NONCOMMUTATIVE OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS IN BANACH ALGEBRAS

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Abstract. Let $\mathcal{B}$ be a complex Banach algebra. Assume that $x, y : [a,b] \to \mathcal{B}$ are continuous and y is strongly differentiable on (a,b). In this paper we show among others that

$$\begin{aligned}&\left\| \int_a^b x(t) y(t) dt - \left(\int_a^b x(s) ds \right) y(u) \right\| \\ &\le \int_u^b \left(\int_t^b \|x(s)\| ds \right) \|y'(t)\| dt + \int_a^u \left(\int_a^t \|x(s)\| ds \right) \|y'(t)\| dt \\ &\le \begin{cases} \sup_{t\in[a,b]} \|y'(t)\| \int_a^b |t-u| \|x(t)\| dt, \\ \\ \max\left\{ \int_u^b \|x(s)\| ds, \int_a^u \|x(s)\| ds \right\} \int_a^b \|y'(t)\| dt \end{cases}\end{aligned}$$

for all $u \in [a,b]$. Some applications for analytic functions of elements in Banach algebras with examples for exponential function are also given.

1. Introduction

Let X be a Banach space and $-\infty < a < b < \infty$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms of vectors or operators acting on X will be denoted by $\|\cdot\|$.

A function $f : [a,b] \to X$ is called *measurable* if there exists a sequence of simple functions $f_n : [a,b] \to X$ which converges punctually almost everywhere on $[a,b]$ at f. We recall also that a measurable function $f : [a,b] \to X$ is *Bochner integrable* if and only if its norm function (i.e. the function $t \mapsto \|f(t)\| : [a,b] \to \mathbb{R}_+$) is Lebesgue integrable on $[a,b]$.

The following generalization of Ostrowski scalar inequality holds [2].

Theorem 1. *Assume that $B : [a,b] \to \mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e.,*

$$\|B(t) - B(s)\| \le H|t-s|^\alpha \quad \text{for all} \quad t,\ s \in [a,b], \tag{1.1}$$

where $H > 0$ and $\alpha \in (0,1]$.

If $f : [a,b] \to X$ is Bochner integrable on $[a,b]$, then we have the inequality:

$$\left\| \int_a^b B(s) f(s) ds - B(t) \int_a^b f(s) ds \right\| \le H \int_a^b |t-s|^\alpha \|f(s)\| ds \tag{1.2}$$

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$$\leq H\times\begin{cases}\dfrac{(b-t)^{\alpha+1}+(t-a)^{\alpha+1}}{\alpha+1}\operatorname{esssup}_{t\in[a,b]}\|f(t)\|, \\ \\ \left[\dfrac{(b-t)^{q\alpha+1}+(t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}}\left(\int_a^b\|f(t)\|^p dt\right)^{1/p}, \ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\dfrac{1}{2}(b-a)+\left|t-\dfrac{a+b}{2}\right|\right]^{\alpha}\int_a^b\|f(t)\|dt\end{cases}$$

for any $t\in[a,b]$*, provided the integrals and* $\operatorname{esssup}_{t\in[a,b]}$ *from the right hand side are finite.*

In order to obtain similar results for two functions with values in Banach algebras, we need the following preparations.

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\|:\mathcal{B}\to[0,\infty)$ such that $(\mathcal{B},\|\cdot\|)$ is a normed space, and, further: $\|ab\|\leq\|a\|\|b\|$ for any $a,b\in\mathcal{B}$. The normed algebra $(\mathcal{B},\|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm*. We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\|=1$.

Let $\mathcal{B}$ be a unital algebra. An element $a\in\mathcal{B}$ is *invertible* if there exists an element $b\in\mathcal{B}$ with $ab=ba=1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by $\operatorname{Inv}(\mathcal{B})$. If $a,b\in\operatorname{Inv}(\mathcal{B})$ then $ab\in\operatorname{Inv}(\mathcal{B})$ and $(ab)^{-1}=b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a\in\mathcal{B}$ and $\lim_{n\to\infty}\|a^n\|^{1/n}<1$, then $1-a\in\operatorname{Inv}(\mathcal{B})$;
(ii) $\{b\in\mathcal{B}:\ \|1-b\|<1\}\subset\operatorname{Inv}(\mathcal{B})$;
(iii) $\operatorname{Inv}(\mathcal{B})$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\operatorname{Inv}(\mathcal{B})\ni a\longmapsto a^{-1}\in\operatorname{Inv}(\mathcal{B})$ is continuous.

For simplicity, we denote $z1$, where $z\in\mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by z. The *resolvent set* of $a\in\mathcal{B}$ is defined by

$$\rho(a):=\{z\in\mathbb{C}:\ z-a\in\operatorname{Inv}(\mathcal{B})\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a:\rho(a)\to\operatorname{Inv}(\mathcal{B})$, $R_a(z):=(z-a)^{-1}$. For each $z,w\in\rho(a)$ we have the identity

$$R_a(w)-R_a(z)=(z-w)R_a(z)R_a(w).$$

We also have that

$$\sigma(a)\subset\{z\in\mathbb{C}:\ |z|\leq\|a\|\}.$$

The *spectral radius* of a is defined as

$$\nu(a)=\sup\{|z|:z\in\sigma(a)\}.$$

Let $\mathcal{B}$ a unital Banach algebra and $a\in\mathcal{B}$. Then

(i) The resolvent set $\rho(a)$ is open in $\mathbb{C}$;
(ii) For any *bounded linear functionals* $\lambda:\mathcal{B}\to\mathbb{C}$, the function $\lambda\circ R_a$ is analytic on $\rho(a)$;
(iii) The spectrum $\sigma(a)$ is compact and nonempty in $\mathbb{C}$;
(iv) For each $n\in\mathbb{N}$ and $r>\nu(a)$, we have $a^n=\frac{1}{2\pi i}\int_{|\xi|=r}\xi^n(\xi-a)^{-1}d\xi$;
(v) We have $\nu(a)=\lim_{n\to\infty}\|a^n\|^{1/n}$.

Let $\mathcal{B}$ be a unital Banach algebra, $a \in \mathcal{B}$ and G be a domain of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic on G, we define an element $f(a)$ in $\mathcal{B}$ by

$$f(a) := \frac{1}{2\pi i} \int_{\delta} f(\xi) (\xi - a)^{-1} d\xi, \tag{1.3}$$

where $\delta \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(a) \subset \text{ins}(\delta)$, the inside of δ.

It is well known (see for instance [5, pp. 201-204]) that $f(a)$ does not depend on the choice of δ and the *Spectral Mapping Theorem* (SMT)

$$\sigma(f(a)) = f(\sigma(a)) \tag{1.4}$$

holds.

Concerning other basic definitions and facts in the theory of Banach algebras, the reader can consult the classical books [18] and [23].

For some recent norm inequalities for functions on Banach algebras, see [10], [3] and [7]-[16].

2. General Ostrowski Inequalities

We have the following weighted version of Ostrowski's inequality for two functions with values in Banach algebras:

Theorem 2. *Assume that $x, y : [a,b] \to \mathcal{B}$, are continuous and y is strongly differentiable on (a,b), then for all $u \in [a,b]$*

$$\left\| \int_a^b x(t) y(t) dt - \left(\int_a^b x(s) ds \right) y(u) \right\| \leq B(x,y,u), \tag{2.1}$$

where

$$B(x,y,u) := \int_u^b \left(\int_t^b \|x(s)\| ds \right) \|y'(t)\| dt + \int_a^u \left(\int_a^t \|x(s)\| ds \right) \|y'(t)\| dt.$$

We also have the bounds

$$\begin{aligned} & B(x,y,u) \\ & \leq \begin{cases} \int_u^b \|x(s)\| ds \int_u^b \|y'(t)\| dt, \\ \left[\int_u^b \left(\int_t^b \|x(s)\| ds \right)^p dt \right]^{1/p} \left(\int_u^b \|y'(t)\|^q dt \right)^{1/q}, \\ \int_u^b \left(\int_t^b \|x(s)\| ds \right) dt \sup_{t\in[u,b]} \|y'(t)\|, \end{cases} \\ & + \begin{cases} \int_a^u \|x(s)\| ds \int_a^u \|y'(t)\| dt, \\ \left[\int_a^u \left(\int_a^t \|x(s)\| ds \right)^p dt \right]^{1/p} \left(\int_a^u \|y'(t)\|^q dt \right)^{1/q}, \\ \int_a^u \left(\int_a^t \|x(s)\| ds \right) dt \sup_{t\in[a,u]} \|y'(t)\|, \end{cases} \end{aligned} \tag{2.2}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $u \in [a,b]$. Using the integration by parts formula for Bochner integral, we have

$$\begin{aligned}
&\int_u^b \left(\int_t^b x(s)\,ds\right) y'(t)\,dt \\
&= \left(\int_t^b x(s)\,ds\right) y(t)\Big]_u^b + \int_u^b x(t)\,y(t)\,dt \\
&= -\left(\int_u^b x(s)\,ds\right) y(u) + \int_u^b x(t)\,y(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^u \left(\int_a^t x(s)\,ds\right) y'(t)\,dt \\
&= \left(\int_a^t x(s)\,ds\right) y(t)\Big]_a^u - \int_a^u x(t)\,y(t)\,dt \\
&= \left(\int_a^u x(s)\,ds\right) y(u) - \int_a^u x(t)\,y(t)\,dt.
\end{aligned}$$

By subtracting the second identity from the first, we get

$$\begin{aligned}
&\int_u^b \left(\int_t^b x(s)\,ds\right) y'(t)\,dt - \int_a^u \left(\int_a^t x(s)\,ds\right) y'(t)\,dt \\
&= \int_u^b x(t)\,y(t)\,dt + \int_a^u x(t)\,y(t)\,dt \\
&- \left(\int_u^b x(s)\,ds\right) y(u) - \left(\int_a^u x(s)\,ds\right) y(u) \\
&= \int_a^b x(t)\,y(t)\,dt - \left(\int_a^b x(s)\,ds\right) y(u).
\end{aligned}$$

Therefore, we get the following identity of interest

$$\begin{aligned}
&\int_a^b x(t)\,y(t)\,dt - \left(\int_a^b x(s)\,ds\right) y(u) \\
&= \int_u^b \left(\int_t^b x(s)\,ds\right) y'(t)\,dt - \int_a^u \left(\int_a^t x(s)\,ds\right) y'(t)\,dt
\end{aligned} \tag{2.3}$$

for all $u \in [a,b]$.

If we take the norm in (2.3), then we get

$$
\begin{aligned}
(2.4)\qquad & \left\| \int_a^b x(t)\, y(t)\, dt - \left(\int_a^b x(s)\, ds \right) y(u) \right\| \\
& \leq \left\| \int_u^b \left(\int_t^b x(s)\, ds \right) y'(t)\, dt \right\| + \left\| \int_a^u \left(\int_a^t x(s)\, ds \right) y'(t)\, dt \right\| \\
& \leq \int_u^b \left\| \int_t^b x(s)\, ds \right\| \|y'(t)\|\, dt + \int_a^u \left\| \int_a^t x(s)\, ds \right\| \|y'(t)\|\, dt \\
& \leq \int_u^b \left(\int_t^b \|x(s)\|\, ds \right) \|y'(t)\|\, dt + \int_a^u \left(\int_a^t \|x(s)\|\, ds \right) \|y'(t)\|\, dt \\
& = B(x, y, u),
\end{aligned}
$$

which proves (2.1).

Using Hölder's inequality, we get for $p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, that

$$
\begin{aligned}
& \int_u^b \left(\int_t^b \|x(s)\|\, ds \right) \|y'(t)\|\, dt \\
& \leq \begin{cases} \sup_{t\in[u,b]} \left(\int_t^b \|x(s)\|\, ds \right) \int_u^b \|y'(t)\|\, dt, \\ \left[\int_u^b \left(\int_t^b \|x(s)\|\, ds \right)^p dt \right]^{1/p} \left(\int_u^b \|y'(t)\|^q\, dt \right)^{1/q}, \\ \int_u^b \left(\int_t^b \|x(s)\|\, ds \right) dt \sup_{t\in[u,b]} \|y'(t)\|, \end{cases} \\
& = \begin{cases} \left(\int_u^b \|x(s)\|\, ds \right) \int_u^b \|y'(t)\|\, dt, \\ \left[\int_u^b \left(\int_t^b \|x(s)\|\, ds \right)^p dt \right]^{1/p} \left(\int_u^b \|y'(t)\|^q\, dt \right)^{1/q}, \\ \int_u^b \left(\int_t^b \|x(s)\|\, ds \right) dt \sup_{t\in[u,b]} \|y'(t)\| \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_a^u \left(\int_a^t \|x(s)\|\, ds \right) \|y'(t)\|\, dt \\
& \leq \begin{cases} \sup_{t\in[a,u]} \left(\int_a^t \|x(s)\|\, ds \right) \int_a^u \|y'(t)\|\, dt, \\ \left[\int_a^u \left(\int_a^t \|x(s)\|\, ds \right)^p dt \right]^{1/p} \left(\int_a^u \|y'(t)\|^q\, dt \right)^{1/q}, \\ \int_a^u \left(\int_a^t \|x(s)\|\, ds \right) dt \sup_{t\in[a,u]} \|y'(t)\|, \end{cases}
\end{aligned}
$$

$$= \begin{cases} \left(\int_a^u \|x(s)\| \, ds\right) \int_a^u \|y'(t)\| \, dt, \\ \\ \left[\int_a^u \left(\int_a^t \|x(s)\| \, ds\right)^p dt\right]^{1/p} \left(\int_a^u \|y'(t)\|^q \, dt\right)^{1/q}, \\ \\ \int_a^u \left(\int_a^t \|x(s)\| \, ds\right) dt \sup_{t\in[a,u]} \|y'(t)\| . \end{cases}$$

By making use of (2.4), we derive (2.2). □

Corollary 1. *With the assumptions of Theorem 2, we have*

$$\left\| \int_a^b x(t) y(t) \, dt - \left(\int_a^b x(s) \, ds\right) y(u) \right\| \tag{2.5}$$
$$\leq \int_u^b \|x(s)\| \, ds \int_u^b \|y'(t)\| \, dt + \int_a^u \|x(s)\| \, ds \int_a^u \|y'(t)\| \, dt$$
$$\leq \begin{cases} \max\left\{\int_u^b \|x(s)\| \, ds, \int_a^u \|x(s)\| \, ds\right\} \int_a^b \|y'(t)\| \, dt \\ \\ \int_a^b \|x(s)\| \, ds \max\left\{\int_u^b \|y'(t)\| \, dt, \int_a^u \|y'(t)\| \, dt\right\} \end{cases}$$
$$\leq \int_a^b \|x(s)\| \, ds \int_a^b \|y'(t)\| \, dt,$$

for all $u \in [a,b]$.

The proof follows by the first branches in the bounds (2.2).

Remark 1. *If* $m \in (a,b)$ *is such that*

$$\int_a^u \|x(s)\| \, ds = \int_u^b \|x(s)\| \, ds = \frac{1}{2} \int_a^b \|x(s)\| \, ds, \tag{2.6}$$

then by (2.5) we get

$$\left\| \int_a^b x(t) y(t) \, dt - \left(\int_a^b x(s) \, ds\right) y(m) \right\| \leq \frac{1}{2} \int_a^b \|x(s)\| \, ds \int_a^b \|y'(t)\| \, dt. \tag{2.7}$$

Corollary 2. *With the assumptions of Theorem 2, we have*

$$\left\| \int_a^b x(t) y(t) \, dt - \left(\int_a^b x(s) \, ds\right) y(u) \right\| \tag{2.8}$$
$$\leq \int_u^b \left(\int_t^b \|x(s)\| \, ds\right) dt \sup_{t\in[u,b]} \|y'(t)\|$$
$$+ \int_a^u \left(\int_a^t \|x(s)\| \, ds\right) dt \sup_{t\in[a,u]} \|y'(t)\|$$
$$\leq \sup_{t\in[a,b]} \|y'(t)\| \int_a^b |t-u| \, \|x(t)\| \, dt,$$

for all $u \in [a,b]$.

Proof. From the third branches in the bounds (2.2) we have

$$\left\| \int_a^b x(t)\, y(t)\, dt - \left(\int_a^b x(s)\, ds \right) y(u) \right\| \tag{2.9}$$
$$\leq \int_u^b \left(\int_t^b \| x(s) \|\, ds \right) dt \sup_{t \in [u,b]} \| y'(t) \|$$
$$+ \int_a^u \left(\int_a^t \| x(s) \|\, ds \right) dt \sup_{t \in [a,u]} \| y'(t) \|$$
$$\leq \sup_{t \in [a,b]} \| y'(t) \| \left[\int_u^b \left(\int_t^b \| x(s) \|\, ds \right) dt + \int_a^u \left(\int_a^t \| x(s) \|\, ds \right) dt \right].$$

Using integration by parts, we have for $u \in [a,b]$ that

$$\int_u^b \left(\int_t^b \| x(s) \|\, ds \right) dt = \left(\int_t^b \| x(s) \|\, ds \right) t \Big|_u^b + \int_u^b t \| x(t) \|\, dt$$
$$= \int_u^b t \| x(t) \|\, dt - \left(\int_u^b \| x(s) \|\, ds \right) u$$
$$= \int_u^b (t-u) \| x(t) \|\, dt = \int_u^b |t-u| \| x(t) \|\, dt$$

and

$$\int_a^u \left(\int_a^t \| x(s) \|\, ds \right) dt = \left(\int_a^t \| x(s) \|\, ds \right) t \Big|_a^u - \int_a^u t \| x(t) \|\, dt$$
$$= \left(\int_a^u \| x(s) \|\, ds \right) u - \int_a^u t \| x(t) \|\, dt$$
$$= \int_a^u (u-t) \| x(t) \|\, dt = \int_a^u |t-u| \| x(t) \|\, dt,$$

which gives that

$$\int_u^b \left(\int_t^b \| x(s) \|\, ds \right) dt + \int_a^u \left(\int_a^t \| x(s) \|\, ds \right) dt$$
$$= \int_u^b |t-u| \| x(t) \|\, dt + \int_a^u |t-u| \| x(t) \|\, dt = \int_a^b |t-u| \| x(t) \|\, dt.$$

By making use of (2.9) we derive (2.8). □

Remark 2. *By making use of Hölder's integral inequality, we have for* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *that*

$$\int_a^b |t-u| \| x(t) \|\, dt \leq \begin{cases} \sup_{t \in [a,b]} |t-u| \int_a^b \| x(t) \|\, dt, \\ \left(\int_a^b |t-u|^q\, dt \right)^{1/q} \left(\int_a^b \| x(t) \|^p\, dt \right)^{1/p}, \\ \int_a^b |t-u|\, dt \sup_{t \in [a,b]} \| x(t) \|. \end{cases}$$

Since

$$\sup_{t\in[a,b]}|t-u| = \max\{u-a, b-u\} = \frac{1}{2}(b-a) + \left|u - \frac{a+b}{2}\right|,$$

$$\left(\int_a^b |t-u|^q\, dt\right)^{1/q} = \left[\frac{(u-a)^{q+1} + (b-u)^{q+1}}{q+1}\right]^{1/q}$$

and

$$\int_a^b |t-u|\, dt = \frac{(u-a)^2 + (b-u)^2}{2} = \frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2}\right)^2.$$

Then by (2.8) we derive the non-commutative Ostrowsky type inequalities for functions in Banach algebras

$$\left\|\int_a^b x(t)\, y(t)\, dt - \left(\int_a^b x(s)\, ds\right) y(u)\right\| \tag{2.10}$$
$$\leq \sup_{t\in[a,b]}\|y'(t)\| \begin{cases} \left[\frac{1}{2}(b-a) + \left|u - \frac{a+b}{2}\right|\right] \int_a^b \|x(t)\|\, dt, \\ \left[\frac{(u-a)^{q+1}+(b-u)^{q+1}}{q+1}\right]^{1/q} \left(\int_a^b \|x(t)\|^p\, dt\right)^{1/p}, \\ \left[\frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2}\right)^2\right] \sup_{t\in[a,b]}\|x(t)\|, \end{cases}$$

for all $u \in [a,b]$.

We also have:

Corollary 3. *With the assumptions of Theorem 2, we have for all* $u \in [a,b]$,

$$\left\|\int_a^b x(t)\, y(t)\, dt - \left(\int_a^b x(s)\, ds\right) y(u)\right\| \tag{2.11}$$
$$\leq \left[\left(\int_u^b \|x(s)\|\, ds\right)^p (b-u) + \left(\int_a^u \|x(s)\|\, ds\right)^p (u-a)\right]^{1/p}$$
$$\times \left(\int_a^b \|y'(t)\|^q\, dt\right)^{1/q}$$
$$\leq (b-a)^{1/p}\left[\left(\int_u^b \|x(s)\|\, ds\right)^p + \left(\int_a^u \|x(s)\|\, ds\right)^p\right]^{1/p}$$
$$\times \left(\int_a^b \|y'(t)\|^q\, dt\right)^{1/q}$$

for $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Observe that, by the elementary inequality for $a,\ b,\ c,\ d \geq 0$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$(ab + cd) \leq (a^p + c^p)^{1/p}(b^q + d^q)^{1/q}$$

we have

$$\begin{aligned}
(2.12)\qquad & \left[\int_u^b \left(\int_t^b \|x(s)\| \, ds\right)^p dt\right]^{1/p} \left(\int_u^b \|y'(t)\|^q \, dt\right)^{1/q} \\
& + \left[\int_a^u \left(\int_a^t \|x(s)\| \, ds\right)^p dt\right]^{1/p} \left(\int_a^u \|y'(t)\|^q \, dt\right)^{1/q} \\
& \leq \left[\int_u^b \left(\int_t^b \|x(s)\| \, ds\right)^p dt + \int_a^u \left(\int_a^t \|x(s)\| \, ds\right)^p dt\right]^{1/p} \\
& \times \left[\int_u^b \|y'(t)\|^q \, dt + \int_a^u \|y'(t)\|^q \, dt\right]^{1/q} \\
& = \left[\int_u^b \left(\int_t^b \|x(s)\| \, ds\right)^p dt + \int_a^u \left(\int_a^t \|x(s)\| \, ds\right)^p dt\right]^{1/p} \\
& \times \left(\int_a^b \|y'(t)\|^q \, dt\right)^{1/q} \\
& \leq \left[\left(\int_u^b \|x(s)\| \, ds\right)^p \int_u^b dt + \left(\int_a^u \|x(s)\| \, ds\right)^p \int_a^u dt\right]^{1/p} \\
& \times \left(\int_a^b \|y'(t)\|^q \, dt\right)^{1/q} \\
& = \left[\left(\int_u^b \|x(s)\| \, ds\right)^p (b-u) + \left(\int_a^u \|x(s)\| \, ds\right)^p (u-a)\right]^{1/p} \\
& \times \left(\int_a^b \|y'(t)\|^q \, dt\right)^{1/q} \\
& \leq (b-a)^{1/p} \left[\left(\int_u^b \|x(s)\| \, ds\right)^p + \left(\int_a^u \|x(s)\| \, ds\right)^p\right]^{1/p} \\
& \times \left(\int_a^b \|y'(t)\|^q \, dt\right)^{1/q},
\end{aligned}$$

which proves (2.11). □

Remark 3. *If $m \in (a,b)$ is such that (2.6) is valid, then by (2.11) we get*

$$\begin{aligned}
(2.13)\qquad & \left\| \int_a^b x(t)\, y(t)\, dt - \left(\int_a^b x(s)\, ds\right) y(m) \right\| \\
& \leq \frac{1}{2} (b-a)^{1/p} \int_a^b \|x(s)\| \, ds \left(\int_a^b \|y'(t)\|^q \, dt\right)^{1/q}.
\end{aligned}$$

Remark 4. *With the assumptions of Theorem 2 we have the mid-point inequality*

$$\left\| \int_a^b x(t)\, y(t)\, dt - \left(\int_a^b x(s)\, ds \right) y\left(\frac{a+b}{2} \right) \right\| \leq M(x,y), \tag{2.14}$$

where

$$M(x,y) := \int_{\frac{a+b}{2}}^b \left(\int_t^b \|x(s)\|\, ds \right) \|y'(t)\|\, dt + \int_a^{\frac{a+b}{2}} \left(\int_a^t \|x(s)\|\, ds \right) \|y'(t)\|\, dt.$$

We also have the bounds

$$M(x,y) \leq \begin{cases} \left(\int_{\frac{a+b}{2}}^b \|x(s)\|\, ds \right) \int_{\frac{a+b}{2}}^b \|y'(t)\|\, dt, \\ \\ \left[\int_{\frac{a+b}{2}}^b \left(\int_t^b \|x(s)\|\, ds \right)^p dt \right]^{1/p} \left(\int_{\frac{a+b}{2}}^b \|y'(t)\|^q\, dt \right)^{1/q}, \\ \\ \int_{\frac{a+b}{2}}^b \left(\int_t^b \|x(s)\|\, ds \right) dt \sup_{t \in \left[\frac{a+b}{2}, b\right]} \|y'(t)\|, \end{cases} \tag{2.15}$$

$$+ \begin{cases} \left(\int_a^{\frac{a+b}{2}} \|x(s)\|\, ds \right) \int_a^{\frac{a+b}{2}} \|y'(t)\|\, dt, \\ \\ \left[\int_a^{\frac{a+b}{2}} \left(\int_a^t \|x(s)\|\, ds \right)^p dt \right]^{1/p} \left(\int_a^{\frac{a+b}{2}} \|y'(t)\|^q\, dt \right)^{1/q}, \\ \\ \int_a^{\frac{a+b}{2}} \left(\int_a^t \|x(s)\|\, ds \right) dt \sup_{t \in \left[a, \frac{a+b}{2}\right]} \|y'(t)\|. \end{cases}$$

Making use of (2.5), we get

$$\begin{aligned} &\left\| \int_a^b x(t)\, y(t)\, dt - \left(\int_a^b x(s)\, ds \right) y\left(\frac{a+b}{2} \right) \right\| \\ &\leq \left(\int_{\frac{a+b}{2}}^b \|x(s)\|\, ds \right) \int_{\frac{a+b}{2}}^b \|y'(t)\|\, dt \\ &+ \left(\int_a^{\frac{a+b}{2}} \|x(s)\|\, ds \right) \int_a^{\frac{a+b}{2}} \|y'(t)\|\, dt \\ &\leq \begin{cases} \max\left\{ \int_{\frac{a+b}{2}}^b \|x(s)\|\, ds, \int_a^{\frac{a+b}{2}} \|x(s)\|\, ds \right\} \int_a^b \|y'(t)\|\, dt \\ \\ \int_a^b \|x(s)\|\, ds \max\left\{ \int_{\frac{a+b}{2}}^b \|y'(t)\|\, dt, \int_a^{\frac{a+b}{2}} \|y'(t)\|\, dt \right\} \end{cases} \\ &\leq \int_a^b \|x(s)\|\, ds \int_a^b \|y'(t)\|\, dt \end{aligned} \tag{2.16}$$

and by (2.8),

$$\left\| \int_a^b x(t)\, y(t)\, dt - \left(\int_a^b x(s)\, ds \right) y\left(\frac{a+b}{2} \right) \right\|$$

$$
\begin{aligned}
&\leq \int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}\|x(s)\|\,ds\right)dt\sup_{t\in\left[\frac{a+b}{2},b\right]}\|y'(t)\| \\
&+\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}\|x(s)\|\,ds\right)dt\sup_{t\in\left[a,\frac{a+b}{2}\right]}\|y'(t)\| \\
&\leq \sup_{t\in[a,b]}\|y'(t)\|\int_{a}^{b}\left|t-\frac{a+b}{2}\right|\|x(t)\|\,dt.
\end{aligned}
$$

From (2.10) we derive the non-commutative mid-point type inequalities for functions in Banach algebras

$$
\begin{aligned}
&\left\|\int_{a}^{b}x(t)\,y(t)\,dt-\left(\int_{a}^{b}x(s)\,ds\right)y\left(\frac{a+b}{2}\right)\right\| \\
&\leq \sup_{t\in[a,b]}\|y'(t)\|\begin{cases}\frac{1}{2}(b-a)\int_{a}^{b}\|x(t)\|\,dt, \\[2ex] \frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}\left(\int_{a}^{b}\|x(t)\|^{p}\,dt\right)^{1/p}, \\[2ex] \frac{1}{4}(b-a)^{2}\sup_{t\in[a,b]}\|x(t)\|.\end{cases}
\end{aligned} \tag{2.17}
$$

By (2.11) we obtain that

$$
\begin{aligned}
&\left\|\int_{a}^{b}x(t)\,y(t)\,dt-\left(\int_{a}^{b}x(s)\,ds\right)y\left(\frac{a+b}{2}\right)\right\| \\
&\leq \frac{(b-a)^{1/p}}{2^{1/p}}\left[\left(\int_{\frac{a+b}{2}}^{b}\|x(s)\|\,ds\right)^{p}+\left(\int_{a}^{\frac{a+b}{2}}\|x(s)\|\,ds\right)^{p}\right]^{1/p} \\
&\times\left(\int_{a}^{b}\|y'(t)\|^{q}\,dt\right)^{1/q}.
\end{aligned} \tag{2.18}
$$

If we consider the case when $x(t)=1$, $t\in[a,b]$, then by (2.1) we get

$$
\left\|\int_{a}^{b}y(t)\,dt-(b-a)\,y(u)\right\|\leq B(y,u), \tag{2.19}
$$

where

$$
B(y,u):=\int_{u}^{b}(b-t)\|y'(t)\|\,dt+\int_{a}^{u}(t-a)\|y'(t)\|\,dt.
$$

Subsequently, by (2.2) we also have the bounds

$$B(x,y,u) \leq \begin{cases} (b-u)\int_u^b \|y'(t)\|\,dt, \\ \frac{1}{(p+1)^{1/p}}(b-u)^{1+1/p}\left(\int_u^b \|y'(t)\|^q\,dt\right)^{1/q}, \\ \frac{1}{2}(b-u)^2 \sup_{t\in[u,b]}\|y'(t)\|, \end{cases} + \begin{cases} (u-a)\int_a^u \|y'(t)\|\,dt, \\ \frac{1}{(p+1)^{1/p}}(u-a)^{1+1/p}\left(\int_a^u \|y'(t)\|^q\,dt\right)^{1/q}, \\ \frac{1}{2}(u-a)^2 \sup_{t\in[a,u]}\|y'(t)\|. \end{cases} \tag{2.20}$$

From (2.5) we get

$$\begin{aligned} &\left\|\int_a^b y(t)\,dt-(b-a)\,y(u)\right\| \\ &\leq (b-u)\int_u^b\|y'(t)\|\,dt+(u-a)\int_a^u\|y'(t)\|\,dt \\ &\leq \begin{cases} \left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_a^b\|y'(t)\|\,dt, \\ (b-a)\max\left\{\int_u^b\|y'(t)\|\,dt,\int_a^u\|y'(t)\|\,dt\right\}. \end{cases} \end{aligned} \tag{2.21}$$

By (2.9) we also have the Ostrowski's inequality

$$\begin{aligned} &\left\|\int_a^b y(t)\,dt-(b-a)\,y(u)\right\| \\ &\leq \left[\frac{1}{4}(b-a)^2+\left(u-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\|y'(t)\| \end{aligned} \tag{2.22}$$

for $u\in[a,b]$.

3. Applications for Analytic Functions

Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and $x,\,y\in\mathcal{B}$, $x\neq y$ with $\sigma(x),\,\sigma(y)\subset G$. By the convexity of G we have that $\sigma((1-t)x+ty)\subset G$ for all $t\in[0,1]$ and we can define the auxiliary function $f_{x,y}:[0,1]\to\mathcal{B}$ by

$$f_{x,y}(t):=f((1-t)x+ty),\ t\in[0,1]. \tag{3.1}$$

Lemma 1. *Assume that $f:G\to\mathbb{C}$ is analytic on the convex domain G and x, $y\in\mathcal{B}$, $x\neq y$ with $\sigma(x)$, $\sigma(y)\subset G$. The function $f_{x,y}:[0,1]\to\mathcal{B}$ is differentiable on $(0,1)$ as a function of t and we have*

$$f'_{x,y}(t)=D(f)((1-t)x+ty)(y-x) \tag{3.2}$$

for all $t\in(0,1)$, where $D(f)(\cdot)(\cdot)$ is the Fréchet derivative of function f as a function defined on the Banach algebra $\mathcal{B}$ by equation (1.3).

We also have the lateral derivatives

$$f_{x,y}^{'}(0+) = D(f)(x)(y-x) \text{ and } f_{x,y}^{'}(1-) = D(f)(y)(y-x). \tag{3.3}$$

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}&\frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \frac{f((1-(t+h))x + (t+h)y) - f((1-t)x + ty)}{h} \\ &= \frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}f_{x,y}^{'}(t) &= \frac{df_{x,y}(t)}{dt} = \lim_{h\to 0} \frac{f_{x,y}(t+h) - f(t)}{h} \\ &= \lim_{h\to 0}\left[\frac{f((1-t)x + ty + h(y-x)) - f((1-t)x + ty)}{h}\right] \\ &= D(f)((1-t)x + ty)(y-x),\end{aligned}$$

which proves (3.2).

The proof is similar for the lateral derivatives. □

Lemma 2. *Assume that $f : G \to \mathbb{C}$ is analytic on the domain G and $x \in \mathcal{B}$, with $\sigma(x) \subset G$, then for $v \in \mathcal{B}$ we have*

$$D(f)(x)(v) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1} v (\xi - x)^{-1} d\xi, \tag{3.4}$$

where $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x) \subset \operatorname{ins}(\gamma)$, the inside of γ.

Proof. Let $v \in \mathcal{B}$. Then there exists a small interval around 0 such that for h in this interval $\sigma(x + hv) \subset \operatorname{ins}(\delta) \subset G$. Then

$$\begin{aligned}&f(x+hv) - f(x) \\ &= \frac{1}{2\pi i}\left(\int_\gamma f(\xi)(\xi - x - hv)^{-1} d\xi - \int_\gamma f(\xi)(\xi - x)^{-1} d\xi\right) \\ &= \frac{1}{2\pi i}\int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} - (\xi - x)^{-1}\right] d\xi \\ &= \frac{1}{2\pi i} h \int_\gamma f(\xi)\left[(\xi - x - hv)^{-1} v (\xi - x)^{-1}\right] d\xi,\end{aligned}$$

which gives for $h \neq 0$ that

$$\frac{f(x+hv) - f(x)}{h} = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x - hv)^{-1} v (\xi - x)^{-1} d\xi.$$

By taking the limit over $h \to 0$ and using the properties of the integral, we get (3.4). □

Lemma 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset G$ while $\gamma \subset G$ is taken to be close rectifiable curve in G and such that $\sigma(x), \sigma(y) \subset \operatorname{ins}(\gamma)$. Then*

$$f_{x,y}^{'}(t) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - (1-t)x - ty)^{-1}(y-x)(\xi - (1-t)x - ty)^{-1}d\xi \tag{3.5}$$

for all $t \in (0,1)$.

We also have the lateral derivatives

$$f_{x,y}^{'}(0+) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - x)^{-1}(y-x)(\xi - x)^{-1}d\xi, \tag{3.6}$$

and

$$f_{x,y}^{'}(1-) = \frac{1}{2\pi i}\int_\gamma f(\xi)(\xi - y)^{-1}(y-x)(\xi - y)^{-1}d\xi. \tag{3.7}$$

The proof is obvious by Lemmas 1 and 2.

Lemma 4. *With the assumptions of Lemma 3 we have the bounds*

$$\begin{aligned}
&\left\|f_{x,y}^{'}(t)\right\| \\
&\le \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\left\|(\xi - (1-t)x - ty)^{-1}\right\|^2|d\xi| \\
&\le \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|(|\xi| - \|(1-t)x + ty\|)^{-2}|d\xi| \\
&\le \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|)]^{-2}|d\xi| \\
&\le \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\left[(1-t)(|\xi| - \|x\|)^{-2} + t(|\xi| - \|y\|)^{-2}\right]|d\xi| \\
&\le \frac{1}{2\pi}\|y-x\|\int_\gamma \frac{|f(\xi)|}{\min\left\{(|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2\right\}}|d\xi|
\end{aligned} \tag{3.8}$$

for all $t \in [0,1]$.

Proof. By taking the norm in (3.5) we get

$$\begin{aligned}
&\left\|f_{x,y}^{'}(t)\right\| \\
&\le \frac{1}{2\pi}\int_\gamma |f(\xi)|\left\|(\xi - (1-t)x - ty)^{-1}(y-x)(\xi - (1-t)x - ty)^{-1}\right\||d\xi| \\
&\le \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)|\left\|(\xi - (1-t)x - ty)^{-1}\right\|^2|d\xi| \\
&= \frac{1}{2\pi}\|y-x\|\int_\gamma |f(\xi)||\xi|^{-2}\left\|\left(1 - (1-t)\frac{x}{\xi} - t\frac{y}{\xi}\right)^{-1}\right\|^2|d\xi|
\end{aligned} \tag{3.9}$$

for all $t \in [0,1]$, which proves the first inequality in (3.8).

Since

$$\left\|(1-t)\frac{x}{\xi} + t\frac{y}{\xi}\right\| \le (1-t)\left\|\frac{x}{\xi}\right\| + t\left\|\frac{y}{\xi}\right\| < 1 - t + t = 1$$

for $\xi \in \gamma$, hence

$$\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}=\sum_{k=0}^{\infty}\left[(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right]^{k}.$$

Therefore

$$\begin{aligned}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\| &\leq \sum_{k=0}^{\infty}\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|^{k}\\ &=\left(1-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=\left(\frac{|\xi|}{|\xi|}-\left\|(1-t)\frac{x}{\xi}+t\frac{y}{\xi}\right\|\right)^{-1}\\ &=|\xi|\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\end{aligned}$$

for $\xi \in \gamma$, which implies that

$$\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2}\leq |\xi|^{2}\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}$$

for $\xi \in \gamma$.

Therefore

$$\begin{aligned}&\int_{\gamma}|f(\xi)|\,|\xi|^{-2}\left\|\left(1-(1-t)\frac{x}{\xi}-t\frac{y}{\xi}\right)^{-1}\right\|^{2}|d\xi|\\ &\leq \int_{\gamma}|f(\xi)|\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}|d\xi|\end{aligned}$$

and we derive the second inequality in (3.8).

By the triangle inequality we have

$$\begin{aligned}|\xi|-\|(1-t)x+ty\| &\geq |\xi|-(1-t)\|x\|-t\|y\|\\ &=(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)>0\end{aligned}$$

for $\xi \in \gamma$.

This implies that

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-1}\leq\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-1}$$

and thus

$$\left(|\xi|-\|(1-t)x+ty\|\right)^{-2}\leq\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2}$$

for $\xi \in \gamma$ and $t \in [0,1]$. This proves the third inequality in (3.8).

By the convexity of the power function $(\cdot)^{-2}$ we also have

$$\begin{aligned}&\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2}\\ &\leq (1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\end{aligned}$$

for $t \in [0,1]$, which proves the fourth inequality in (3.8).

Finally, observe that

$$(1-t)(|\xi|-\|x\|)^{-2}+t(|\xi|-\|y\|)^{-2}\leq \max\left\{(|\xi|-\|x\|)^{-2},(|\xi|-\|y\|)^{-2}\right\}$$

and the last part of (3.8) is thus proved. □

We have the following bounds for the p-norm of $f_{x,y}^{'}$.

Proposition 1. *Assume that* $f:G\to\mathbb{C}$ *is analytic on the convex domain* G *and* $x,\ y\in\mathcal{B},\ x\neq y$ *with* $\sigma(x),\ \sigma(y)\subset G$ *while* $\gamma\subset G$ *is taken to be close rectifiable curve in* G *and such that* $\sigma(x),\ \sigma(y)\subset\operatorname{ins}(\gamma)$. *Then*

$$\sup_{t\in[0,1]}\left\Vert f_{x,y}^{'}(t)\right\Vert \leq \frac{1}{2\pi}\left\Vert y-x\right\Vert \int_{\gamma}\frac{\left\vert f(\xi)\right\vert}{\min\left\{(|\xi|-\|x\|)^{2},(|\xi|-\|y\|)^{2}\right\}}\left\vert d\xi\right\vert, \tag{3.10}$$

$$\int_{0}^{1}\left\Vert f_{x,y}^{'}(t)\right\Vert dt \leq \frac{1}{2\pi}\left\Vert y-x\right\Vert \int_{\gamma}\frac{\left\vert f(\xi)\right\vert}{(|\xi|-\|y\|)(|\xi|-\|x\|)}\left\vert d\xi\right\vert \tag{3.11}$$

and

$$\begin{aligned}&\left(\int_{0}^{1}\left\Vert f_{x,y}^{'}(t)\right\Vert^{p}dt\right)^{1/p} \\ &\leq \frac{1}{2\pi}\left\Vert y-x\right\Vert\left(\int_{\gamma}\left\vert f(\xi)\right\vert^{q}\left\vert d\xi\right\vert\right)^{1/q} \\ &\times\left(\frac{1}{(2p-1)(\|y\|-\|x\|)}\int_{\gamma}\frac{(|\xi|-\|x\|)^{2p-1}-(|\xi|-\|y\|)^{2p-1}}{(|\xi|-\|x\|)^{2p-1}(|\xi|-\|y\|)^{2p-1}}\left\vert d\xi\right\vert\right)^{1/p},\end{aligned} \tag{3.12}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. The inequality (3.10) is obvious by (3.8).

From (3.8) we get, by taking the integral and by using Fubini's theorem, that

$$\begin{aligned}&\int_{0}^{1}\left\Vert f_{x,y}^{'}(t)\right\Vert dt \\ &\leq \frac{1}{2\pi}\left\Vert y-x\right\Vert\int_{\gamma}\left\vert f(\xi)\right\vert\left(\int_{0}^{1}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2}dt\right)\left\vert d\xi\right\vert.\end{aligned} \tag{3.13}$$

Observe that

$$\begin{aligned}&\int_{0}^{1}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-2}dt \\ &=-\frac{1}{\|x\|-\|y\|}\int_{0}^{1}\frac{d}{dt}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-1}dt \\ &=-\frac{1}{\|x\|-\|y\|}\left[(1-t)(|\xi|-\|x\|)+t(|\xi|-\|y\|)\right]^{-1}\Big|_{0}^{1} \\ &=\frac{1}{\|y\|-\|x\|}\left[(|\xi|-\|y\|)^{-1}-(|\xi|-\|x\|)^{-1}\right] \\ &=\frac{1}{\|y\|-\|x\|}\frac{\|y\|-\|x\|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}=\frac{1}{(|\xi|-\|y\|)(|\xi|-\|x\|)},\end{aligned}$$

for $\|y\|\neq\|x\|$, which, by (3.13), proves (3.11).

If $\|y\|=\|x\|$, then (3.11) also holds.

From (3.8) we also have for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
&\left\| f_{x,y}^{\prime}(t) \right\| \\
&\le \frac{1}{2\pi} \|y - x\| \int_{\gamma} |f(\xi)| \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2} |d\xi| \\
&\le \frac{1}{2\pi} \|y - x\| \left(\int_{\gamma} |f(\xi)|^{q} |d\xi| \right)^{1/q} \\
&\times \left(\int_{\gamma} \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2p} \right)^{1/p}
\end{aligned}$$

and by taking the power p we get

$$\begin{aligned}
\left\| f_{x,y}^{\prime}(t) \right\|^{p} &\le \left[\frac{1}{2\pi} \|y - x\| \left(\int_{\gamma} |f(\xi)|^{q} |d\xi| \right)^{1/q} \right]^{p} \\
&\times \left(\int_{\gamma} \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2p} \right)
\end{aligned}$$

for $t \in [0,1]$.

Integrating this inequality on $[0,1]$, we get by Fubini's theorem that

$$\begin{aligned}
\int_0^1 \left\| f_{x,y}^{\prime}(t) \right\|^{p} dt &\le \left[\frac{1}{2\pi} \|y - x\| \left(\int_{\gamma} |f(\xi)|^{q} |d\xi| \right)^{1/q} \right]^{p} \\
&\times \int_0^1 \left(\int_{\gamma} \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2p} |d\xi| \right) dt \\
&= \left[\frac{1}{2\pi} \|y - x\| \left(\int_{\gamma} |f(\xi)|^{q} |d\xi| \right)^{1/q} \right]^{p} \\
&\times \int_{\gamma} \left(\int_0^1 \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2p} dt \right) |d\xi| \\
&= \left[\frac{1}{2\pi} \|y - x\| \left(\int_{\gamma} |f(\xi)|^{q} |d\xi| \right)^{1/q} \right]^{p} \\
&\int_{\gamma} \left(\int_0^1 \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2p} dt \right) |d\xi| .
\end{aligned} \tag{3.14}$$

Observe that

$$\begin{aligned}
&\int_{\gamma} \left(\int_0^1 \left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2p} dt \right) |d\xi| \\
&= \int_{\gamma} \left(\left. \frac{\left[(1-t)(|\xi| - \|x\|) + t(|\xi| - \|y\|) \right]^{-2p+1}}{(1-2p)(\|x\| - \|y\|)} \right|_0^1 \right) |d\xi|
\end{aligned}$$

$$= \int_\gamma \frac{(|\xi| - \|y\|)^{-2p+1} - (|\xi| - \|x\|)^{-2p+1}}{(1-2p)(\|x\| - \|y\|)} |d\xi|$$
$$= \int_\gamma \frac{\frac{1}{(|\xi|-\|y\|)^{2p-1}} - \frac{1}{(|\xi|-\|x\|)^{2p-1}}}{(1-2p)(\|x\| - \|y\|)} |d\xi|$$
$$= \frac{1}{(2p-1)(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^{2p-1} - (|\xi| - \|y\|)^{2p-1}}{(|\xi| - \|x\|)^{2p-1} (|\xi| - \|y\|)^{2p-1}} |d\xi| ,$$

then by (3.14) we get

$$\int_0^1 \left\| f_{x,y}^{'}(t) \right\|^p dt$$
$$\leq \left[\frac{1}{2\pi} \|y-x\| \left(\int_\gamma |f(\xi)|^q |d\xi| \right)^{1/q} \right]^p$$
$$\times \frac{1}{(2p-1)(\|y\| - \|x\|)} \int_\gamma \frac{(|\xi| - \|x\|)^{2p-1} - (|\xi| - \|y\|)^{2p-1}}{(|\xi| - \|x\|)^{2p-1} (|\xi| - \|y\|)^{2p-1}} |d\xi| ,$$

which proves (3.12) □

We can state now the main result of this section:

Theorem 3. *Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and x, $y \in \mathcal{B}$, $x \neq y$ with $\sigma(x)$, $\sigma(y) \subset G$. If $g : [0,1] \to \mathcal{B}$ is continuous, then for all $u \in [0,1]$ we have*

$$(3.15) \quad \left\| \int_0^1 g(t) f((1-t)x + ty) dt - \left(\int_0^1 g(s) ds \right) f((1-u)x + uy) \right\|$$
$$\leq \frac{1}{2\pi} \|y-x\| \max \left\{ \int_u^1 \|g(s)\| ds, \int_0^u \|g(s)\| ds \right\}$$
$$\times \int_\gamma \frac{|f(\xi)|}{(|\xi| - \|y\|)(|\xi| - \|x\|)} |d\xi| ,$$

$$(3.16) \quad \left\| \int_0^1 g(t) f((1-t)x + ty) dt - \left(\int_0^1 g(s) ds \right) f((1-u)x + uy) \right\|$$
$$\leq \frac{1}{2\pi} \|y-x\| \int_0^1 |t-u| \|g(t)\| dt$$
$$\times \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi|$$
$$\leq \frac{1}{2\pi} \|y-x\| \left[\frac{1}{4} + \left(u - \frac{1}{2} \right)^2 \right] \sup_{t\in[0,1]} \|g(t)\|$$
$$\times \int_\gamma \frac{|f(\xi)|}{\min\left\{ (|\xi| - \|x\|)^2, (|\xi| - \|y\|)^2 \right\}} |d\xi|$$

and

$$\left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right) f((1-u)x+uy) \right\| \tag{3.17}$$
$$\leq \frac{1}{2\pi}\|y-x\| \left(\int_\gamma |f(\xi)|^p |d\xi|\right)^{1/p}$$
$$\times \left[\left(\int_u^1 \|g(s)\|\,ds\right)^p (1-u) + \left(\int_0^u \|g(s)\|\,ds\right)^p u\right]^{1/p}$$
$$\times \left(\frac{1}{(2q-1)(\|y\|-\|x\|)} \int_\gamma \frac{(|\xi|-\|x\|)^{2q-1}-(|\xi|-\|y\|)^{2q-1}}{(|\xi|-\|x\|)^{2q-1}(|\xi|-\|y\|)^{2q-1}} |d\xi|\right)^{1/q}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If we use (2.5) for the interval $[0,1]$ and the functions $x(t)=g(t)$, $y(t)=f_{x,y}(t)$, $t\in[0,1]$, then we get

$$\left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right) f((1-u)x+uy) \right\|$$
$$\leq \max\left\{\int_u^1 \|g(s)\|\,ds, \int_0^u \|g(s)\|\,ds\right\} \int_0^1 \left\|f'_{x,y}(t)\right\| dt$$

for all $u\in[0,1]$.

By making use of the inequality (3.11) we then deduce (3.15).

The first inequality (3.16) follows similarly by (2.8) and (3.10). The second part is obvious.

The inequality (3.17) follows by (2.11) and (3.12) for q instead of p. □

Remark 5. *From (3.16) we get the mid-point inequality*

$$\left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right) f\left(\frac{x+y}{2}\right) \right\| \tag{3.18}$$
$$\leq \frac{1}{2\pi}\|y-x\| \int_0^1 \left|t-\frac{1}{2}\right| \|g(t)\|\,dt$$
$$\times \int_\gamma \frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2\right\}} |d\xi|$$
$$\leq \frac{1}{8\pi}\|y-x\| \sup_{t\in[0,1]} \|g(t)\| \int_\gamma \frac{|f(\xi)|}{\min\left\{(|\xi|-\|x\|)^2, (|\xi|-\|y\|)^2\right\}} |d\xi|,$$

for any continuous $g:[0,1]\to\mathcal{B}$.

If g *is symmetric in norm, namely* $\|g(1-t)\|=\|g(t)\|$, $t\in[0,1]$, *then*

$$\int_0^{1/2} \|g(t)\|\,dt = \int_{1/2}^1 \|g(t)\|\,dt = \frac{1}{2}\int_0^1 \|g(t)\|\,dt$$

and by (3.15) we get

$$\left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right) f\left(\frac{x+y}{2}\right)\right\| \le \frac{1}{4\pi}\|y-x\| \int_0^1 \|g(t)\|\,dt \int_\gamma \frac{|f(\xi)|}{(|\xi|-\|y\|)(|\xi|-\|x\|)}|d\xi|. \tag{3.19}$$

4. The Case of Circular Paths

We consider the circular path $\xi(s) = Re^{2\pi i s}$ where $s \in [0,1]$, then $d\xi(s) = 2\pi i Re^{2\pi i s} ds$, $|d\xi(s)| = 2\pi R ds$ and $|\xi| = R$.

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset G$. Then by Proposition 1 we derive the simpler inequalities

$$\sup_{t\in[0,1]} \left\|f'_{x,y}(t)\right\| \le \frac{R\|y-x\|}{\min\left\{(R-\|x\|)^2, (R-\|y\|)^2\right\}} \int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| ds, \tag{4.1}$$

$$\int_0^1 \left\|f'_{x,y}(t)\right\| dt \le \frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)} \int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| ds, \tag{4.2}$$

and

$$\left(\int_0^1 \left\|f'_{x,y}(t)\right\|^p dt\right)^{1/p} \le R\|y-x\| \left(\int_0^1 \left|f\left(Re^{2\pi i s}\right)\right|^q ds\right)^{1/q} \times \left(\frac{1}{(2p-1)(\|y\|-\|x\|)} \frac{(R-\|x\|)^{2p-1} - (R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right)^{1/p}, \tag{4.3}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Assume that $f : G \to \mathbb{C}$ is analytic on the convex domain G and $x, y \in \mathcal{B}$, $x \neq y$ with $\sigma(x), \sigma(y) \subset D(0,R) \subset G$. If $g : [0,1] \to \mathcal{B}$ is continuous, then for all $u \in [0,1]$ we have by Theorem 3

$$\left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right) f((1-u)x+uy)\right\| \le \frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)} \max\left\{\int_u^1 \|g(s)\|\,ds, \int_0^u \|g(s)\|\,ds\right\} \times \int_0^1 \left|f\left(Re^{2\pi i s}\right)\right| ds, \tag{4.4}$$

$$\begin{aligned}
(4.5)\qquad & \left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right) f((1-u)x+uy) \right\| \\
& \le \frac{R\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \int_0^1 |t-u| \|g(t)\|\,dt \\
& \times \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds \\
& \le \frac{R\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \left[\frac{1}{4}+\left(u-\frac{1}{2}\right)^2\right] \sup_{t\in[0,1]} \|g(t)\| \\
& \times \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds
\end{aligned}$$

and

$$\begin{aligned}
(4.6)\qquad & \left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right) f((1-u)x+uy) \right\| \\
& \le R\|y-x\| \left(\int_0^1 \left|f\left(Re^{2\pi is}\right)\right|^q ds\right)^{1/q} \\
& \times \left[\left(\int_u^1 \|g(s)\|\,ds\right)^p (1-u) + \left(\int_0^u \|g(s)\|\,ds\right)^p u\right]^{1/p} \\
& \times \left(\frac{1}{(2p-1)(\|y\|-\|x\|)} \frac{(R-\|x\|)^{2p-1}-(R-\|y\|)^{2p-1}}{(R-\|x\|)^{2p-1}(R-\|y\|)^{2p-1}}\right)^{1/p},
\end{aligned}$$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

From (4.5) we get the midpoint inequality

$$(4.7)\qquad \left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right) f\left(\frac{x+y}{2}\right) \right\|$$

$$\begin{aligned}
& \le \frac{R\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \int_0^1 \left|t-\frac{1}{2}\right| \|g(t)\|\,dt \\
& \times \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds \\
& \le \frac{1}{4}\frac{R\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \sup_{t\in[0,1]} \|g(t)\| \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds
\end{aligned}$$

for any continuous $g:[0,1]\to\mathcal{B}$.

If g is *symmetric in norm*, then by (4.4) we also have

$$\begin{aligned}
(4.8)\qquad & \left\| \int_0^1 g(t) f((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right) f\left(\frac{x+y}{2}\right) \right\| \\
& \le \frac{1}{2}\frac{R\|y-x\|}{(R-\|y\|)(R-\|x\|)} \int_0^1 \|g(s)\|\,ds \int_0^1 \left|f\left(Re^{2\pi is}\right)\right| ds.
\end{aligned}$$

The *modified Bessel function of the first kind* $I_\nu(z)$ for real number ν can be defined by the power series as [1, p. 376]

$$I_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\nu+k+1)},$$

where Γ is the *gamma function.* For $n=0$ we have $I_0(z)$ given by

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}.$$

An integral formula for real number ν is

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)\,d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}d\theta.$$

Consider the exponential function $f(a) = \exp a$, $a \in \mathcal{B}$. Assume that $x, y \in \mathcal{B}$ and $\|x\|, \|y\| < R$ for some $R > 0$. Observe that

$$\left|\exp\left(Re^{2\pi it}\right)\right| = \left|\exp\left[R\left(\cos(2\pi t) + i\sin(2\pi t)\right)\right]\right| = \exp\left[R\cos(2\pi t)\right]$$

and, by changing the variable $\theta = 2\pi t$, we get $dt = \frac{1}{2\pi}d\theta$ and

$$\begin{aligned} &\int_0^1 \exp\left[R\cos(2\pi t)\right]dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\exp\left[R\cos\theta\right]d\theta \\ &= \frac{1}{2}\left(\frac{1}{\pi}\int_0^\pi \exp\left[R\cos\theta\right]d\theta + \frac{1}{\pi}\int_\pi^{2\pi}\exp\left[R\cos\theta\right]d\theta\right) \\ &= \frac{1}{2}\left(I_0(R) + I_0(-R)\right) = I_0(R). \end{aligned} \tag{4.9}$$

If $g : [0,1] \to \mathcal{B}$ is continuous, then for all $u \in [0,1]$ we have by (4.4) and (4.5) for the exponential function, that

$$\begin{aligned} &\left\|\int_0^1 g(t)\exp((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right)\exp((1-u)x+uy)\right\| \\ &\le \frac{RI_0(R)\,\|y-x\|}{(R-\|y\|)(R-\|x\|)}\max\left\{\int_u^1 \|g(s)\|\,ds, \int_0^u \|g(s)\|\,ds\right\} \end{aligned} \tag{4.10}$$

and

$$\begin{aligned}(4.11)\quad & \left\| \int_0^1 g(t)\exp((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right)\exp((1-u)x+uy) \right\| \\ & \leq \frac{RI_0(R)\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \int_0^1 |t-u|\,\|g(t)\|\,dt \\ & \leq \frac{RI_0(R)\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \left[\frac{1}{4}+\left(u-\frac{1}{2}\right)^2\right] \sup_{t\in[0,1]}\|g(t)\|,\end{aligned}$$

where $x,\ y \in \mathcal{B}$, $x \neq y$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$.

In particular, we have from (4.11) the midpoint inequality

$$\begin{aligned}(4.12)\quad & \left\| \int_0^1 g(t)\exp((1-t)x+ty)\,dt - \left(\int_0^1 g(s)\,ds\right)\exp\left(\frac{x+y}{2}\right) \right\| \\ & \leq \frac{RI_0(R)\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \int_0^1 \left|t-\frac{1}{2}\right|\|g(t)\|\,dt \\ & \leq \frac{1}{4}\frac{RI_0(R)\|y-x\|}{\min\left\{(R-\|x\|)^2,(R-\|y\|)^2\right\}} \sup_{t\in[0,1]}\|g(t)\|,\end{aligned}$$

where $x,\ y \in \mathcal{B}$, $x \neq y$ with $\sigma(x),\ \sigma(y) \subset D(0,R)$.

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