4.18) by $v_1(\sigma)v_2(\sigma)q^2(\sigma)$, we have

$$u_2(\tau)p(\tau)v_2(\sigma)q(\sigma)+u_1(\tau)p(\tau)v_1(\sigma)q(\sigma) \geq v_1(\sigma)v_2(\sigma)p^2(\tau)+u_1(\tau)u_2(\tau)q^2(\sigma), \tag{4.19}$$

multiplying both sides of (4.19) by $\psi(x,\tau)$ remain positive because for all $\tau \in (1,x)$, $(x>1), \alpha, \beta > 0$, then integrating resulting identity with respect to τ from 1 to x, we get

$$\begin{aligned} &v_2(\sigma)q(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[u_1 p(x)] + v_1(\sigma)q(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[u_1 p(x)] \\ &\geq v_1(\sigma)v_2(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[p^2(x)] + q^2(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[u_1 u_2(x)]. \end{aligned} \tag{4.20}$$

Multiplying both sides of (4.20) by $\psi(x,\sigma)$ remain positive because for all $\sigma \in (1,x)$, $(x>1), \alpha, \beta > 0$. Then integrating resulting identity with respect to σ from 1 to x, we get

$$\begin{aligned} &\mathcal{H}_{1,x}^{\phi,\varphi}[v_2 q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[u_1 p(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[v_1 q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[u_2 p(x)] \\ &\geq \mathcal{H}_{1,x}^{\phi,\varphi}[v_1 v_2(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[p^2(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[q^2(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[u_1 u_2(x)]. \end{aligned} \tag{4.21}$$

By $a+b \geq 2\sqrt{ab}$, where $a, b \in \mathbb{R}^+$, we have

$$\begin{aligned} &\mathcal{H}_{1,x}^{\phi,\varphi}[v_2 q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[u_1 p(x)] + \mathcal{H}_{1,x}^{\phi,\varphi}[v_1 q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[u_2 p(x)] \\ &\geq 2\sqrt{\mathcal{H}_{1,x}^{\phi,\varphi}[v_1 v_2(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[p^2(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[q^2(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[u_1 u_2(x)]}, \end{aligned} \tag{4.22}$$

which gives the required inequality (4.13). This completes the proof.

Theorem 4.3 *Let p and q be two integrable functions on $[1,\infty)$. Assume that there exist four positive integrable functions u_1,u_2, v_1 and v_2 on $[1,\infty)$ such that*

$$0 < u_1(\tau) \leq p(\tau) \leq u_2(\tau),\ 0 < v_1(\tau) \leq q(\tau) \leq v_2(\tau),\ (\tau \in (1,x), x > 1).$$

Then for $x > 1$ and $\alpha > 0$, $\beta \in (0,1]$ $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$, the following inequality holds

$$\mathcal{H}_{1,x}^{\alpha,\beta}\{p^2\}(x)\mathcal{H}_{1,x}^{\alpha,\beta}\{q^2\}(x) \leq \mathcal{H}_{1,x}^{\alpha,\beta}\{\frac{u_2 pq}{v_1}\}(x)\mathcal{H}_{1,x}^{\alpha,\beta}\{\frac{v_2 pq}{u_1}\}(x). \tag{4.23}$$

Proof:- Multiplying (4.2) by $p(\tau)$, we have

$$p^2(\tau) \leq \frac{u_2(\tau)}{v_1(\tau)}p(\tau)q(\tau). \tag{4.24}$$

Multiplying both sides of (4.24) by $\psi(x,\tau)$, $(\tau \in (1,x),\ x>1)$, where $\psi(x,\tau)$ is defined by (3.4). Then integrating resulting identity with respect to τ from 1 to x, we have

$$\mathcal{H}_{1,x}^{\alpha,\beta} p^2(x) \leq \mathcal{H}_{1,x}^{\alpha,\beta}\{\frac{u_2 pq}{v_1}\}(x). \tag{4.25}$$

Analogously, we obtain

$$\mathcal{H}_{1,x}^{\alpha,\beta} q^2(x) \leq \mathcal{H}_{1,x}^{\alpha,\beta}\{\frac{v_2 pq}{u_1}\}(x), \tag{4.26}$$

multiplying the inequality (4.25) and (4.26), we obtain the required inequality (4.23). This completes the proof.

Here we present some special case of above theorem which is as below

Proposition 4.1 *Let p and q be two integrable functions on $[1,\infty)$ such that*

$$0 < m \leq p(\tau) \leq M < \infty,\ 0 < n \leq q(\tau) \leq N < \infty,\ (\tau \in (1,x), x > 1). \tag{4.27}$$

Then for $x > 1$ and $\alpha > 0$, $\beta \in (0,1]$ $\alpha \in \mathbb{C}$ and $\Re(\alpha) > 0$, the following inequality holds

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}\{p^2\}(x)\mathcal{H}_{1,x}^{\alpha,\beta}\{q^2\}(x)}{(\mathcal{H}_{1,x}^{\alpha,\beta}\{pq\}(x))^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2. \tag{4.28}$$

5 Concluding Remarks

Nale etal. and Rahaman et al. [31, 43] investigated some Minkowski-type inequalities and other integral inequalities by considering generalized proportional Hadamard fractional integral operator. In [44, 45] Rahaman et al. and S. Rashid et al. have studied tempered fractional Pólya-Szegö and Chebyshev-type inequalities using different fractional integral operators. In [17, 13, 15], authors have established some fractional integral inequalities for extended Chebyshev functional using Hadamard and generalized K-fractional integral operators. Motivated by the above work, here we studied fractional Pólya-Szegö and some fractional integral inequalities for extended Chebyshev function and by considering generalized proportional Hadamard fractional integral operator. The inequalities investigated in this paper give some contribution in the fields of fractional calculus and generalized proportional Hadamard fractional integral operators. Moreover, they are expected to led to some application for finding uniqueness of solutions in fractional differential equations.

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