On fractional inequalities using generalized proportional Hadamard fractional integral operator

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Abstract

In this paper, we use generalized proportional Hadamard fractional integral operator to establish some new fractional integral inequalities for extended Chebyshev functional. In addition, we investigate some fractional integral inequalities for positive continuous functions by employing generalized proportional Hadamard fractional integral operator.

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1 Introduction

Fractional calculus is the study of generalization of traditional calculus into non-integer differential and integral order. Fractional derivative and integrals have become more popular and helpful due to it's various application in the field of science and technology. Fractional integral inequalities play a vital role in obtaining uniqueness of solution of fractional ordinary differential equations, fractional partial differential equations and fractional boundary value problems.
In [7], the Chebyshev for two integrable functions f and g on $[a, b]$ is defined as

$$T(u,v) = \frac{1}{b-a}\int_a^b u(x)v(x)dx - \frac{1}{b-a}\left(\int_a^b u(x)dx\right)\frac{1}{b-a}\left(\int_a^b v(x)dx\right). \tag{1.1}$$

Many applications and several inequalities related to Chebyshev functional are found in [2, 6, 20, 21]. Let us consider the extended Chebyshev functional, [17]

$$
\begin{aligned}
T(u,v,p,q) &= \int_a^b q(x)dx \int_a^b p(x)u(x)v(x)dx + \int_a^b p(x)dx \int_a^b q(x)u(x)v(x)dx \\
&\quad - \left(\int_a^b p(x)u(x)dx\right)\left(\int_a^b q(x)v(x)dx\right) \\
&\quad - \left(\int_a^b q(x)u(x)dx\right)\left(\int_a^b p(x)v(x)dx\right),
\end{aligned} \tag{1.2}
$$

where f and g are two integrable functions on $[a,b]$ and p and q are positive integrable functions on $[a,b]$. If f an g are synchronous on $[a,b]$, then $T(u,v,p,q) \geq 0$.

Recently, many mathematicians have been work with slightly different fractional integral formulas, for example, Riemann-Liouville, Hadamard, Saigo, generalized Katugamapola, Erd*é*lyi-Kober, Riemann-Liouville k-fractional, Hadamard k-fractional, (k,s)-Riemann-Liouville and k-generalized (in terms hypergeometric function) fractional integral operators, see [1, 9, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 22, 25, 26, 28, 29, 35, 36, 37, 43, 45, 47]. In [17], Dahmani investigated fractional integral inequalities for extended Chebyshev functional by employing Riemann-Liouville fractional integral. V. L. Chinchane et al. [13, 15] proposed fractional inequalities for extended Chebyshev functional via Hadamard fractional integral operators and generalized k-fractional integral (in terms of hypergeometric function). A. Anber and et al. [3] presented some fractional integral inequalities which is similar to Minkowski fractional integral inequality using Riemann-Liouville fractional integral. In [34], S. K. Panchal et al. investigated weighted fractional integral inequalities using generalized Katugampola fractional integral operator. In [4] M. Andric et al. proposed the reverse fractional Minkowski integral inequality using extended Mittag-Leffler function with the corresponding fractional integral operator is proved, as well as several related Minkowski-type inequalities. G. Rahman et al. [38, 39, 40] investigated Minkowski inequality and some other fractional inequalities for convex functions by employing fractional proportional integral operators. Atangana and Baleanu proposed a new fractional derivative operator with the non-local and non-singular kernel, see [5]. In [24], F. Jarad et al. proposed the fractional conformable integral and derivative operators. In [23, 41, 42], F. Jarad et al. and G. Rahman presented concepts of non-local fractional proportional and generalized Hadamard proportional integrals involving exponential functions in their ker-

nels. In [31, 32, 33, 46], authors investigated various integral inequalities by employing conformable and generalized conformable fractional integrals. M. Caputo and M. Fabrizio [8] introduced new fractional derivative and integral without singular kernel. Later on, Lasada and Niteto proposed certain properties of fraction derivative without a singular kernel, see [27].
Motivated from [3, 14, 17, 39, 40, 41, 42], our purpose in this paper is to obtain fractional integral inequalities for extended Chebyshev functional and other some fractional inequalities using generalized Hadamard proportional integral. The paper has been organized as follows, in Section 2, we recall basic definitions, remarks and lemma related to generalized Hadamard proportional integrals. In Section 3, we obtain fractional integral inequalities for extended Chebyshev functional using generalized Hadamard proportional integrals, in Section 4, we present some other some fractional integral inequalities using generalized Hadamard proportional integrals. In section 5, we give the concluding remarks.

2 Preliminary

Here, we present some important definition, remarks and lemma of generalised proportional Hadamard fractional integral operator which will be used throughout this paper.

Definition 2.1 *The left and right sided generalized proportional fractional integrals are respectively defined by*

$$({}_a\mathfrak{J}^{\alpha,\beta}z)(x)=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_a^x e^{[\frac{\beta-1}{\beta}(x-t)]}(x-t)^{\alpha-1}z(t)dt, a<x \tag{2.1}$$

and

$$(\mathfrak{J}_b^{\alpha,\beta}z)(x)=\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_x^b e^{[\frac{\beta-1}{\beta}(t-x)]}(t-x)^{\alpha-1}z(t)dt, x<b, \tag{2.2}$$

where the proportionality index $\beta\in(0,1]$ *and* $\alpha\in\mathbb{C}$ *with* $\Re(\alpha)>0$.

Remark 2.1 *If we consider* $\beta=1$ *in*(2.1) *and* (2.2)*, then we get the well known left and right Riemann-Liouville integrals which are respectively defined by*

$$({}_a\mathfrak{J}^{\alpha}z)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}z(t)dt, a<x \tag{2.3}$$

and

$$(\mathfrak{J}_b^{\alpha}z)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}z(t)dt, x<b, \tag{2.4}$$

where $\alpha\in\mathbb{C}$ *with* $\Re(\alpha)>0$.

Recently, Rahman et al.[42] proposed the following generalized Hadamard proportional fractional integrals.

Definition 2.2 *The left sided generalized Hadamard proportional fractional integral of order* $\alpha > 0$ *and proportional index* $\beta \in (0, 1]$ *is defined by*

$$({}_a\mathcal{H}^{\alpha,\beta} z)(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_a^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln t)]} (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, a < x. \quad (2.5)$$

Definition 2.3 *The right sided generalized Hadamard proportional fractional integral of order* $\alpha > 0$ *and proportional index* $\beta \in (0, 1]$ *is defined by*

$$(\mathcal{H}_b^{\alpha,\beta} z)(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_x^b e^{[\frac{\beta-1}{\beta}(\ln t - \ln x)]} (\ln t - \ln x)^{\alpha-1} \frac{z(t)}{t} dt, x < b. \quad (2.6)$$

Definition 2.4 *The one sided generalized Hadamard proportional fractional integral of order* $\alpha > 0$ *and proportional index* $\beta \in (0, 1]$ *is defined by*

$$(\mathcal{H}_{1,x}^{\alpha,\beta} z)(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln t)]} (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, t > 1, \quad (2.7)$$

where $\Gamma(\alpha)$ *is the classical well known gamma function.*

Remark 2.2 *If we consider* $\beta = 1$*, then* (2.5)-(2.7) *will led to the following well known Hadamard fractional integrals*

$$({}_a\mathcal{H}^{\alpha} z)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, a < x, \quad (2.8)$$

$$(\mathcal{H}_b^{\alpha} z)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln t - \ln x)^{\alpha-1} \frac{z(t)}{t} dt, x < b, \quad (2.9)$$

and

$$(\mathcal{H}_{1,x}^{\alpha} z)(x) = \frac{1}{\Gamma(\alpha)} \int_1^x (\ln x - \ln t)^{\alpha-1} \frac{z(t)}{t} dt, x > 1. \quad (2.10)$$

One can easily prove the following results:

Lemma 2.1

$$(\mathcal{H}_{1,x}^{\alpha,\beta} e^{[\frac{\beta-1}{\beta}(\ln x)]} (\ln x)^{\lambda-1})(x) = \frac{\Gamma(\lambda)}{\beta^\alpha \Gamma(\alpha+\lambda)} e^{[\frac{\beta-1}{\beta}(\ln x)]} (\ln x)^{\alpha+\lambda-1}, \quad (2.11)$$

and the semigroup property

$$(\mathcal{H}_{1,x}^{\alpha,\beta})(\mathcal{H}_{1,x}^{\lambda,\beta}) z(x) = (\mathcal{H}_{1,x}^{\alpha+\lambda,\beta}) z(x). \quad (2.12)$$

Remark 2.3 *If* $\beta = 1$*, then* (2.11) *will reduce to the result of [44] as defined by*

$$(\mathcal{H}_{1,x}^{\alpha} (\ln x)^{\lambda-1})(x) = \frac{\Gamma(\lambda)}{\Gamma(\alpha+\lambda)} (\ln x)^{\alpha+\lambda-1}. \quad (2.13)$$

3 Fractional integral inequalities for extended Chebyshev functional

In this section, we establish fractional integral inequality involving generalized proportional Hadamard fractional integral operators. We now prove the following Lemma.

Lemma 3.1 *Let f and g be two integrable and synchronous functions on $[1,\infty)$. and $u, v : [1,\infty) \to [1,\infty)$. then for all $x > 1$, $\alpha > 0$, $\beta \in (0,1]$ $\alpha \in \mathbb{C}$ and $\mathbb{R}(\alpha) > 0$, we have*

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[vfg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[v(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)] \geq \\ &\mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[vg(x)] + \mathcal{H}_{1,x}^{\alpha,\beta}[vf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]. \end{aligned} \tag{3.1}$$

Proof: Since f and g are synchronous functions on $[1,\infty)$ for all $\tau \geq 0$, $\sigma \geq 0$, we have

$$\Big(f(\tau) - f(\sigma)\Big)\Big(g(\tau) - g(\sigma)\Big) \geq 0. \tag{3.2}$$

From (3.2),

$$f(\tau)g(\tau) + f(\sigma)g(\sigma) \geq f(\tau)g(\sigma) + f(\sigma)g(\tau). \tag{3.3}$$

Consider

$$\psi(x,\tau) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)\tau} e^{[\frac{\beta-1}{\beta}(lnx - ln\tau)]}(lnx - ln\tau)^{\alpha-1}. \tag{3.4}$$

Clearly, we can say that the function $\psi(x,\tau)u(\tau)$ remain positive because for all $\tau \in (1,x)$, $(x > 1), \alpha, \beta > 0$. Multiplying both side of (3.3) by $\psi(x,\tau)$, then integrating resulting identity with respect to τ from 1 to x, we get

$$\begin{aligned} &\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln\tau)]}(\ln x - \ln\tau)^{\alpha-1}u(\tau)f(\tau)g(\tau)\frac{d\tau}{\tau} \\ &+ \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln\tau)]}(\ln x - \ln\tau)^{\alpha-1}u(\tau)f(\sigma)g(\sigma)\frac{d\tau}{\tau} \\ &\geq \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln\tau)]}(\ln x - \ln\tau)^{\alpha-1}u(\tau)f(\tau)g(\sigma)\frac{d\tau}{\tau} \\ &+ \frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(\ln x - \ln\tau)]}(\ln x - \ln\tau)^{\alpha-1}u(\tau)f(\sigma)g(\tau)\frac{d\tau}{\tau}, \end{aligned} \tag{3.5}$$

consequently,

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)] + f(\sigma)g(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)] \\ &\geq g(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)] + f(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]. \end{aligned} \tag{3.6}$$

Multiplying both sides of (3.6) by $\psi(x,\sigma)v(\sigma)$ remain positive because for all $\sigma \in (1,x)$, $(x>1), \alpha, \beta > 0$. Then integrating resulting identity with respect to σ from 1 to x, we get

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)]\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(lnx-ln\sigma)]}(lnx-ln\sigma)^{\alpha-1}v(\sigma)\frac{d\sigma}{\sigma} \\
&+\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(lnx-ln\sigma)]}(lnx-ln\sigma)^{\alpha-1}v(\sigma)f(\sigma)g(\sigma)\frac{d\sigma}{\sigma} \\
&\geq \mathcal{H}_{1,x}^{\alpha,\beta}[uf(t)]\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(lnx-ln\sigma)]}(lnx-ln\sigma)^{\alpha-1}v(\sigma)g(\sigma)\frac{d\sigma}{\sigma} \\
&+\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_1^x e^{[\frac{\beta-1}{\beta}(lnx-ln\sigma)]}(lnx-ln\sigma)^{\alpha-1}v(\sigma)f(\sigma)\frac{d\sigma}{\sigma}.
\end{aligned} \tag{3.7}$$

This completes the proof of inequality (3.1).
Here, we present our main result.

Theorem 3.2 *Let f and g be two integrable and synchronous functions on $[1,\infty)$, and $r,p,q : [1,\infty) \to [1,\infty)$, then for all $x>1$ $\alpha>0$, $\beta\in(0,1]$ $\alpha\in\mathbb{C}$, $\Re(\alpha)>0$, we have*

$$\begin{aligned}
&2\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]\right]+ \\
&2\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]\right]+ \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right]+ \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right].
\end{aligned} \tag{3.8}$$

Proof: To prove Theorem, put $u=p,\ v=q$, and using Lemma 3.1, we get

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]\geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)].
\end{aligned} \tag{3.9}$$

Now, multiplying both sides by (3.9) $\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]$, we have

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]\right]\geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]\right],
\end{aligned} \tag{3.10}$$

again, put $u=r, v=q$, and using Lemma 3.1, we get

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)],
\end{aligned} \tag{3.11}$$

multiplying both sides of (3.11) by $\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]$, we have

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qfg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\right]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[qg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[qf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right],
\end{aligned}\tag{3.12}$$

with the same arguments as in equation (3.10) and (3.12), we can write

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rfg](x)\right]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]+\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right],
\end{aligned}\tag{3.13}$$

adding the inequalities (3.10), (3.12) and (3.13), we get required inequality (3.8).

Lemma 3.3 *Let f and g be two integrable and synchronous functions on $[0,\infty)$, and $x,y:[0,\infty[\to[0,\infty)$, then for all $x>1$ $\alpha,\phi>0$, $\beta,\varphi\in(0,1]$ α, $\phi\in\mathbb{C}$, $\Re(\alpha)>0$ and $\Re(\phi)>0$, we have*

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[vfg(x)]+\mathcal{H}_{1,x}^{\phi,\varphi}[v(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)]\geq\\
&\mathcal{H}_{1,x}^{\alpha,\beta}[uf(x)]\mathcal{H}_{1,x}^{\phi,\varphi}[vg(x)]+\mathcal{H}_{1,x}^{\phi,\varphi}[vf(x)]\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)].
\end{aligned}\tag{3.14}$$

Proof:- Multiplying both sides of (3.6) by $\frac{1}{\varphi^{\phi}\Gamma(\phi)\sigma}e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1},\sigma\in(1,x),x>1,\phi,\varphi>0$ which (in view of the argument mentioned above in proof of Lemma 3.1) remain positive. Then integrating resulting identity with respect to σ from 1 to x, we have

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[ufg(x)]\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}v(\sigma)\frac{d\sigma}{\sigma}\\
&+\mathcal{H}_{1,x}^{\alpha,\beta}[u(x)]\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}v(\sigma)f(\sigma)g(\sigma)\frac{d\sigma}{\sigma}\\
&\geq\mathcal{H}_{1,x}^{\alpha,\beta}[uf(t)]\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}v(\sigma)g(\sigma)\frac{d\sigma}{\sigma}\\
&+\mathcal{H}_{1,x}^{\alpha,\beta}[ug(x)]\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}v(\sigma)f(\sigma)\frac{d\sigma}{\sigma}.
\end{aligned}\tag{3.15}$$

This completes the proof of inequality (3.14).

Theorem 3.4 *Let f and g be two integrable and synchronous functions on $[1,\infty)$, and $r,p,q:[1,\infty)\to[1,\infty)$, then for all $\alpha,\phi>0$, $\beta,\phi\in(0,1]$*

$\alpha, \phi \in \mathbb{C}$, $\Re(\alpha) > 0$ *and* $\Re(\phi) > 0$, *we have*

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)][\,\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[pfg(t)] + 2\,\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg](\mathcal{H}_{1,x}^{\phi,\varphi}) \\
&+\ \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]] \\
&+\left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[q(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[p(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\right]\ \mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)] \geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]\right] + \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right] + \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[pg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[pf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right].
\end{aligned} \tag{3.16}$$

Proof: To prove Theorem, we put $u = p$, $v = q$ and using Lemma 3.3 we get

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)] \geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)].
\end{aligned} \tag{3.17}$$

Now, multiplying both sides by (3.17) $\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]$, we have

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pfg(x)]\right] \geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[pf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[pg(x)]\right],
\end{aligned} \tag{3.18}$$

putting $u = r,\ v = q$, and using Lemma 3.3, we get

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)] \geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)],
\end{aligned} \tag{3.19}$$

multiplying both sides by (3.19) $\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]$, we have

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qfg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[q(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\right] \geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[p(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[qg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[qf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right],
\end{aligned} \tag{3.20}$$

with the same argument as in equation (3.18) and (3.20), we obtain

$$\begin{aligned}
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[r(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[pfg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[p(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rfg(x)]\right] \geq \\
&\mathcal{H}_{1,x}^{\alpha,\beta}[q(x)]\left[\mathcal{H}_{1,x}^{\alpha,\beta}[rf(x)]\,\mathcal{H}_{1,x}^{\phi,\varphi}[pg(x)] +\ \mathcal{H}_{1,x}^{\phi,\varphi}[pf(x)]\,\mathcal{H}_{1,x}^{\alpha,\beta}[rg(x)]\right].
\end{aligned} \tag{3.21}$$

Adding the inequalities (3.18), (3.20) and (3.21), we get the inequality (3.16).

Remark 3.1 *If f, g, r, p and q satisfy the following conditions,*

1. *The functions f and g is asynchronous on $[1, \infty)$.*
2. *The functions r, p, q are negative on $[1, \infty)$.*
3. *Two of the functions r, p, q are positive and the third is negative on $[1, \infty)$.*

then the inequalities (3.8) and (3.16) are reversed.

4 Other some fractional integral inequalities

Now, we give some other fractional integral inequalities using generalized proportional Hadamard fractional integral operators.

Theorem 4.1 *Suppose that f, g and h be positive and continuous functions on $[0, \infty[$, such that*

$$(g(\tau) - g(\sigma)) \left(\frac{f(\sigma)}{h(\sigma)} - \frac{f(\tau)}{h(\tau)} \right) \geq 0; \ \tau, \sigma \in (0, x) \quad x > 0, \tag{4.1}$$

then for all $x > 1$, $\alpha > 0$, $\beta \in (0, 1]$ $\alpha \in \mathbb{C}$ and $\mathbb{R}(\alpha) > 0$, we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta} f(x)}{\mathcal{H}_{1,x}^{\alpha,\beta} h(x)} \geq \frac{\mathcal{H}_{1,x}^{\alpha,\beta} (gf)(x)}{\mathcal{H}_{1,x}^{\alpha,\beta} (gh)(x)}. \tag{4.2}$$

Proof: Since f, g and h be three positive and continuous functions on $[0, \infty[$ by (4.1), we have

$$g(\tau)\frac{f(\sigma)}{h(\sigma)} + g(\sigma)\frac{f(\tau)}{h(\tau)} - g(\sigma)\frac{f(\sigma)}{h(\sigma)} - g(\tau)\frac{f(\tau)}{h(\tau)} \geq 0; \tau, \sigma \in (0, x) x > 0. \tag{4.3}$$

Multiplying both sides of (4.3) by $h(\sigma)h(\tau)$, we have

$$g(\tau)f(\sigma)h(\tau) - g(\tau)f(\tau)h(\sigma) - g(\sigma)f(\sigma)h(\tau) + g(\sigma)f(\tau)h(\sigma) \geq 0. \tag{4.4}$$

Now multiplying equation (4.4) by $\psi(x, \tau)$ which is defined by (3.4), then integrating resulting identity with respect to τ from 1 to x, we get

$$\begin{aligned}
&\frac{f(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} g(\tau) h(\tau) \frac{d\tau}{\tau} \\
&- \frac{h(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} g(\tau) f(\tau) \frac{d\tau}{\tau} \\
&- \frac{f(\sigma)g(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} h(\tau) \frac{d\tau}{\tau} \\
&+ \frac{h(\sigma)g(\sigma)}{\beta^\alpha \Gamma(\alpha)} \int_1^x e^{\left[\frac{\beta-1}{\beta}(\ln x - \ln \tau)\right]} (\ln x - \ln \tau)^{\alpha-1} f(\tau) \frac{d\tau}{\tau} \geq 0,
\end{aligned} \tag{4.5}$$

from (4.5)

$$\begin{aligned} & f(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}(gh)(x)+g(\sigma)h(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}f(x)- \\ & g(\sigma)f(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}h(x)-h(\sigma)\mathcal{H}_{1,x}^{\alpha,\beta}(gf)(x)\geq 0. \end{aligned} \tag{4.6}$$

Again multiplying (4.6) by $\psi(x,\sigma)$ which is defined in view of (3.4) and remain positive because for all $\sigma \in (1,x)$, $(x>1), \alpha, \beta > 0$. Then integrating resulting identity with respect to σ from 1 to x, we get

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}f(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gh)(x)-\mathcal{H}_{1,x}^{\alpha,\beta}h(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gf)(x) \\ & -\mathcal{H}_{1,x}^{\alpha,\beta}(gf)(x)\mathcal{H}_{1,x}^{\alpha,\beta}h(x)+\mathcal{H}_{1,x}^{\alpha,\beta}(gh)(x)\mathcal{H}_{1,x}^{\alpha,\beta}f(x)\geq 0, \end{aligned} \tag{4.7}$$

which implies that

$$\mathcal{H}_{1,x}^{\alpha,\beta}f(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gh)(x)\geq \mathcal{H}_{1,x}^{\alpha,\beta}h(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gf)(x). \tag{4.8}$$

This completes the proof of Theorem.

Theorem 4.2 *Suppose that f, g and h be positive and continuous functions on $[0,\infty[$, such that*

$$(g(\tau)-g(\sigma))\left(\frac{f(\sigma)}{h(\sigma)}-\frac{f(\tau)}{h(\tau)}\right)\geq 0, \tau,\sigma\in(0,x) \quad x>0, \tag{4.9}$$

then for all $x>1$ $\alpha,\phi>0$, $\beta,\varphi\in(0,1]$ α, $\phi\in\mathbb{C}$, $\Re(\alpha)>0$ and $\Re(\phi)>0$, we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}f(x)\mathcal{H}_{1,x}^{\phi,\varphi}(gf)(x)+\mathcal{H}_{1,x}^{\phi,\varphi}f(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gh)(x)}{\mathcal{H}_{1,x}^{\alpha,\beta}h(x)\mathcal{H}_{1,x}^{\phi,\varphi}(gf)(x)+\mathcal{H}_{1,x}^{\phi,\varphi}h(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gf)(x)}\geq 1, \tag{4.10}$$

hold.

Proof:- Multiplying equation (4.6) by $\frac{1}{\varphi^{\phi}\Gamma(\phi)\sigma}e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}, \sigma\in(1,x), x>1, \phi,\varphi>0$ which (in view of the argument mentioned above in proof of lemma 3.1) remain positive. Then integrating resulting identity with respect to σ from 1 to x, we have

$$\begin{aligned} & \mathcal{H}_{1,x}^{\alpha,\beta}(hg)(x)\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}f(\sigma)\frac{d\sigma}{\sigma} \\ & -\mathcal{H}_{1,x}^{\alpha,\beta}(gf)(x)\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}h(\sigma)\frac{d\sigma}{\sigma} \\ & -\mathcal{H}_{1,x}^{\alpha,\beta}h(x)\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}gf(\sigma)\frac{d\sigma}{\sigma} \\ & +\mathcal{H}_{1,x}^{\alpha,\beta}f(x)\frac{1}{\varphi^{\phi}\Gamma(\phi)}\int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x-\ln\sigma)]}(\ln x-\ln\sigma)^{\phi-1}fh(\sigma)\frac{d\sigma}{\sigma}\geq 0, \end{aligned} \tag{4.11}$$

which gives

$$\mathcal{H}_{1,x}^{\phi,\varphi}f(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gh)(x)-\mathcal{H}_{1,x}^{\phi,\varphi}h(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gf)(x)$$
$$-\mathcal{H}_{1,x}^{\phi,\varphi}(gf)(x)\mathcal{H}_{1,x}^{\alpha,\beta}h(x)+\mathcal{H}_{1,x}^{\phi,\varphi}(gh)(x)\mathcal{H}_{1,x}^{\alpha,\beta}f(x)\geq 0, \tag{4.12}$$

from (4.12), we get

$$\mathcal{H}_{1,x}^{\phi,\varphi}f(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gh)(x)+\mathcal{H}_{1,x}^{\phi,\varphi}(gh)(x)\mathcal{H}_{1,x}^{\alpha,\beta}f(x)$$
$$\geq\mathcal{H}_{1,x}^{\phi,\varphi}h(x)\mathcal{H}_{1,x}^{\alpha,\beta}(gf)(x)+\mathcal{H}_{1,x}^{\phi,\varphi}(gf)(x)\mathcal{H}_{1,x}^{\alpha,\beta}h(x), \tag{4.13}$$

this gives the required inequality (4.10). This completes the proof of Theorem.

Remark 4.1 *If we take $\alpha=\phi$ and $\beta=\varphi$ in Theorem 4.2 then we get the Theorem 4.1.*

Theorem 4.3 *Suppose that f and h are two positive continuous functions such that $f\leq h$ on $[0,\infty[$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0,\infty[$, then for any $p\geq 0$, $x>1$, $\alpha>0$, $\beta\in(0,1]$ $\alpha\in\mathbb{C}$ and $\mathbb{R}(\alpha)>0$, we have*

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}f(x)}{\mathcal{H}_{1,x}^{\alpha,\beta}h(x)}\geq\frac{\mathcal{H}_{1,x}^{\alpha,\beta}f^{p}(x)}{\mathcal{H}_{1,x}^{\alpha,\beta}h^{p}(x)}. \tag{4.14}$$

Proof: Now by taking $g=f^{p-1}$ in Theorem 4.1, we get

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}f(x)}{\mathcal{H}_{1,x}^{\alpha,\beta}h(x)}\geq\frac{\mathcal{H}_{1,x}^{\alpha,\beta}(ff^{p-1})(x)}{\mathcal{H}_{1,x}^{\alpha,\beta}(hf^{p-1})(x)}. \tag{4.15}$$

Since $f\leq h$ on $[0,\infty[$, then we have

$$hf^{p-1}(t)\leq h^{p}. \tag{4.16}$$

Multiplying equation (4.16) by $\psi(x,\tau)$ which is defined by (3.4), then integrating resulting identity with respect to τ from 1 to x, we get

$$\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_{1}^{x}e^{[\frac{\beta-1}{\beta}(\ln x-\ln\tau)]}(\ln x-\ln\tau)^{\alpha-1}[f^{p-1}h](\tau)\frac{d\tau}{\tau} \tag{4.17}$$
$$\leq\frac{1}{\beta^{\alpha}\Gamma(\alpha)}\int_{1}^{x}e^{[\frac{\beta-1}{\beta}(\ln x-\ln\tau)]}(\ln x-\ln\tau)^{\alpha-1}[h^{p}](\tau)\frac{d\tau}{\tau}, \tag{4.18}$$

implies that

$$\mathcal{H}_{1,x}^{\alpha,\beta}(hf^{p-1})(x)\leq\mathcal{H}_{1,x}^{\alpha,\beta}(h^{p})(x), \tag{4.19}$$

and thus we have

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta}(ff^{p-1})(x)}{\mathcal{H}_{1,x}^{\alpha,\beta}(hf^{p-1})(x)}\geq\frac{\mathcal{H}_{1,x}^{\alpha,\beta}f^{p}(x)}{\mathcal{H}_{1,x}^{\alpha,\beta}h^{p}(x)}, \tag{4.20}$$

then from equation (4.15) and (4.20), we obtain (4.14).

Theorem 4.4 *Suppose that f and h are two positive continuous functions such that $f \leq h$ on $[0, \infty)$. If $\frac{f}{h}$ is decreasing and f is increasing on $[0, \infty)$, then for any $p \geq 1$, $x > 1$ $\alpha, \phi > 0$, $\beta, \varphi \in (0, 1]$ α, $\phi \in \mathbb{C}$, $\Re(\alpha) > 0$ and $\Re(\phi) > 0$, we have*

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta} f(x) \mathcal{H}_{1,x}^{\phi,\varphi} h^p(x) + \mathcal{H}_{1,x}^{\phi,\varphi} f(x) \mathcal{H}_{1,x}^{\alpha,\beta} h^p(x)}{\mathcal{H}_{1,x}^{\alpha,\beta} h(x) \mathcal{H}_{1,x}^{\phi,\varphi} f^p(x) + \mathcal{H}_{1,x}^{\phi,\varphi} h(x) \mathcal{H}_{1,x}^{\alpha,\beta} f^p(x)} \geq 1. \tag{4.21}$$

Proof: Now by taking $g = f^{p-1}$ in Theorem 4.2, then we obtain

$$\frac{\mathcal{H}_{1,x}^{\alpha,\beta} f(x) \mathcal{H}_{1,x}^{\phi,\varphi} (hf^{p-1})(x) + \mathcal{H}_{1,x}^{\phi,\varphi} f(x) \mathcal{H}_{1,x}^{\alpha,\beta} (hf^{p-1})(x)}{\mathcal{H}_{1,x}^{\alpha,\beta} h(x) \mathcal{H}_{1,x}^{\phi,\varphi} f^p(x) + \mathcal{H}_{1,x}^{\phi,\varphi} h(x) \mathcal{H}_{1,x}^{\alpha,\beta} f^p(x)} \geq 1, \tag{4.22}$$

then by hypothesis, $f \leq h$ on $[0, \infty[$, which implies that

$$hf^{p-1} \leq h^p. \tag{4.23}$$

Now, multiplying both sides of (4.23) by $\frac{1}{\varphi^\phi \Gamma(\phi)\sigma} e^{[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)]} (\ln x - \ln \sigma)^{\phi-1}, \sigma \in (1, x), x > 1, \phi, \varphi > 0$ which (in view of the argument mentioned above in proof of Lemma 3.1) remain positive. Then integrating resulting identity with respect to σ from 1 to x, we have

$$\begin{aligned} &\frac{1}{\varphi^\phi \Gamma(\phi)} \int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)]} (\ln x - \ln \sigma)^{\phi-1} hf^{p-1}(\sigma) \frac{d\sigma}{\sigma} \\ &\leq \frac{1}{\varphi^\phi \Gamma(\phi)} \int_1^x e^{[\frac{\varphi-1}{\varphi}(\ln x - \ln \sigma)]} (\ln x - \ln \sigma)^{\phi-1} h^p(\sigma) \frac{d\sigma}{\sigma}. \end{aligned} \tag{4.24}$$

Integrating both sides of (4.24) with respect to σ over 1 to x, we have

$$\mathcal{H}_{1,x}^{\phi,\varphi} (hf^{p-1})(x) \leq \mathcal{H}_{1,x}^{\phi,\varphi} h^p(x), \tag{4.25}$$

multiplying on both sides of ((4.25) by $\mathcal{H}_{1,x}^{\alpha,\beta} f(x)$, we obtain

$$\mathcal{H}_{1,x}^{\alpha,\beta} f(x) \mathcal{H}_{1,x}^{\phi,\varphi} (hf^{p-1})(x) \leq \mathcal{H}_{1,x}^{\alpha,\beta} f(x) \mathcal{H}_{1,x}^{\phi,\varphi} h^p(x). \tag{4.26}$$

Hence from ((4.19) and (4.26), we obtain

$$\begin{aligned} &\mathcal{H}_{1,x}^{\alpha,\beta} f(x) \mathcal{H}_{1,x}^{\phi,\varphi} (hf^{p-1})(x) + \mathcal{H}_{1,x}^{\phi,\varphi} f(x) \mathcal{H}_{1,x}^{\alpha,\beta} (hf^{p-1})(x) \\ &\leq \mathcal{H}_{1,x}^{\alpha,\beta} f(x) \mathcal{H}_{1,x}^{\phi,\varphi} h^p(x) + \mathcal{H}_{1,x}^{\phi,\varphi} f(x) \mathcal{H}_{1,x}^{\alpha,\beta} h^p(x). \end{aligned} \tag{4.27}$$

From (4.22) and (4.27), we obtain (4.21). This completes the proof of Theorem.

5 Concluding Remarks

Nale et al. and Rahaman et al. [30, 42] investigated some Minkowski-type inequalities and other integral inequalities by considering generalized proportional Hadamard fractional integral operator. In [13, 15, 17], authors have established some fractional integral inequalities for extended Chebyshev functional using Hadamard and generalized K-fractional integral operators. Motivated by the above work, here we proposed some fractional integral inequalities for extended Chebyshev function and by considering generalized proportional Hadamard fractional integral operator. The inequalities investigated in this paper give some contribution in the fields of fractional calculus and generalized proportional Hadamard fractional integral operators. Moreover, they are expected to led to some application for finding uniqueness of solutions in fractional differential equations.

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