SOME NORM INEQUALITIES FOR DIFFERENCE BETWEEN WEIGHTED AND INTEGRAL MEANS RELATED TO ANALYTIC FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Assume that $f: G \to \mathbb{C}$ is analytic on the convex domain G and $x,\, y \in \mathcal{B}$, $x \neq y$ with spectra $\sigma(x),\, \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x),\, \sigma(y) \subset \operatorname{ins}(\gamma)$. In this paper we show among others that, if $g:[0,1] \to \mathbb{C}$ is Lebesgue integrable, then

$$\left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \int_0^1 g(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \right\|$$
$$\leq \frac{1}{2\pi}\|y-x\|$$
$$\times \begin{cases} \frac{1}{12}\sup_{t\in(0,1)}|M_g(t)| \int_\gamma |f(\xi)| \left((|\xi| - \|x\|)^{-2} + (|\xi| - \|y\|)^{-2} \right) |d\xi| \\ \\ \int_0^1 t(1-t)|M_g(t)|\,dt \int_\gamma \frac{|f(\xi)||d\xi|}{\min\left\{(|\xi|-\|x\|)^2,(|\xi|-\|y\|)^2\right\}}, \end{cases}$$

where

$$M_g(t) := \frac{1}{1-t}\int_t^1 g(s)\,ds - \frac{1}{t}\int_0^t g(s)\,ds,\ t \in (0,1).$$

Some examples for exponential function and functions defined by power series in Banach algebras are also given.

1. Introduction

In 1906, Fejér [18], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1. *Consider the integral* $\int_a^b f(t)\,p(t)\,dt$*, where* f *is a convex function in the interval* (a,b) *and* p *is a positive function in the same interval such that*

$$p(a+b-t) = p(t),\ \textit{for } a \leq t \leq b,$$

i.e., $y = p(t)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b), 0\right)$ *and is normal to the* t*-axis. Under those conditions the following inequalities are valid:*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \leq \int_a^b f(t)\,p(t)\,dt \leq \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt. \tag{1.1}$$

If f *is concave on* (a,b)*, then the inequalities reverse in (1.1)*

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Motivated by the above result, it is natural to ask for upper and lower bounds for the difference

$$\int_a^b p(t) f(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \int_a^b p(t) dt$$

under various assumptions for the functions involved.

In the recent paper [11] we obtained the following scalar inequality:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be convex and $p : [a,b] \to \mathbb{R}$ a Lebesgue integrable function such that*

$$\frac{1}{t-a} \int_a^t p(s) ds \le \frac{1}{b-t} \int_t^b p(s) ds \text{ for all } t \in (a,b).$$

Then we have the inequalities

$$\begin{aligned} f'_+(a) \int_a^b \left(t - \frac{a+b}{2}\right) p(t) dt \\ \le \int_a^b p(t) f(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \int_a^b p(t) dt \\ \le f'_-(b) \int_a^b \left(t - \frac{a+b}{2}\right) p(t) dt. \end{aligned} \tag{1.2}$$

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be convex and $p : [a,b] \to \mathbb{R}$ a monotonic nondecreasing function, then we have the inequalities (1.2).*

In order to extend this results for norms and functions with values in Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further: $\|ab\| \le \|a\| \|b\|$ for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.* We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a, b \in \operatorname{Inv}(\mathcal{B})$ then $ab \in \operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in \operatorname{Inv}(\mathcal{B})$;
(ii) $\{b \in \mathcal{B}: \|1 - b\| < 1\} \subset \operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B}) \ni a \longmapsto a^{-1} \in \operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{z \in \mathbb{C} : z - a \in \operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \operatorname{Inv}(\mathcal{B})$, $R_a(z) := (z-a)^{-1}$. For each $z, w \in \rho(a)$ we have the identity

$$R_a(w) - R_a(z) = (z-w) R_a(z) R_a(w).$$

We also have that

$$\sigma(a) \subset \{z \in \mathbb{C} : \ |z| \leq \|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a) = \sup\{|z| : z \in \sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a \in \mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda : \mathcal{B} \to \mathbb{C}$, the function $\lambda \circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n \in \mathbb{N}$ and $r > \nu(a)$, we have $a^n = \frac{1}{2\pi i}\int_{|\xi|=r} \xi^n (\xi - a)^{-1} d\xi$;
(v) We have $\nu(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i}\int_\delta f(\xi)(\xi - a)^{-1} d\xi, \tag{1.3}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \text{ins}(\delta)$, the inside of δ.

It is well known (see for instance [4, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.4}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [17] and [22].

For some recent norm inequalities for functions on Banach algebras, see [9], [2] and [6]-[15].

2. Some Preliminary Facts

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. By the convexity of G we have that $\sigma((1-t)x + ty) \subset G$ for all $t \in [0,1]$ and we can define the auxiliary function $f_{x,y} : [0,1] \to \mathcal{B}$ by

$$f_{x,y}(t) := f((1-t)x + ty), \ t \in [0,1]. \tag{2.1}$$

Lemma 1. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$. The function $f_{x,y} : [0,1] \to \mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f'_{x,y}(t) = D(f)((1-t)x + ty)(y - x) \tag{2.2}$$

for all $t \in (0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f'_{x,y}(0+) = D(f)(x)(y - x) \ \text{ and } \ f'_{x,y}(1-) = D(f)(y)(y - x). \tag{2.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} & \frac{f_{x,y}(t+h) - f(t)}{h} \\ & = \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\ & = \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f'_{x,y}(t) = \frac{df_{x,y}(t)}{dt} &= \lim_{h \to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \lim_{h \to 0} \left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h} \right] \\ &= D(f)((1-t)x + ty)(y-x), \end{aligned}$$

which proves (2.5).

The proof is similar for the lateral derivatives. □

Lemma 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{2.4}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x + hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned} & f(x+hv) - f(x) \\ & = \frac{1}{2\pi i} \left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi \right) \\ & = \frac{1}{2\pi i} \int_\gamma f(\xi) \left[(\xi - x - hv)^{-1} - (\xi - x)^{-1} \right] d\xi \\ & = \frac{1}{2\pi i} h \int_\gamma f(\xi) \left[(\xi - x - hv)^{-1} v (\xi - x)^{-1} \right] d\xi, \end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x+hv) - f(x)}{h} = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - x - hv)^{-1} v (\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (2.4). □

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f'_{x,y}(t) = \frac{1}{2\pi i} \int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1} (y-x)(\xi - (1-t)x - ty)^{-1} d\xi \tag{2.5}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f_{x,y}^{'}(0+) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - x)^{-1}(y-x)(\xi - x)^{-1} d\xi, \tag{2.6}$$

and

$$f_{x,y}^{'}(1-) = \frac{1}{2\pi i}\int_{\gamma} f(\xi)(\xi - y)^{-1}(y-x)(\xi - y)^{-1} d\xi. \tag{2.7}$$

The proof is obvious by Lemmas 1 and 2.

3. General Bounds

We start with the following representation result that is of interest in itself as well:

Lemma 4. *Assume that* $f : G \to \mathbb{C}$ *is analytic on the convex domain* G *and* x, $y \in \mathcal{B}$, $x \neq y$ *with* $\sigma(x)$, $\sigma(y) \subset G$ *while* $\gamma \subset G$ *is taken to be close rectifiable curve in* G *and such that* $\sigma(x)$, $\sigma(y) \subset \operatorname{ins}(\gamma)$. *If* $g : [0,1] \to \mathbb{C}$ *is Lebesgue integrable and*

$$M_g(t) := \frac{1}{1-t}\int_t^1 g(s)\, ds - \frac{1}{t}\int_0^t g(s)\, ds,\ t \in (0,1), \tag{3.1}$$

then

$$\begin{aligned} &\int_0^1 g(t) f((1-t)x + ty)\, dt - \int_0^1 g(t)\, dt \int_0^1 f((1-t)x + ty)\, dt \\ &= \frac{1}{2\pi i}\int_{\gamma} f(\xi)\left(\int_0^1 t(1-t) M_g(t)\right. \\ &\times \left.(\xi - (1-t)x - ty)^{-1}(y-x)(\xi - (1-t)x - ty)^{-1} d\xi\right) dt. \end{aligned} \tag{3.2}$$

Proof. Integrating by parts in the Bochner's integral, we have

$$\begin{aligned} &\int_0^t s f_{x,y}'(s)\, ds + \int_t^1 (s-1) f_{x,y}'(s)\, ds \\ &= t f_{x,y}(t) - \int_0^t f_{x,y}(s)\, ds - (t-1) f_{x,y}(t) - \int_t^1 f_{x,y}(s)\, ds \\ &= f_{x,y}(t) - \int_0^1 f_{x,y}(s)\, ds \end{aligned}$$

that holds for all $t \in [0,1]$.

If we multiply this identity by $g(t)$ and integrate over t in $[0,1]$, then we get

$$\begin{aligned} &\int_0^1 g(t) f_{x,y}(t)\, dt - \int_0^1 g(t)\, dt \int_0^1 f_{x,y}(s)\, ds \\ &= \int_0^1 g(t)\left(\int_0^t s f_{x,y}'(s)\, ds\right) dt + \int_0^1 g(t)\left(\int_t^1 (s-1) f_{x,y}'(s)\, ds\right) dt. \end{aligned} \tag{3.3}$$

Using integration by parts, we get

$$\begin{aligned}
(3.4)\qquad & \int_0^1 g(t)\left(\int_0^t s f'_{x,y}(s)\,ds\right)dt \\
&= \int_0^1 \left(\int_0^t s f'_{x,y}(s)\,ds\right) d\left(\int_0^t g(s)\,ds\right) \\
&= \left(\int_0^t g(s)\,ds\right)\left(\int_0^t s f'_{x,y}(s)\,ds\right)\Big|_0^1 - \int_0^1 \left(\int_0^t g(s)\,ds\right) t f'_{x,y}(t)\,dt \\
&= \left(\int_0^1 g(s)\,ds\right)\left(\int_0^1 s f'_{x,y}(s)\,ds\right) - \int_0^1 \left(\int_0^t g(s)\,ds\right) t f'_{x,y}(t)\,dt \\
&= \int_0^1 \left(\int_0^1 g(s)\,ds - \int_0^t g(s)\,ds\right) t f'_{x,y}(t)\,dt \\
&+ \int_0^1 \left(\int_t^1 g(s)\,ds\right) t f'_{x,y}(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
(3.5)\qquad & \int_0^1 g(t)\left(\int_t^1 (s-1) f'_{x,y}(s)\,ds\right)dt \\
&= \int_0^1 \left(\int_t^1 (s-1) f'_{x,y}(s)\,ds\right) d\left(\int_0^t g(s)\,ds\right) \\
&= \left(\int_t^1 (s-1) f'_{x,y}(s)\,ds\right)\left(\int_0^t g(s)\,ds\right)\Big|_0^1 \\
&+ \int_0^1 \left(\int_0^t g(s)\,ds\right)(t-1) f'_{x,y}(t)\,dt \\
&= \int_0^1 \left(\int_0^t g(s)\,ds\right)(t-1) f'_{x,y}(t)\,dt,
\end{aligned}$$

which proves the identity

$$\begin{aligned}
(3.6)\qquad & \int_0^1 g(t) f_{x,y}(t)\,dt - \int_0^1 g(t)\,dt \int_0^1 f_{x,y}(t)\,dt \\
&= \int_0^1 \left(\int_t^1 g(s)\,ds\right) t f'_{x,y}(t)\,dt + \int_0^1 \left(\int_0^t g(s)\,ds\right)(t-1) f'_{x,y}(t)\,dt \\
(3.7)\qquad &= \int_0^1 \left(\frac{1}{1-t}\int_t^1 g(s)\,ds - \frac{1}{t}\int_0^t g(s)\,ds\right) t(1-t) f'_{x,y}(t)\,dt \\
&= \int_0^1 t(1-t) M_g(t) f'_{x,y}(t)\,dt.
\end{aligned}$$

Using (2.5), we get

$$\int_0^1 g(t)\, f_{x,y}(t)\, dt - \int_0^1 g(t)\, dt \int_0^1 f_{x,y}(t)\, dt$$
$$= \int_0^1 t(1-t)\, M_g(t)$$
$$\times \left(\frac{1}{2\pi i} \int_\gamma f(\xi)\, (\xi - (1-t)x - ty)^{-1} (y-x)\, (\xi - (1-t)x - ty)^{-1} d\xi \right) dt$$

and by Fubini's theorem we derive (3.2). □

We have the following bounds:

Theorem 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. If $g : [0,1] \to \mathbb{C}$ is Lebesgue integrable, then*

$$\left\| \int_0^1 g(t)\, f((1-t)x + ty)\, dt - \int_0^1 g(t)\, dt \int_0^1 f((1-t)x + ty)\, dt \right\| \tag{3.8}$$
$$\leq \frac{1}{2\pi} \|y - x\| \int_\gamma |f(\xi)|$$
$$\times \left(\int_0^1 t(1-t)\, |M_g(t)| \left\| (\xi - (1-t)x - ty)^{-1} \right\|^2 dt \right) |d\xi|$$
$$\leq \frac{1}{2\pi} \|y - x\| \int_\gamma |f(\xi)|$$
$$\times \left(\int_0^1 t(1-t)\, |M_g(t)|\, (|\xi| - \|(1-t)x + ty\|)^{-2} dt \right) |d\xi|$$
$$\leq \frac{1}{2\pi} \|y - x\| \int_\gamma |f(\xi)|$$
$$\times \left(\int_0^1 t(1-t)\, |M_g(t)| \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2} dt \right) |d\xi|$$
$$\leq \frac{1}{2\pi} \|y - x\| \int_\gamma |f(\xi)|$$
$$\times \left(\int_0^1 t(1-t)\, |M_g(t)| \left[(1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2} \right] dt \right) |d\xi| .$$

Proof. By taking the norm and using the integral's properties, we get

$$\begin{aligned}
(3.9)\quad & \left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \int_0^1 g(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \right\| \\
&\le \frac{1}{2\pi} \int_\gamma |f(\xi)| \left(\int_0^1 t(1-t)|M_g(t)| \right. \\
&\times \left. \left\| (\xi-(1-t)x-ty)^{-1}(y-x)(\xi-(1-t)x-ty)^{-1} \right\| dt \right) |d\xi| \\
&\le \frac{1}{2\pi} \|y-x\| \\
&\times \int_\gamma |f(\xi)| \left(\int_0^1 t(1-t)|M_g(t)| \left\| (\xi-(1-t)x-ty)^{-1} \right\|^2 dt \right) |d\xi|,
\end{aligned}$$

which proves the first inequality in (3.8).

Observe that

(3.10)

$$\begin{aligned}
& \int_\gamma |f(\xi)| \left(\int_0^1 t(1-t)|M_g(t)| \left\| (\xi-(1-t)x-ty)^{-1} \right\|^2 dt \right) |d\xi| \\
&= \int_\gamma |f(\xi)| |\xi|^{-2} \left(\int_0^1 t(1-t)|M_g(t)| \left\| \left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} \right\|^2 dt \right) |d\xi|.
\end{aligned}$$

Since

$$\left\| (1-t)\frac{x}{\xi}+t\frac{y}{\xi} \right\| \le (1-t)\left\|\frac{x}{\xi}\right\| + t\left\|\frac{y}{\xi}\right\| < 1-t+t = 1$$

for $\xi \in \gamma$, hence

$$\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} = \sum_{k=0}^{\infty} \left[(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right]^k.$$

Therefore

$$\begin{aligned}
\left\| \left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} \right\| &\le \sum_{k=0}^{\infty} \left\| (1-t)\frac{x}{\xi}+t\frac{y}{\xi} \right\|^k \\
&= \left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1} \\
&= \left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1} \\
&= |\xi|\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}
\end{aligned}$$

for $\xi \in \gamma$, which implies that

$$\left\| \left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1} \right\|^2 \le |\xi|^2 \left(|\xi|-\|(1-t)x+ty\|\right)^{-2}$$

for $\xi \in \gamma$.

Therefore

$$\begin{aligned}&\int_\gamma |f(\xi)|\,|\xi|^{-2}\left(\int_0^1 t(1-t)\,|M_g(t)|\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^2 dt\right)|d\xi|\\ &\le \int_\gamma |f(\xi)|\left(\int_0^1 t(1-t)\,|M_g(t)|\,(|\xi|-\|(1-t)x+ty\|)^{-2}\,dt\right)|d\xi|\end{aligned}$$

and the second inequality in (3.8) is proved.

By the triangle inequality we have

$$\begin{aligned}|\xi|-\|(1-t)x+ty\| &\ge |\xi|-(1-t)\|x\|-t\|y\|\\ &=(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)>0\end{aligned}$$

for $\xi\in\gamma$.

This implies that

$$(|\xi|-\|(1-t)x+ty\|)^{-1}\le[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-1}$$

and thus

$$(|\xi|-\|(1-t)x+ty\|)^{-2}\le[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2}$$

for $\xi\in\gamma$ and $t\in[0,1]$. This proves the third inequality in (3.8).

By the convexity of the power function $(\cdot)^{-2}$ we also have

$$\begin{aligned}&[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2}\\ &\le(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\end{aligned}$$

for $t\in[0,1]$, which proves the fourth inequality in (3.8). □

Remark 1. *Since $t(1-t)\le\frac{1}{4}$ for all $t\in[0,1]$, then we also have the chain of inequalities*

$$\begin{aligned}&\left\|\int_0^1 g(t)f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \qquad (3.11)\\ &\le\frac{1}{8\pi}\|y-x\|\int_\gamma|f(\xi)|\left(\int_0^1|M_g(t)|\left\|(\xi-(1-t)x-ty)^{-1}\right\|^2dt\right)|d\xi|\\ &\le\frac{1}{8\pi}\|y-x\|\int_\gamma|f(\xi)|\left(\int_0^1|M_g(t)|(|\xi|-\|(1-t)x+ty\|)^{-2}dt\right)|d\xi|\\ &\le\frac{1}{8\pi}\|y-x\|\int_\gamma|f(\xi)|\\ &\times\left(\int_0^1|M_g(t)|[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)]^{-2}dt\right)|d\xi|\\ &\le\frac{1}{8\pi}\|y-x\|\int_\gamma|f(\xi)|\\ &\times\left(\int_0^1|M_g(t)|\left[(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\right]dt\right)|d\xi|\\ &=:T(f,g,x,y,\gamma),\end{aligned}$$

provided that $f : G \to \mathbb{C}$ *is analytic on the convex domain* G *and* $x, y \in \mathcal{B}$, $x \neq y$ *with* $\sigma(x), \sigma(y) \subset G$ *while* $\gamma \subset G$ *is taken to be close rectifiable curve in* G *and such that* $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$.

By utilising Hölder's integral inequality, we have for $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
&\int_0^1 |M_g(t)| \left[(1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2}\right] dt \\
&\leq \begin{cases} \sup_{t\in(0,1)} |M_g(t)| \int_0^1 \left[(1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2}\right] dt, \\ \left(\int_0^1 |M_g(t)|^p dt\right)^{1/p} \left(\int_0^1 \left[(1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2}\right]^q dt\right)^{1/q}, \\ \int_0^1 |M_g(t)| dt \sup_{t\in(0,1)} \left[(1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2}\right], \end{cases} \\
&= \begin{cases} \frac{1}{2} \sup_{t\in(0,1)} |M_g(t)| \left[(|\xi| - \|x\|)^{-2} + (|\xi| - \|y\|)^{-2}\right], \\ \left(\int_0^1 |M_g(t)|^p dt\right)^{1/p} \left(\frac{(|\xi|-\|y\|)^{-2(q+1)} - (|\xi|-\|x\|)^{-2(q+1)}}{(q+1)\left[(|\xi|-\|y\|)^{-2} - (|\xi|-\|x\|)^{-2}\right]}\right)^{1/q}, \\ \int_0^1 |M_g(t)| dt \max\left[(|\xi| - \|x\|)^{-2}, (|\xi| - \|y\|)^{-2}\right]. \end{cases}
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left\|\int_0^1 g(t) f((1-t)x + ty) dt - \int_0^1 g(t) dt \int_0^1 f((1-t)x + ty) dt\right\| \\
&\leq T(f, g, x, y, \gamma) \\
&\leq \frac{1}{8\pi} \|y - x\| \\
&\times \begin{cases} \frac{1}{2} \sup_{t\in(0,1)} |M_g(t)| \int_\gamma |f(\xi)| \left[(|\xi| - \|x\|)^{-2} + (|\xi| - \|y\|)^{-2}\right] |d\xi|, \\ \left(\int_0^1 |M_g(t)|^p dt\right)^{1/p} \\ \times \int_\gamma |f(\xi)| \left(\frac{(|\xi|-\|y\|)^{-2(q+1)} - (|\xi|-\|x\|)^{-2(q+1)}}{(q+1)\left[(|\xi|-\|y\|)^{-2} - (|\xi|-\|x\|)^{-2}\right]}\right)^{1/q} |d\xi|, \\ \int_0^1 |M_g(t)| dt \int_\gamma \frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2\right\}} |d\xi| \end{cases}
\end{aligned} \tag{3.12}$$

for $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

By taking γ parametrized by $\xi(s) = Re^{2\pi is}$ where $s \in [0,1]$, then $d\xi(s) = 2\pi iRe^{2\pi is}ds$, $|d\xi(s)| = 2\pi Rds$, $|\xi| = R$ and by (3.12) we get for $p, q > 1$ with

$\frac{1}{p}+\frac{1}{q}=1$,

$$\left\|\int_0^1 g(t) f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{3.13}$$
$$\le$$
$$\le \frac{R}{4}\|y-x\|\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds$$
$$\times\begin{cases}\frac{1}{2}\left[(R-\|x\|)^{-2}+(R-\|y\|)^{-2}\right]\sup_{t\in(0,1)}|M_g(t)|,\\ \\ \left(\frac{(R-\|y\|)^{-2(q+1)}-(R-\|x\|)^{-2(q+1)}}{(q+1)\left[(R-\|y\|)^{-2}-(R-\|x\|)^{-2}\right]}\right)^{1/q}\left(\int_0^1|M_g(t)|^p\,dt\right)^{1/p}\\ \\ \frac{1}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}\int_0^1|M_g(t)|\,dt\end{cases}$$

for $x,\ y\in\mathcal{B}$, $x\neq y$ with $\sigma(x),\ \sigma(y)\subset D(0,R)\subset G$.

Corollary 2. *With the assumptions of Theorem 3, we have*

$$\left\|\int_0^1 g(t) f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \le \frac{1}{24\pi}\|y-x\|\sup_{t\in(0,1)}|M_g(t)|\int_\gamma|f(\xi)|\left((|\xi|-\|x\|)^{-2}+(|\xi|-\|y\|)^{-2}\right) \tag{3.14}$$

Proof. By Theorem 3 we have

$$\begin{aligned}&\left\|\int_0^1 g(t) f((1-t)x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \\ &\le \frac{1}{2\pi}\|y-x\|\int_\gamma|f(\xi)| \\ &\times\left(\int_0^1 t(1-t)|M_g(t)|\left[(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\right]dt\right)|d\xi| \\ &\le \frac{1}{2\pi}\|y-x\|\sup_{t\in(0,1)}|M_g(t)|\int_\gamma|f(\xi)| \\ &\times\left(\int_0^1 t(1-t)\left[(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\right]dt\right)|d\xi|.\end{aligned} \tag{3.15}$$

Since, after some computations, we have

$$\begin{aligned}&\int_0^1 t(1-t)\left[(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\right]dt \\ &=\frac{1}{12}\left((|\xi|-\|x\|)^{-2}+(|\xi|-\|y\|)^{-2}\right),\end{aligned}$$

then by (3.15) we derive (3.14). □

We observe that the inequality (3.14) is better than the first inequality in (3.12).

If $x,\, y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)\,,\,\sigma(y)\subset D(0,R)\subset G$, then by (3.14) we get

$$\left\|\int_0^1 g(t)\, f((1-t)\,x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)\,x+ty)\,dt\right\| \leq \frac{R}{12}\left\|y-x\right\|\left((R-\|x\|)^{-2}+(R-\|y\|)^{-2}\right)\sup_{t\in(0,1)}|M_g(t)| \times\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds. \tag{3.16}$$

Corollary 3. *With the assumptions of Theorem 3, we have*

$$\left\|\int_0^1 g(t)\, f((1-t)\,x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)\,x+ty)\,dt\right\| \leq \frac{1}{2\pi}\left\|y-x\right\|\int_0^1 t(1-t)\,|M_g(t)|\,dt \times\int_\gamma\frac{|f(\xi)|\,|d\xi|\,.}{\min\left\{(|\xi|-\|x\|)^2,(|\xi|-\|y\|)^2\right\}}. \tag{3.17}$$

If $x,\, y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)\,,\,\sigma(y)\subset D(0,R)\subset G$, then by (3.17) we get

$$\left\|\int_0^1 g(t)\, f((1-t)\,x+ty)\,dt-\int_0^1 g(t)\,dt\int_0^1 f((1-t)\,x+ty)\,dt\right\| \leq R\frac{\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}\int_0^1 t(1-t)\,|M_g(t)|\,dt \times\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds. \tag{3.18}$$

4. The Case of Real Weights

For $g=p$, where p is a real Lebesgue integrable function $p:[0,1]\to\mathbb{R}$, assume that

$$M_p(t):=\frac{1}{1-t}\int_t^1 p(s)\,ds-\frac{1}{t}\int_0^t p(s)\,ds\geq 0 \tag{P}$$

for all $t\in(0,1)\,.$

If $p:[0,1]\to\mathbb{R}$ is a monotonic nondecreasing function, then

$$\frac{1}{t}\int_0^t p(s)\,ds\leq p(t)\leq\frac{1}{1-t}\int_t^1 p(s)\,ds$$

for $t\in(0,1)\,,$ which shows that nondecreasing functions satisfy condition (P).

We say that the function $p:[0,1]\to\mathbb{R}$ is asymmetric if

$$p(1-t)=-p(t)\ \text{ for all } t\in[0,1]\,.$$

If $p:[0,1]\to\mathbb{R}$ is asymmetric and Lebesgue integrable, then $\int_0^1 p(s)\,ds=0$. If $t\in[0,1]$ then $\int_0^t p(s)\,ds+\int_t^1 p(s)\,ds=0$, which implies that $\int_t^1 p(s)\,ds=-\int_0^t p(s)\,ds$.

If $p:[0,1]\to\mathbb{R}$ is an asymmetric Lebesgue integrable function such that

$$\int_0^t p(s)\,ds\leq 0 \text{ for all } t\in[0,1]\,, \tag{4.1}$$

or, equivalently

$$0 \le \int_t^1 p(s)\,ds \text{ for all } t \in [0,1], \tag{4.2}$$

then

$$\frac{1}{1-t}\int_t^1 p(s)\,ds - \frac{1}{t}\int_0^t p(s)\,ds = \frac{1}{1-t}\int_t^1 p(s)\,ds + \frac{1}{t}\int_t^1 p(s)\,ds$$
$$= \frac{1}{(1-t)\,t}\int_t^1 p(s)\,ds \ge 0,$$

which shows that asymmetric Lebesgue integrable functions that satisfy condition (4.1) have the property (P). For instance $p(s) = s - \frac{1}{2}$ is a such example.

Theorem 4. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \ne y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. If $p : [0,1] \to \mathbb{R}$ is Lebesgue integrable and satisfy property (P), then*

$$\left\| \int_0^1 p(t)\, f((1-t)x + ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt \right\| \tag{4.3}$$
$$\le \frac{1}{2\pi}\|y-x\| \int_0^1 \left(t - \frac{1}{2}\right) p(t)\,dt$$
$$\times \int_\gamma \frac{|f(\xi)|\,|d\xi|\,.}{\min\left\{(|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2\right\}}.$$

Proof. Since p satisfy property (P), then

$$\begin{aligned}\int_0^1 t(1-t)\,|M_p(t)|\,dt &= \int_0^1 t(1-t)\,M_p(t)\,dt \\ &= \int_0^1 \left(\frac{1}{1-t}\int_t^1 p(s)\,ds - \frac{1}{t}\int_0^t p(s)\,ds\right) t(1-t)\,dt \\ &= \int_0^1 \left(t\int_t^1 p(s)\,ds - (1-t)\int_0^t p(s)\,ds\right) dt.\end{aligned}$$

Using the integration by parts formula, we have

$$\begin{aligned} 0 &\le \int_0^1 \left(t\int_t^1 p(s)\,ds - (1-t)\int_0^t p(s)\,ds\right) dt \\ &= \left(t\int_t^1 p(s)\,ds - (1-t)\int_0^t p(s)\,ds\right) t\Big|_0^1 \\ &\quad - \int_0^1 t\left(t\int_t^1 p(s)\,ds - (1-t)\int_0^t p(s)\,ds\right)' dt \\ &= -\int_0^1 t\left(t\int_t^1 p(s)\,ds - (1-t)\int_0^t p(s)\,ds\right)' dt.\end{aligned}$$

Observe that, for all $t\in[0,1]$

$$\begin{aligned}&\left(t\int_t^1 p(s)\,ds-(1-t)\int_0^t p(s)\,ds\right)'\\&=\int_t^1 p(s)\,ds-tp(t)+\int_0^t p(s)\,ds-(1-t)\,p(t)\\&=\int_0^1 p(s)\,ds-p(t).\end{aligned}$$

Therefore

$$\begin{aligned}0&\le\int_0^1\left(t\int_t^1 p(s)\,ds-(1-t)\int_0^t p(s)\,ds\right)dt\\&=-\int_0^1 t\left(\int_0^1 p(s)\,ds-p(t)\right)dt=\int_0^1 tp(t)\,dt-\int_0^1 tdt\int_0^1 p(s)\,ds\\&=\int_0^1 tp(t)\,dt-\frac{1}{2}\int_0^1 p(s)\,ds=\int_0^1\left(t-\frac{1}{2}\right)p(t)\,dt\end{aligned}$$

and by (3.17) we deduce (4.3). □

Corollary 4. *Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and $x,\ y\in\mathcal{B},\ x\neq y$ with $\sigma(x),\ \sigma(y)\subset D(0,R)\subset G$. If $p:[0,1]\to\mathbb{R}$ is Lebesgue integrable and satisfy property (P), then*

$$\begin{aligned}&\left\|\int_0^1 p(t)\,f((1-t)\,x+ty)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)\,x+ty)\,dt\right\|\\&\le R\frac{\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}}\int_0^1\left(t-\frac{1}{2}\right)p(t)\,dt\\&\times\int_0^1\left|f\left(Re^{2\pi is}\right)\right|ds.\end{aligned}\tag{4.4}$$

Remark 2. *If $p:[0,1]\to\mathbb{R}$ is Lebesgue integrable satisfy property (P) and $m\le p(t)\le M$ for almost $t\in[0,1]$, then*

$$\begin{aligned}&\left\|\int_0^1 p(t)\,f((1-t)\,x+ty)\,dt-\int_0^1 p(t)\,dt\int_0^1 f((1-t)\,x+ty)\,dt\right\|\\&\le\frac{1}{16\pi}\|y-x\|\,(M-m)\\&\times\int_\gamma\frac{|f(\xi)|\,|d\xi|\,.}{\min\left\{(|\xi|-\|x\|)^2,(|\xi|-\|y\|)^2\right\}}.\end{aligned}\tag{4.5}$$

This follows by the fact that

$$\int_0^1\left(t-\frac{1}{2}\right)p(t)\,dt=\int_0^1\left(t-\frac{1}{2}\right)\left(p(t)-\frac{m+M}{2}\right)dt$$

and

$$\begin{aligned}0 &\leq \int_0^1 \left(t-\frac{1}{2}\right) p(t)\,dt = \left|\int_0^1 \left(t-\frac{1}{2}\right) p(t)\,dt\right| \\ &\leq \int_0^1 \left|t-\frac{1}{2}\right| \left|p(t)-\frac{m+M}{2}\right| dt \leq \frac{1}{2}(M-m)\int_0^1 \left|t-\frac{1}{2}\right| dt \\ &= \frac{1}{8}(M-m).\end{aligned}$$

Moreover, if $x,\ y \in \mathcal{B}$, $x \neq y$ *with* $\sigma(x), \sigma(y) \subset D(0,R) \subset G$, *then*

$$\begin{aligned}&\left\|\int_0^1 p(t) f((1-t)x+ty)\,dt - \int_0^1 p(t)\,dt \int_0^1 f((1-t)x+ty)\,dt\right\| \\ &\leq \frac{1}{8} R(M-m) \frac{\|y-x\|}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \\ &\times \int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| ds.\end{aligned} \tag{4.6}$$

5. Examples

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function*. For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t - \nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta} d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi i t}\right)\right| = \left|\exp\left[R(\cos(2\pi t) + i\sin(2\pi t))\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi} d\theta$ and

$$\begin{aligned}
&\int_0^1 \exp\left[R\cos\left(2\pi t\right)\right] dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \exp\left[R\cos\theta\right] d\theta \\
&= \frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi} \exp\left[R\cos\theta\right] d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} \exp\left[R\cos\theta\right] d\theta\right) \\
&= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R).
\end{aligned}$$

Assume that x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset D(0,R) \subset G$. If $p : [0,1] \to \mathbb{R}$ is Lebesgue integrable and satisfy property (P), then by (4.4) for the exponential function we get

$$\begin{aligned}
&\left\|\int_0^1 p(t)\exp\left((1-t)x + ty\right) dt - \int_0^1 p(t)\, dt \int_0^1 \exp\left((1-t)x + ty\right) dt\right\| \\
&\leq RI_0(R)\frac{\|y-x\|}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}}\int_0^1 \left(t - \frac{1}{2}\right) p(t)\, dt.
\end{aligned} \tag{5.1}$$

Let f be an analytic functions on the open disk $D(0,\rho)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $(|\lambda| < \rho)$. If $\nu(a) < \rho$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. We define the associated function

$$\lambda \mapsto f_A(\lambda) : D(0,\rho) \to \mathbb{C},\ f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned}
f(\lambda) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\lambda^n = \ln\frac{1}{1+\lambda},\ \lambda \in D(0,1); \\
g(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\lambda^{2n} = \cos\lambda,\ \lambda \in \mathbb{C}; \\
h(\lambda) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\lambda^{2n+1} = \sin\lambda,\ \lambda \in \mathbb{C}; \\
l(\lambda) &= \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda},\ \lambda \in D(0,1);
\end{aligned} \tag{5.2}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \ \lambda \in D(0,1); \tag{5.3}$$

$$g_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \ \lambda \in \mathbb{C};$$

$$h_A(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \ \lambda \in \mathbb{C};$$

$$l_A(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{5.4}$$

$$\frac{1}{2} \ln\left(\frac{1+\lambda}{1-\lambda}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!} \lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$${}_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)} \lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function*.

Lemma 5. *Let f be an analytic functions on the open disk $D(0,\rho)$ given by the power series $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$. We have*

$$|f(\lambda)| \le f_A(|\lambda|) \ \ \textit{for } \lambda \in D(0,\rho). \tag{5.5}$$

Assume that $x,\, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset G$. If $p : [0,1] \to \mathbb{R}$ is Lebesgue integrable and satisfy property (P), then by (4.4) we have

$$\left\| \int_0^1 p(t)\, f((1-t)\,x + ty)\, dt - \int_0^1 p(t)\, dt \int_0^1 f((1-t)\,x + ty)\, dt \right\| \tag{5.6}$$
$$\le R \frac{f_A(R)\, \|y-x\|}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \int_0^1 \left(t - \frac{1}{2}\right) p(t)\, dt.$$

If in this inequality we take $f(z) = \ln(1 \pm z)^{-1}$ with $z \in D(0,1)$, then we get for $0 < R < 1$

$$\left\| \int_0^1 p(t) \ln\left[1 \pm ((1-t)x + ty)\right]^{-1} dt - \int_0^1 p(t)\,dt \int_0^1 \ln\left[1 \pm ((1-t)x + ty)\right]^{-1} dt \right\| \leq R \frac{\ln(1-R)^{-1} \|y - x\|}{\min\left\{(R - \|x\|)^2, (R - \|y\|)^2\right\}} \int_0^1 \left(t - \frac{1}{2}\right) p(t)\,dt, \tag{5.7}$$

where x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset D(0,R)$ and $p : [0,1] \to \mathbb{R}$ is Lebesgue integrable and satisfy property (P).

As more examples, one can consider the other functions f and f_A listed above.

References

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