NORM INEQUALITIES FOR DIFFERENCE BETWEEN WEIGHTED AND INTEGRAL MEANS RELATED TO ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with spectra $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. In this paper we show among others that, if $g : [0,1] \to \mathbb{C}$ is Lebesgue integrable, then

$$\left\| \int_0^1 g(t) f((1-t)x + ty)\, dt - \int_0^1 g(t)\, dt \int_0^1 f((1-t)x + ty)\, dt \right\|$$
$$\leq \frac{1}{2\pi} \|y - x\|$$
$$\times \begin{cases} \int_0^1 |P_g(t)|\, dt \int_\gamma |f(\xi)| \max\left\{ (|\xi| - \|x\|)^{-2}, (|\xi| - \|y\|)^{-2} \right\} |d\xi|, \\ \\ \sup_{t\in[0,1]} |P_g(t)| \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|x\|)(|\xi| - \|y\|)} |d\xi|, \end{cases}$$

where

$$P_g(t) := \left(\int_t^1 g(s)\, ds \right) t - \left(\int_0^t g(s)\, ds \right)(1 - t),\ t \in [0,1].$$

Some examples for exponential function and functions defined by power series in Banach algebras are also given.

1. Introduction

In 1906, Fejér [18], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral* $\int_a^b f(t) p(t)\, dt$, *where* f *is a convex function in the interval* (a,b) *and* p *is a positive function in the same interval such that*

$$p(a + b - t) = p(t),\ \text{for } a \leq t \leq b,$$

i.e., $y = p(t)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b), 0\right)$ *and is normal to the* t*-axis. Under those conditions the following inequalities are valid:*

$$f\left(\frac{a+b}{2}\right) \int_a^b p(t)\, dt \leq \int_a^b f(t) p(t)\, dt \leq \frac{f(a) + f(b)}{2} \int_a^b p(t)\, dt. \tag{1.1}$$

If f *is concave on* (a,b), *then the inequalities reverse in (1.1)*

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Motivated by the above result, it is natural to ask for upper and lower bounds for the difference

$$\int_a^b p(t) f(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \int_a^b p(t)\,dt$$

under various assumptions for the functions involved.

It is well known that if p and f have the same monotonicity on $[a,b]$, the above difference is nonnegative. This is known in the literature as *Chebyshev's inequality*. Also, if

$$n \le p \le N \text{ and } m \le f \le M$$

for some constants n, N, m, M, then

$$\left|\int_a^b p(t) f(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \int_a^b p(t)\,dt\right| \le \frac{1}{4}(b-a)(N-n)(M-m),$$

which is well known in the literature as *Grüss' inequality.*

In order to extend this results for norms and functions with values in Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\| \le \|a\|\,\|b\|$ for any $a,b \in \mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a,b \in \operatorname{Inv}(\mathcal{B})$ then $ab \in \operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1-a \in \operatorname{Inv}(\mathcal{B})$;
(ii) $\{b \in \mathcal{B}\colon\ \|1-b\| < 1\} \subset \operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} :\ z - a \in \operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z-w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} :\ |z| \le \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to\mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;

(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_{\delta} f(\xi)(\xi - a)^{-1} d\xi, \tag{1.2}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \operatorname{ins}(\delta)$, the inside of δ.

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.3}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [17] and [22].

For some recent norm inequalities for functions on Banach algebras, see [9], [2] and [6]-[15].

2. Some Preliminary Facts

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \ t \in [0,1]. \tag{2.1}$$

Lemma 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f'_{x,y}(t) = D(f)((1-t)x + ty)(y - x) \tag{2.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.2).

We also have the lateral derivatives

$$f'_{x,y}(0+) = D(f)(x)(y - x) \text{ and } f'_{x,y}(1-) = D(f)(y)(y - x). \tag{2.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f'_{x,y}(t) = \frac{df_{x,y}(t)}{dt} &= \lim_{h \to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \right] \\ &= D(f)((1-t)x + ty)(y-x), \end{aligned}$$

which proves (2.5).

The proof is similar for the lateral derivatives. □

Lemma 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{2.4}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x + hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned} & f(x + hv) - f(x) \\ &= \frac{1}{2\pi i} \left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi \right) \\ &= \frac{1}{2\pi i} \int_\gamma f(\xi) \left[(\xi - x - hv)^{-1} - (\xi - x)^{-1} \right] d\xi \\ &= \frac{1}{2\pi i} h \int_\gamma f(\xi) \left[(\xi - x - hv)^{-1} v (\xi - x)^{-1} \right] d\xi, \end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x + hv) - f(x)}{h} = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x - hv)^{-1} v (\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (2.4). □

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f'_{x,y}(t) = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1} (y - x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{2.5}$$

for all $t \in (0, 1)$.

We also have the lateral derivatives

$$f'_{x,y}(0+) = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x)^{-1} (y - x)(\xi - x)^{-1} d\xi, \tag{2.6}$$

and

$$f'_{x,y}(1-) = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - y)^{-1} (y - x)(\xi - y)^{-1} d\xi. \tag{2.7}$$

The proof is obvious by Lemmas 1 and 2.

3. General Bounds

We start with the following representation result that is of interest in itself as well:

Lemma 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. If $g : [0,1] \to \mathbb{C}$ is Lebesgue integrable and*

$$P_g(t) := \left(\int_t^1 g(s)\,ds\right) t - \left(\int_0^t g(s)\,ds\right)(1-t), \; t \in [0,1], \tag{3.1}$$

then

$$\begin{aligned} &\int_0^1 g(t) f((1-t)x + ty)\,dt - \int_0^1 g(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \\ &= \frac{1}{2\pi i}\int_\gamma f(\xi)\left(\int_0^1 P_g(t)\right. \\ &\times \left.(\xi - (1-t)x - ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\,d\xi\right) dt. \end{aligned} \tag{3.2}$$

Proof. Integrating by parts in the Bochner's integral, we have

$$\begin{aligned} &\int_0^t s f'_{x,y}(s)\,ds + \int_t^1 (s-1) f'_{x,y}(s)\,ds \\ &= t f_{x,y}(t) - \int_0^t f_{x,y}(s)\,ds - (t-1) f_{x,y}(t) - \int_t^1 f_{x,y}(s)\,ds \\ &= f_{x,y}(t) - \int_0^1 f_{x,y}(s)\,ds \end{aligned}$$

that holds for all $t \in [0,1]$.

If we multiply this identity by $g(t)$ and integrate over t in $[0,1]$, then we get

$$\begin{aligned} &\int_0^1 g(t) f_{x,y}(t)\,dt - \int_0^1 g(t)\,dt \int_0^1 f_{x,y}(s)\,ds \\ &= \int_0^1 g(t)\left(\int_0^t s f'_{x,y}(s)\,ds\right) dt + \int_0^1 g(t)\left(\int_t^1 (s-1) f'_{x,y}(s)\,ds\right) dt. \end{aligned} \tag{3.3}$$

Using integration by parts, we get

$$\begin{aligned}
(3.4)\qquad & \int_0^1 g(t)\left(\int_0^t s f'_{x,y}(s)\,ds\right)dt \\
&= \int_0^1 \left(\int_0^t s f'_{x,y}(s)\,ds\right) d\left(\int_0^t g(s)\,ds\right) \\
&= \left(\int_0^t g(s)\,ds\right)\left(\int_0^t s f'_{x,y}(s)\,ds\right)\Big|_0^1 - \int_0^1 \left(\int_0^t g(s)\,ds\right) t f'_{x,y}(t)\,dt \\
&= \left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 s f'_{x,y}(s)\,ds\right) - \int_0^1 \left(\int_0^t g(s)\,ds\right) t f'_{x,y}(t)\,dt \\
&= \int_0^1 \left(\int_0^1 g(s)\,ds - \int_0^t g(s)\,ds\right) t f'_{x,y}(t)\,dt \\
&+ \int_0^1 \left(\int_t^1 g(s)\,ds\right) t f'_{x,y}(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
(3.5)\qquad & \int_0^1 g(t)\left(\int_t^1 (s-1) f'_{x,y}(s)\,ds\right)dt \\
&= \int_0^1 \left(\int_t^1 (s-1) f'_{x,y}(s)\,ds\right) d\left(\int_0^t g(s)\,ds\right) \\
&= \left(\int_t^1 (s-1) f'_{x,y}(s)\,ds\right)\left(\int_0^t g(s)\,ds\right)\Big|_0^1 \\
&+ \int_0^1 \left(\int_0^t g(s)\,ds\right)(t-1) f'_{x,y}(t)\,dt \\
&= \int_0^1 \left(\int_0^t g(s)\,ds\right)(t-1) f'_{x,y}(t)\,dt,
\end{aligned}$$

which proves the identity

$$\begin{aligned}
(3.6)\qquad & \int_0^1 g(t) f_{x,y}(t)\,dt - \int_0^1 g(t)\,dt \int_0^1 f_{x,y}(t)\,dt \\
&= \int_0^1 \left(\int_t^1 g(s)\,ds\right) t f'_{x,y}(t)\,dt + \int_0^1 \left(\int_0^t g(s)\,ds\right)(t-1) f'_{x,y}(t)\,dt \\
&= \int_0^1 P_g(t) f'_{x,y}(t)\,dt.
\end{aligned}$$

Using (2.5), we get

$$\begin{aligned}
& \int_0^1 g(t) f_{x,y}(t)\,dt - \int_0^1 g(t)\,dt \int_0^1 f_{x,y}(t)\,dt \\
&= \int_0^1 P_g(t) \\
&\times \left(\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\,d\xi\right)dt
\end{aligned}$$

and by Fubini's theorem we derive (3.2). □

We have the following bounds:

Theorem 2. *Assume that* $f:G\to\mathbb{C}$ *is analytic on the convex domain* G *and* x, $y\in\mathcal{B}$, $x\neq y$ *with* $\sigma(x)$, $\sigma(y)\subset G$ *while* $\gamma\subset G$ *is taken to be close rectifiable curve in* G *and such that* $\sigma(x)$, $\sigma(y)\subset\operatorname{ins}(\gamma)$. *If* $g:[0,1]\to\mathbb{C}$ *is Lebesgue integrable, then*

$$\begin{aligned}
&\left\|\int_0^1 g(t) f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \qquad (3.7)\\
&\leq\frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\left(\int_0^1 |P_g(t)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^2 dt\right)|d\xi|\\
&\leq\frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\left(\int_0^1 |P_g(t)|\left(|\xi|-\|(1-t)x+ty\|\right)^{-2} dt\right)|d\xi|\\
&\leq\frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\\
&\times\left(\int_0^1 |P_g(t)|\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2} dt\right)|d\xi|\\
&\leq\frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\\
&\times\left(\int_0^1 |P_g(t)|\left[(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\right] dt\right)|d\xi|\\
&\leq\frac{1}{2\pi}\|y-x\|\int_0^1 |P_g(t)|\,dt\\
&\times\int_\gamma |f(\xi)|\max\left\{(|\xi|-\|x\|)^{-2},(|\xi|-\|y\|)^{-2}\right\}|d\xi|.
\end{aligned}$$

Proof. By taking the norm and using the integral's properties, we get

$$\begin{aligned}
&\left\|\int_0^1 g(t) f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \qquad (3.8)\\
&\leq\frac{1}{2\pi}\int_\gamma |f(\xi)|\left(\int_0^1 |P_g(t)|\right.\\
&\times\left.\left\|(\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1}\right\| dt\right)|d\xi|\\
&\leq\frac{1}{2\pi}\|y-x\|\\
&\times\int_\gamma |f(\xi)|\left(\int_0^1 |P_g(t)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^2 dt\right)|d\xi|,
\end{aligned}$$

which proves the first inequality in (3.7).

Observe that

$$\begin{aligned}
&\int_\gamma |f(\xi)|\left(\int_0^1 |P_g(t)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^2 dt\right)|d\xi| \qquad (3.9)\\
&=\int_\gamma |f(\xi)|\,|\xi|^{-2}\left(\int_0^1 |P_g(t)|\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^2 dt\right)|d\xi|.
\end{aligned}$$

Since

$$\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\leq(1-t)\left\|\frac{x}{\xi}\right\|+t\left\|\frac{y}{\xi}\right\|<1-t+t=1$$

for $\xi\in\gamma$, hence

$$\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}=\sum_{k=0}^{\infty}\left[(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right]^{k}.$$

Therefore

$$\begin{aligned}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\| &\leq\sum_{k=0}^{\infty}\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|^{k}\\ &=\left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=\left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=|\xi|\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\end{aligned}$$

for $\xi\in\gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2}\leq|\xi|^{2}\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}$$

for $\xi\in\gamma$.

Therefore

$$\begin{aligned}&\int_{\gamma}|f(\xi)||\xi|^{-2}\left(\int_{0}^{1}|P_{g}(t)|\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2}dt\right)|d\xi|\\ &\leq\int_{\gamma}|f(\xi)|\left(\int_{0}^{1}|P_{g}(t)|\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}dt\right)|d\xi|\end{aligned}$$

and the second inequality in (3.7) is proved.

By the triangle inequality we have

$$\begin{aligned}|\xi|-\|(1-t)x+ty\| &\geq|\xi|-(1-t)\|x\|-t\|y\|\\ &=(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)>0\end{aligned}$$

for $\xi\in\gamma$.

This implies that

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\leq\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-1}$$

and thus

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}\leq\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}$$

for $\xi\in\gamma$ and $t\in[0,1]$. This proves the third inequality in (3.7).

By the convexity of the power function $(\cdot)^{-2}$ we also have

$$\begin{aligned}&\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}\\ &\leq(1-t)\left(|\xi|-\|x\|\right)^{-2}+t\left(|\xi|-\|y\|\right)^{-2}\end{aligned}$$

for $t\in[0,1]$, which proves the fourth inequality in (3.7).

Finally, observe that

$$(1-t)\left(|\xi|-\|x\|\right)^{-2}+t\left(|\xi|-\|y\|\right)^{-2}\leq\max\left\{\left(|\xi|-\|x\|\right)^{-2},\left(|\xi|-\|y\|\right)^{-2}\right\}$$

and the last part of (3.7) is thus proved. □

We define

$$\begin{aligned}(3.10)\qquad & M\left(f,g,x,y,\gamma\right)\\ & :=\frac{1}{2\pi}\|y-x\|\int_{\gamma}|f\left(\xi\right)|\\ & \times\left(\int_{0}^{1}|P_{g}\left(t\right)|\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}dt\right)|d\xi|.\end{aligned}$$

We have the following bounds for $B\left(f,g,x,y,\gamma\right):$

Lemma 5. *With the assumptions of Theorem 2, we have*

$$\begin{aligned}(3.11)\qquad & M\left(f,g,x,y,\gamma\right)\\ & \leq\frac{1}{2\pi}\|y-x\|\\ & \times\begin{cases}\int_{0}^{1}|P_{g}\left(t\right)|dt\int_{\gamma}|f\left(\xi\right)|\max\left\{\left(|\xi|-\|x\|\right)^{-2},\left(|\xi|-\|y\|\right)^{-2}\right\}|d\xi|,\\ \\ \frac{1}{(2q-1)^{1/q}}\left(\int_{0}^{1}|P_{g}\left(t\right)|^{p}dt\right)^{1/p}\\ \times\left(\int_{\gamma}\frac{|f(\xi)|\left[(|\xi|-\|x\|)^{2q-1}-(|\xi|-\|y\|)^{2q-1}\right]}{(\|y\|-\|x\|)(|\xi|-\|x\|)^{2q-1}(|\xi|-\|y\|)^{2q-1}}|d\xi|\right)^{1/q},\\ \\ \sup_{t\in[0,1]}|P_{g}\left(t\right)|\int_{\gamma}\frac{|f(\xi)|}{(|\xi|-\|x\|)(|\xi|-\|y\|)}|d\xi|\end{cases}\end{aligned}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1.$

Proof. By Hölder's inequality we have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, that

$$\begin{aligned}(3.12)\qquad & \int_{0}^{1}|P_{g}\left(t\right)|\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}dt\\ & \leq\begin{cases}\int_{0}^{1}|P_{g}\left(t\right)|dt\sup_{t\in[0,1]}\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2},\\ \\ \left(\int_{0}^{1}|P_{g}\left(t\right)|^{p}dt\right)^{1/p}\\ \times\left(\int_{0}^{1}\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2q}dt\right)^{1/q},\\ \\ \sup_{t\in[0,1]}|P_{g}\left(t\right)|\int_{0}^{1}\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}dt.\end{cases}\end{aligned}$$

Observe that

$$\begin{aligned}
&\sup_{t\in[0,1]}\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}\\
&=\sup_{t\in[0,1]}\frac{1}{\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{2}}\\
&=\frac{1}{\inf_{t\in[0,1]}\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{2}}\\
&=\frac{1}{\min\left\{\left(|\xi|-\|x\|\right)^{2},\left(|\xi|-\|y\|\right)^{2}\right\}}\\
&=\max\left\{\left(|\xi|-\|x\|\right)^{-2},\left(|\xi|-\|y\|\right)^{-2}\right\},
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2q}dt\\
&=\frac{\left(|\xi|-\|y\|\right)^{-2q+1}-\left(|\xi|-\|x\|\right)^{-2q+1}}{\left(|\xi|-\|y\|-|\xi|+\|x\|\right)\left(-2q+1\right)}\\
&=\frac{\left(|\xi|-\|y\|\right)^{-2q+1}-\left(|\xi|-\|x\|\right)^{-2q+1}}{\left(\|x\|-\|y\|\right)\left(-2q+1\right)}\\
&=\frac{\left(|\xi|-\|y\|\right)^{-2q+1}-\left(|\xi|-\|x\|\right)^{-2q+1}}{\left(\|y\|-\|x\|\right)\left(2q-1\right)}\\
&=\frac{\left(|\xi|-\|x\|\right)^{2q-1}-\left(|\xi|-\|y\|\right)^{2q-1}}{\left(2q-1\right)\left(\|y\|-\|x\|\right)\left(|\xi|-\|x\|\right)^{2q-1}\left(|\xi|-\|y\|\right)^{2q-1}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\left[(1-t)\left(|\xi|-\|x\|\right)+t\left(|\xi|-\|y\|\right)\right]^{-2}dt\\
&=\frac{\left(|\xi|-\|x\|\right)-\left(|\xi|-\|y\|\right)}{\left(\|y\|-\|x\|\right)\left(|\xi|-\|x\|\right)\left(|\xi|-\|y\|\right)}\\
&=\frac{1}{\left(|\xi|-\|x\|\right)\left(|\xi|-\|y\|\right)},
\end{aligned}$$

then by (3.12) we derive (3.11). □

Corollary 1. *Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and x, $y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)$, $\sigma(y)\subset G$ while $\gamma\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x)$, $\sigma(y)\subset\operatorname{ins}(\gamma)$. If $g:[0,1]\to\mathbb{C}$ is Lebesgue integrable,* *then for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$\left\|\int_0^1 g(t)\,f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.13}$$
$$\leq \frac{1}{2\pi}\|y-x\|$$
$$\times\begin{cases}\int_0^1|P_g(t)|\,dt\int_\gamma|f(\xi)|\max\left\{(|\xi|-\|x\|)^{-2},(|\xi|-\|y\|)^{-2}\right\}|d\xi|,\\[2ex] \frac{1}{(2q-1)^{1/q}}\left(\int_0^1|P_g(t)|^p\,dt\right)^{1/p}\\ \times\left(\int_\gamma\frac{|f(\xi)|\left[(|\xi|-\|x\|)^{2q-1}-(|\xi|-\|y\|)^{2q-1}\right]}{(\|y\|-\|x\|)(|\xi|-\|x\|)^{2q-1}(|\xi|-\|y\|)^{2q-1}}|d\xi|\right)^{1/q},\\[2ex] \sup_{t\in[0,1]}|P_g(t)|\int_\gamma\frac{|f(\xi)|}{(|\xi|-\|x\|)(|\xi|-\|y\|)}|d\xi|.\end{cases}$$

Remark 1. *By taking* γ *parametrized by* $\xi(s)=Re^{2\pi is}$ *where* $s\in[0,1]$, *then* $d\xi(s)=2\pi iRe^{2\pi is}ds$, $|d\xi(s)|=2\pi Rds$, $|\xi|=R$ *and by (3.13) we get for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$\left\|\int_0^1 g(t)\,f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.14}$$
$$\leq R\|y-x\|$$
$$\times\begin{cases}\max\left\{(R-\|x\|)^{-2},(R-\|y\|)^{-2}\right\}\\ \times\int_0^1|P_g(t)|\,dt\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds,\\[2ex] \frac{1}{(2\pi R)^{1/p}(2q-1)^{1/q}}\left(\frac{\left[(R-\|x\|)^{2q-1}-(R-\|y\|)^{2q-1}\right]}{(\|y\|-\|x\|)(R-\|x\|)^{2q-1}(R-\|y\|)^{2q-1}}\right)^{1/q}\\ \times\left(\int_0^1|P_g(t)|^p\,dt\right)^{1/p}\left(\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds\right)^{1/q},\\[2ex] \frac{1}{(R-\|x\|)(R-\|y\|)}\sup_{t\in[0,1]}|P_g(t)|\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds,\end{cases}$$

where $x,\ y\in\mathcal{B}$, $x\neq y$ *with* $\sigma(x),\ \sigma(y)\subset D(0,R)\subset G$.

4. Bounds for Real Weights

When $g=p$, a real valued function, we have some simple bounds as follows:

Corollary 2. *Assume that* $f:G\to\mathbb{C}$ *is analytic on the convex domain* G *and* x, $y\in\mathcal{B}$, $x\neq y$ *with* $\sigma(x),\ \sigma(y)\subset G$ *while* $\gamma\subset G$ *is taken to be close rectifiable curve in* G *and such that* $\sigma(x),\ \sigma(y)\subset\operatorname{ins}(\gamma)$. *If* $p:[0,1]\to\mathbb{R}$ *is Lebesgue integrable that satisfies the condition*

$$0\leq\int_0^t p(s)\,ds\leq\int_0^1 p(s)\,ds \text{ for } t\in[0,1], \tag{4.1}$$

then

$$\left\|\int_0^1 p(t) f((1-t)x+ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt\right\| \tag{4.2}$$
$$\le \frac{1}{4\pi}\|y-x\| \int_0^1 \left[t^2+(1-t)^2\right] p(t)\,dt$$
$$\times \int_\gamma |f(\xi)| \max\left\{(|\xi|-\|x\|)^{-2}, (|\xi|-\|y\|)^{-2}\right\} |d\xi|.$$

If p is nonnegative on $[0,1]$, then the inequality (4.2) also holds.

Proof. From the second inequality in (4.1) we observe that

$$\int_0^t p(s)\,ds \le \int_0^t p(s)\,ds + \int_t^1 p(s)\,ds$$

that also gives that $\int_t^1 p(s)\,ds \ge 0$ for $t \in [0,1]$.

We then have

$$\int_0^1 |P_p(t)|\,dt \le \int_0^1 \left[\left|\int_t^1 p(s)\,ds\right| t + \left|\int_0^t p(s)\,ds\right| (1-t)\right] dt$$
$$= \int_0^1 \left[\left(\int_t^1 p(s)\,ds\right) t + \left(\int_0^t p(s)\,ds\right)(1-t)\right] dt.$$

Using integration by parts, we have

$$\int_0^1 \left(\int_t^1 p(s)\,ds\right) tdt = \left(\int_t^1 p(s)\,ds\right) \frac{t^2}{2}\bigg|_0^1 + \int_0^1 \frac{t^2}{2} p(t)\,dt$$
$$= \frac{1}{2}\int_0^1 t^2 p(t)\,dt$$

and

$$\int_0^1 \left(\int_0^t p(s)\,ds\right)(1-t)\,dt$$
$$= -\frac{1}{2}\left(\int_0^t p(s)\,ds\right)(1-t)^2\bigg|_0^1 + \frac{1}{2}\int_0^1 (1-t)^2 p(t)\,dt$$
$$= \frac{1}{2}\int_0^1 (1-t)^2 p(t)\,dt.$$

Therefore

$$\int_0^1 |P_p(t)|\,dt \le \frac{1}{2}\int_0^1 t^2 p(t)\,dt + \frac{1}{2}\int_0^1 (1-t)^2 p(t)\,dt$$
$$= \frac{1}{2}\int_0^1 \left[t^2+(1-t)^2\right] p(t)\,dt$$

and by the first inequality in (3.13) we deduce (4.2). □

Remark 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset G$. If p is as in Corollary 2, then*

$$\begin{aligned}(4.3)\qquad &\left\| \int_0^1 p(t) f((1-t)x+ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \right\| \\ &\le \frac{1}{2} R \frac{\|y-x\|}{\min\left\{ (R-\|x\|)^2, (R-\|y\|)^2 \right\}} \\ &\times \int_0^1 \left[t^2 + (1-t)^2 \right] p(t)\,dt \int_0^1 \left| f\left(Re^{2\pi i s}\right)\right| ds.\end{aligned}$$

If p is nonnegative on $[0,1]$, then the inequality (4.3) also holds.

We also have:

Corollary 3. *With the assumptions of Corollary 2, we have*

$$\begin{aligned}(4.4)\qquad &\left\| \int_0^1 p(t) f((1-t)x+ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \right\| \\ &\le \frac{1}{4\pi} \|y-x\| \left[\int_0^1 p(s)\,ds + \sup_{t\in[0,1]} \left| \int_t^1 p(s)\,ds - \int_0^t p(s)\,ds \right| \right] \\ &\times \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|x\|)(|\xi| - \|y\|)} |d\xi|.\end{aligned}$$

Proof. Observe that

$$|P_p(t)| \le \left(\int_t^1 p(s)\,ds \right) t + \left(\int_0^t p(s)\,ds \right)(1-t)$$

for all $t \in [0,1]$, which implies that

$$\begin{aligned}&\sup_{t\in[0,1]} |P_p(t)| \\ &\le \sup_{t\in[0,1]} \left[\left(\int_t^1 p(s)\,ds \right) t + \left(\int_0^t p(s)\,ds \right)(1-t) \right] \\ &= \sup_{t\in[0,1]} \left[\max\left\{ \int_t^1 p(s)\,ds, \int_0^t p(s)\,ds \right\} \right] \\ &= \sup_{t\in[0,1]} \frac{1}{2} \left[\int_t^1 p(s)\,ds + \int_0^t p(s)\,ds + \left| \int_t^1 p(s)\,ds - \int_0^t p(s)\,ds \right| \right] \\ &= \frac{1}{2} \left[\int_0^1 p(s)\,ds + \sup_{t\in[0,1]} \left| \int_t^1 p(s)\,ds - \int_0^t p(s)\,ds \right| \right]\end{aligned}$$

and by the third inequality in (3.13) we deduce (4.4). □

Remark 3. *With the assumptions of Remark 2 we have*

$$\begin{aligned}(4.5)\quad & \left\|\int_0^1 p(t)\, f((1-t)x+ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt\right\| \\ & \le \frac{1}{2}\frac{R\|y-x\|}{(R-\|x\|)(R-\|y\|)} \\ & \times \left[\int_0^1 p(s)\,ds + \sup_{t\in[0,1]}\left|\int_t^1 p(s)\,ds - \int_0^t p(s)\,ds\right|\right] \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds.\end{aligned}$$

5. Some Examples

We consider the weight $g(t)=\ell(t)=t$, $t\in[0,1]$ and observe that

$$\begin{aligned}P_\ell(t) &:= t\left(\int_t^1 \ell(s)\,ds\right) - (1-t)\left(\int_0^t \ell(s)\,ds\right) \\ &= t\left(\int_t^1 sds\right) - (1-t)\left(\int_0^t sds\right) = \frac{1}{2}t\left(1-t^2\right) - \frac{1}{2}(1-t)\,t^2 \\ &= \frac{1}{2}t(1-t).\end{aligned}$$

Then

$$\int_0^1 |P_\ell(t)|\,dt = \frac{1}{12},\quad \int_0^1 |P_\ell(t)|^p\,dt = \frac{1}{2^p}\frac{\Gamma^2(p+1)}{\Gamma(2p+2)}$$

and

$$\sup_{t\in[0,1]} |P_\ell(t)| = \frac{1}{8}.$$

Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and $x, y\in\mathcal{B}$, $x\neq y$ with $\sigma(x), \sigma(y)\subset G$ while $\gamma\subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y)\subset \operatorname{ins}(\gamma)$. Then by (3.13) for $g(t)=\ell(t)=t$, $t\in[0,1]$ we get for $p, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}(5.1)\quad & \left\|\int_0^1 tf((1-t)x+ty)\,dt - \frac{1}{2}\int_0^1 f((1-t)x+ty)\,dt\right\| \\ & \le \frac{1}{4\pi}\|y-x\| \\ & \times \begin{cases} \frac{1}{6}\int_\gamma |f(\xi)| \max\left\{(|\xi|-\|x\|)^{-2}, (|\xi|-\|y\|)^{-2}\right\} |d\xi|, \\ \\ \frac{\Gamma^{2/p}(p+1)}{(2q-1)^{1/q}\Gamma^{1/p}(2p+2)} \\ \times\left(\int_\gamma \frac{|f(\xi)|\left[(|\xi|-\|x\|)^{2q-1}-(|\xi|-\|y\|)^{2q-1}\right]}{(\|y\|-\|x\|)(|\xi|-\|x\|)^{2q-1}(|\xi|-\|y\|)^{2q-1}}|d\xi|\right)^{1/q}, \\ \\ \frac{1}{4}\int_\gamma \frac{|f(\xi)|}{(|\xi|-\|x\|)(|\xi|-\|y\|)}|d\xi|. \end{cases}\end{aligned}$$

If $x,\,y\in\mathcal{B}$, $x\neq y$ with $\sigma(x),\,\sigma(y)\subset D(0,R)\subset G$, then

$$\left\|\int_0^1 tf\left((1-t)\,x+ty\right)dt-\frac{1}{2}\int_0^1 f\left((1-t)\,x+ty\right)dt\right\| \tag{5.2}$$
$$\leq\frac{1}{2}R\,\|y-x\|$$
$$\times\begin{cases}\frac{1}{6}\max\left\{(R-\|x\|)^{-2},(R-\|y\|)^{-2}\right\}\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds,\\[2ex] \frac{\Gamma^{2/p}(p+1)}{(2\pi R)^{1/p}(2q-1)^{1/q}\Gamma^{1/p}(2p+2)(\|y\|-\|x\|)^{1/q}}\\ \times\left(\frac{\left[(R-\|x\|)^{2q-1}-(R-\|y\|)^{2q-1}\right]}{(R-\|x\|)^{2q-1}(R-\|y\|)^{2q-1}}\right)^{1/q}\left(\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds\right)^{1/q},\\[2ex] \frac{1}{4}\frac{1}{(R-\|x\|)(R-\|y\|)}\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds,\end{cases}$$

where $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z)=\left(\frac{1}{2}z\right)^\nu\sum_{k=0}^\infty\frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z)=\sum_{k=0}^\infty\frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a)=\exp a$, $a\in\mathcal{B}$. Assume that $x,\,y\in\mathcal{B}$ and $\|x\|,\,\|y\|<R$ for some $R>0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right|=\left|\exp\left[R\left(\cos(2\pi t)+i\sin(2\pi t)\right)\right]\right|=\exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi} d\theta$ and

$$\begin{aligned}
&\int_0^1 \exp\left[R\cos\left(2\pi t\right)\right] dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \exp\left[R\cos\theta\right] d\theta \\
&= \frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi} \exp\left[R\cos\theta\right] d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} \exp\left[R\cos\theta\right] d\theta\right) \\
&= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).
\end{aligned}$$

Assume that $x,\, y \in \mathcal{B}$, $x \neq y$ with $\sigma\left(x\right), \sigma\left(y\right) \subset D\left(0, R\right) \subset G$ and $g : [0,1] \to \mathbb{C}$ is Lebesgue integrable, then by taking the exponential function in (3.14), we get for $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$(5.3)\quad \begin{aligned}
&\left\| \int_0^1 p\left(t\right) \exp\left(\left(1-t\right)x + ty\right) dt - \left(\int_0^1 p\left(t\right) dt\right) \int_0^1 \exp\left(\left(1-t\right)x + ty\right) dt \right\| \\
&\leq R I_0(R) \left\| y - x \right\| \\
&\times \begin{cases}
\max\left\{\left(R - \left\|x\right\|\right)^{-2}, \left(R - \left\|y\right\|\right)^{-2}\right\} \int_0^1 \left|P_p\left(t\right)\right| dt, \\[2ex]
\frac{1}{(2\pi R)^{1/p} I_0^{1/p}(R)(2q-1)^{1/q}} \left(\frac{\left[(R-\|x\|)^{2q-1} - (R-\|y\|)^{2q-1}\right]}{(\|y\|-\|x\|)(R-\|x\|)^{2q-1}(R-\|y\|)^{2q-1}}\right)^{1/q} \\
\times \left(\int_0^1 \left|P_p\left(t\right)\right|^p dt\right)^{1/p}, \\[2ex]
\frac{1}{(R-\|x\|)(R-\|y\|)} \sup_{t\in[0,1]} \left|P_p\left(t\right)\right|.
\end{cases}
\end{aligned}$$

Let f be an analytic functions on the open disk $D\left(0, \rho\right)$ given by the *power series* $f\left(\lambda\right) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $\left(\left|\lambda\right| < \rho\right)$. If $\nu\left(a\right) < \rho$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} \left|\alpha_j\right| \left\|a^j\right\| < \infty$, and we can define $f\left(a\right)$ to be its sum. Clearly $f\left(a\right)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. We define the associated function

$$\lambda \mapsto f_A(\lambda) : D(0,\rho) \to \mathbb{C},\;\; f_A(\lambda) := \sum_{n=0}^{\infty} \left|\alpha_n\right| \lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1); \tag{5.4}$$

$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};$$

$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};$$

$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \lambda\in D(0,1); \tag{5.5}$$

$$g_A(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \lambda\in\mathbb{C};$$

$$h_A(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \lambda\in\mathbb{C};$$

$$l_A(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n\qquad\lambda\in\mathbb{C}, \tag{5.6}$$

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1);$$

$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\qquad\lambda\in D(0,1);$$

$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n,\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1);$$

where Γ is *Gamma function*.

Lemma 6. *Let f be an analytic functions on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. We have*

$$|f(\lambda)| \leq f_A(|\lambda|) \text{ for } \lambda \in D(0,\rho). \tag{5.7}$$

We can state the following result:

Proposition 1. *Let f be an analytic function on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. If $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$ and $0 < R < \rho$, then for p satisfying condition (4.1) we have*

$$\begin{aligned} &\left\| \int_0^1 p(t) f((1-t)x + ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \right\| \\ &\leq \frac{1}{2} \frac{R f_A(R) \|y-x\|}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \int_0^1 \left[t^2 + (1-t)^2\right] p(t)\,dt \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &\left\| \int_0^1 p(t) f((1-t)x + ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \right\| \\ &\leq \frac{1}{2} \frac{R f_A(R) \|y-x\|}{(R-\|x\|)(R-\|y\|)} \\ &\times \left[\int_0^1 p(s)\,ds + \sup_{t\in[0,1]} \left| \int_t^1 p(s)\,ds - \int_0^t p(s)\,ds \right| \right]. \end{aligned} \tag{5.9}$$

The proof follows by Lemma 6 and the inequalities (4.3) and (4.5).

As examples, one can consider the functions f and f_A listed above. For instance, if we take $f(z) = (1 \pm z)^{-1}$ and $x,\ y \in \mathcal{B}$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$ and $0 < R < 1$, then for p satisfying condition (4.1) we have

$$\begin{aligned} &\left\| \int_0^1 p(t) (1 \pm [(1-t)x+ty])^{-1}\,dt - \int_0^1 p(t)\,dt \int_0^1 (1 \pm [(1-t)x+ty])^{-1}\,dt \right\| \\ &\leq \frac{1}{2} \frac{R\|y-x\|}{(1-R)\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \int_0^1 \left[t^2 + (1-t)^2\right] p(t)\,dt \end{aligned}$$

and

$$\begin{aligned} &\left\| \int_0^1 p(t) (1 \pm [(1-t)x+ty])^{-1}\,dt - \int_0^1 p(t)\,dt \int_0^1 (1 \pm [(1-t)x+ty])^{-1}\,d \right\| \\ &\leq \frac{1}{2} \frac{R\|y-x\|}{(1-R)(R-\|x\|)(R-\|y\|)} \\ &\times \left[\int_0^1 p(s)\,ds + \sup_{t\in[0,1]} \left| \int_t^1 p(s)\,ds - \int_0^t p(s)\,ds \right| \right]. \end{aligned}$$

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