REVERSES OF THE FIRST HERMITE-HADAMARD TYPE INEQUALITY FOR THE SQUARE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$ and $\operatorname{Re}A:=\frac{1}{2}(A^*+A)$. In this paper we show among other that, if $A,B\in\mathcal{B}(H)$ with $0\le m\le|(1-t)A+tB|^2\le M$ for all $t\in[0,1]$, then

$$0\le\int_0^1 f\left(|(1-t)A+tB|^2\right)dt-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)$$
$$\le 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_H$$

for operator convex functions $f:[0,\infty)\to\mathbb{R}$. Applications for power and logarithmic functions are also provided.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I\subseteq\mathbb{R}$ be an interval. A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$f(tx+(1-t)t)\le tf(x)+(1-t)f(y)$$

for all x, $y\in I$ and $t\in[0,1]$.

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ and $a,b\in I$ with $a<b$. Then the inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if f is convex, and is known in the literature as *Hermite-Hadamard inequality*, after the name of C. Hermite and J. Hadamard (see [17]). The inequalities in (1.1) hold in reversed direction if f is a concave function.

A vast literature related to (1.1) have been produced by a large number of mathematicians [13] since it is considered to be one of the most famous inequality for convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [3], Information Theory [2], Operator Theory [9], [10] and others.

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Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X,\ x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0,1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0,1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [4, p. 2], [5, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x) + f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0,1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, then for any $x, y \in X$ we have the following norm inequality from (1.2) (see [16, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}. \tag{1.3}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f\left((1-\lambda)A + \lambda B\right) \leq (\geq) (1-\lambda) f(A) + \lambda f(B) \tag{1.4}$$

in the operator order, for all $\lambda \in [0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

In the recent paper [12] we obtained among others the following Hermite-Hadamard type inequalities for operator convex functions:

Theorem 1. *Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I and for any $\lambda \in [0,1]$ we* *have the inequalities*

$$
\begin{aligned}
(1.5)\qquad & f\left(\frac{A+B}{2}\right)\\
&\leq(1-\lambda)f\left[\frac{(1-\lambda)A+(1+\lambda)B}{2}\right]+\lambda f\left[\frac{(2-\lambda)A+\lambda B}{2}\right]\\
&\leq\int_0^1 f((1-s)A+sB)\,ds\\
&\leq\frac{1}{2}\left[f((1-\lambda)A+\lambda B)+(1-\lambda)f(B)+\lambda f(A)\right]\\
&\leq\frac{f(A)+f(B)}{2}.
\end{aligned}
$$

A similar result and with a different proof was obtained by B. Li in [15]. For $\lambda=\frac{1}{2}$ in (1.5) we recapture the result obtained in the earlier paper [11] by the author. For other similar inequalities for operator convex functions see [1] and [18]-[22].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used.

In the recent paper [14] we obtained among others the following Hermite-Hadamard type inequalities for operator convex (concave) functions $f:[0,\infty)\to\mathbb{R}$ and A, $B\in\mathcal{B}(H)$ with $\operatorname{Re}(B^*A)\geq 0$,

$$
\begin{aligned}
f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)&\leq(\geq)\int_0^1 f\left(|(1-t)A+tB|^2\right)dt\\
&\leq(\geq)\frac{1}{3}\left[f\left(|A|^2\right)+f\left[\operatorname{Re}(B^*A)\right]+f\left(|B|^2\right)\right].
\end{aligned}
$$

Some examples for power functions and logarithm were also provided.

In this paper we show among other that, if A, $B\in\mathcal{B}(H)$ with $0\leq m\leq|(1-t)A+tB|^2\leq M$ for all $t\in[0,1]$, then

$$
\begin{aligned}
0&\leq\int_0^1 f\left(|(1-t)A+tB|^2\right)dt-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_H
\end{aligned}
$$

for operator convex functions $f:[0,\infty)\to\mathbb{R}$. Applications for power and logarithmic functions are also provided.

2. Main Results

The first main result is as follows:

Theorem 2. *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex on $[0,\infty)$. If $A,\ B\in\mathcal{B}(H)$ with $0\leq m\leq|(1-t)A+tB|^2\leq M$ for all $t\in[0,1]$, then*

$$\begin{aligned}(2.1)\quad 0&\leq\int_0^1 f\left(|(1-t)A+tB|^2\right)dt-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\\&\leq f(m)\frac{M-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]}{M-m}\\&\quad+f(M)\frac{\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m}{M-m}-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\\&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].\end{aligned}$$

Proof. We use the double inequality, see [6]

$$\begin{aligned}(2.2)\qquad &2\min\{\alpha,1-\alpha\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\leq(1-\alpha)f(m)+\alpha f(M)-f((1-\alpha)m+\alpha M)\\&\leq 2\max\{\alpha,1-\alpha\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],\end{aligned}$$

for $\alpha\in[0,1]$.

Let $x\in[m,M]$ and take $\alpha=\frac{x-m}{M-m}\in[0,1]$ to get

$$\begin{aligned}&2\min\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\leq\frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M)-f\left(\frac{M-x}{M-m}m+\frac{x-m}{M-m}M\right)\\&\leq 2\max\left\{\frac{x-m}{M-m},\frac{M-x}{M-m}\right\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],\end{aligned}$$

namely

$$\begin{aligned}(2.3)\qquad 0&\leq\frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)-\left|x-\frac{m+M}{2}\right|\right\}\\&\quad\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\\&\leq\frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M)-f(x)\\&\leq\frac{2}{M-m}\min\left\{\frac{1}{2}(M-m)+\left|x-\frac{m+M}{2}\right|\right\}\\&\quad\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].\end{aligned}$$

If we use the continuous functional calculus for selfadjoint operators and the inequality (2.3) we get for $m1_H \leq T \leq M1_H$

$$0 \leq \frac{2}{M-m} \min\left\{\frac{1}{2}(M-m)1_H - \left|T - \frac{m+M}{2}\right|\right\} \times \left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \leq f(m)\frac{M-T}{M-m} + f(M)\frac{T-m}{M-m} - f(T) \leq \frac{2}{M-m} \min\left\{\frac{1}{2}(M-m)1_H + \left|T - \frac{m+M}{2}\right|\right\} \times \left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]. \tag{2.4}$$

So, if we take $T = \frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}$ in (2.4), then we get the inequality of interest

$$0 \leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \times \min\left\{\frac{1}{2}(M-m)1_H - \left|\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3} - \frac{m+M}{2}\right|\right\} \leq f(m)\frac{M - \frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right]}{M-m} + f(M)\frac{\frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right] - m}{M-m} - f\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right) \leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \times \min\left\{\frac{1}{2}(M-m)1_H + \left|\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3} - \frac{m+M}{2}\right|\right\} \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]. \tag{2.5}$$

From (2.4) we also get

$$f(T) \leq f(m)\frac{M-T}{M-m} + f(M)\frac{T-m}{M-m},$$

which gives for $T = |(1-t)A + tB|^2$, $t \in [0,1]$ that

$$f\left(|(1-t)A+tB|^2\right) \leq f(m)\frac{M - |(1-t)A+tB|^2}{M-m} + f(M)\frac{|(1-t)A+tB|^2 - m}{M-m},$$

$t \in [0,1]$.

By taking the integral and by observing also that

$$\begin{aligned}
&\int_0^1 |(1-t)A+tB|^2 \, dt \\
&= \int_0^1 \left[(1-t)^2 |A|^2 + 2t(1-t)\operatorname{Re}(B^*A) + t^2 |B|^2\right] dt \\
&= \left(\int_0^1 (1-t)^2 \, dt\right) |A|^2 + 2\left(\int_0^1 t(1-t)\, dt\right) \operatorname{Re}(B^*A) + \left(\int_0^1 t^2 dt\right) |B|^2 \\
&= \frac{1}{3}\left[|A|^2 + \operatorname{Re}(B^*A) + |B|^2\right],
\end{aligned}$$

we get

$$\begin{aligned}
\int_0^1 f\left(|(1-t)A+tB|^2\right) dt \le{}& f(m) \frac{M - \frac{1}{3}\left[|A|^2 + \operatorname{Re}(B^*A) + |B|^2\right]}{M-m} \\
&+ f(M) \frac{\frac{1}{3}\left[|A|^2 + \operatorname{Re}(B^*A) + |B|^2\right] - m}{M-m}.
\end{aligned} \tag{2.6}$$

By subtracting in (2.6) $f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)$, then we get

$$\begin{aligned}
&\int_0^1 f\left(|(1-t)A+tB|^2\right) dt - f\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right) \\
&\le f(m) \frac{M - \frac{1}{3}\left[|A|^2 + \operatorname{Re}(B^*A) + |B|^2\right]}{M-m} \\
&\quad + f(M) \frac{\frac{1}{3}\left[|A|^2 + \operatorname{Re}(B^*A) + |B|^2\right] - m}{M-m} - f\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right).
\end{aligned} \tag{2.7}$$

By (2.7) and (2.5) we derive the desired result (2.1). □

Corollary 1. *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex on $[0,\infty)$ with $f(0)=0$. If $A, B\in\mathcal{B}(H)$ with $|A|^2, |B|^2 \le M$, where $M>0$, then*

$$\begin{aligned}
0 &\le \int_0^1 f\left(|(1-t)A+tB|^2\right) dt - f\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right) \\
&\le \frac{f(M)}{M}\left[\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right] - f\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right) \\
&\le 2\left[\frac{1}{2} f(M) - f\left(\frac{1}{2}M\right)\right].
\end{aligned} \tag{2.8}$$

Proof. The following Cauchy-Bunyakowsky-Schwarz inequality holds

$$\sum_{k=1}^n w_k |z_k|^2 \sum_{k=1}^n w_k |A_k|^2 \ge \left|\sum_{k=1}^n w_k z_k A_k\right|^2, \tag{2.9}$$

where $z_k\in\mathbb{C}$, $A_k\in\mathcal{B}(H)$, $w_k\ge 0$ for $k\in\{1,...,n\}$ and $\sum_{k=1}^n w_k = 1$.

Let A, $B\in\mathcal{B}(H)$ and $t\in[0,1]$. Then by (2.9), we get

$$\begin{aligned}\left|(1-t)A+tB\right|^{2} &= \left|(1-t)^{1/2}(1-t)^{1/2}A+t^{1/2}t^{1/2}B\right|^{2}\\ &\leq\left[\left((1-t)^{1/2}\right)^{2}+\left(t^{1/2}\right)^{2}\right]\left[\left|(1-t)^{1/2}A\right|^{2}+\left|t^{1/2}B\right|^{2}\right]\\ &=(1-t+t)\left[(1-t)\left|A\right|^{2}+t\left|B\right|^{2}\right]\\ &=(1-t)\left|A\right|^{2}+t\left|B\right|^{2}.\end{aligned}$$

So, if $|A|^2\leq M$ and $|B|^2\leq M$, then $|(1-t)A+tB|^2\leq(1-t)M+tM=M$ and by writing the inequality (2.1) for $m=0$ we derive (2.8). □

We also have:

Theorem 3. *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex on $[0,\infty)$. If A, $B\in\mathcal{B}(H)$ with $0\leq m\leq|(1-t)A+tB|^2\leq M$ for all $t\in[0,1]$, then*

$$\begin{aligned}(2.10)\qquad 0&\leq\int_0^1 f\left(\left|(1-t)A+tB\right|^{2}\right)dt-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\\ &\leq f(m)\frac{M1_H-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]}{M-m}\\ &\quad+f(M)\frac{\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m1_H}{M-m}\\ &\quad-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)\\ &\leq\frac{f'_-(M)-f'_+(m)}{M-m}\left(\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m1_H\right)\\ &\quad\times\left(M1_H-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]\right)\\ &\leq\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_H.\end{aligned}$$

Proof. We use the inequality, see [8]

$$\begin{aligned}&(1-\alpha)f(m)+\alpha f(M)-f((1-\alpha)m+\alpha M)\\ &\leq\alpha(1-\alpha)(M-m)\left[f'_-(M)-f'_+(m)\right]\end{aligned}$$

for $\alpha\in[0,1]$.

Let $x\in[m,M]$ and take $\alpha=\frac{x-m}{M-m}\in[0,1]$ to get

$$(2.11)\qquad \frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M)-f(x)\leq\frac{f'_-(M)-f'_+(m)}{M-m}(x-m)(M-x).$$

If we use the continuous functional calculus for selfadjoint operators and the inequality (2.11) we get

$$f(m)\frac{M1_H - T}{M-m} + f(M)\frac{T-m1_H}{M-m} - f(T) \le \frac{f'_-(M)-f'_+(m)}{M-m}(T-m1_H)(M1_H-T). \tag{2.12}$$

So, if we take $T=\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}$ in (2.12), then we get the inequality of interest

$$\begin{aligned} & f(m)\frac{M1_H - \frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right]}{M-m} \\ & + f(M)\frac{\frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right]-m1_H}{M-m} \\ & - f\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right) \\ & \le \frac{f'_-(M)-f'_+(m)}{M-m}\left(\frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right]-m1_H\right) \\ & \times\left(M1_H-\frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right]\right) \\ & \le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_H \end{aligned} \tag{2.13}$$

From (2.7) and (2.13) we derive (2.10). □

Corollary 2. *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex on $[0,\infty)$ with $f(0)=0$. If A, $B\in\mathcal{B}(H)$ with $|A|^2, |B|^2\le M$, where $M>0$, then*

$$\begin{aligned} 0 & \le \int_0^1 f\left(|(1-t)A+tB|^2\right)dt - f\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right) \\ & \le \frac{f(M)}{M}\left[\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right] - f\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right) \\ & \le \frac{f'_-(M)-f'_+(0)}{M}\left(\frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right]\right) \\ & \times\left(M1_H-\frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right]\right) \\ & \le \frac{1}{4}M\left[f'_-(M)-f'_+(0)\right]1_H, \end{aligned} \tag{2.14}$$

provided that $f'_+(0)$ is finite.

Theorem 4. *Let $f:[0,\infty)\to\mathbb{R}$ be twice differentiable and so that there exists the constants $0\le d$, D such that $d\le f''(t)\le D$ for any $t\in(m,M)\subset[0,\infty)$. If A, $B\in\mathcal{B}(H)$ with $0\le m\le|(1-t)A+tB|^2\le M$ for all $t\in[0,1]$, then*

$$0\le\int_0^1 f\left(|(1-t)A+tB|^2\right)dt - f\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right) \tag{2.15}$$

$$\leq f(m)\frac{M1_H-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]}{M-m}$$
$$+f(M)\frac{\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]-m1_H}{M-m}-f\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)$$

$$\leq\frac{1}{2}D\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}-m1_H\right)\left(M1_H-\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)$$
$$\leq\frac{1}{8}(M-m)^2D.$$

Proof. If there exists the constants $0\leq d$, D such that

$$d\leq f''(t)\leq D \text{ for any } t\in(m,M),$$

then, see for instance [7],

$$\frac{1}{2}\nu(1-\nu)d(M-m)^2\leq(1-\nu)f(m)+\nu f(M)-f((1-\nu)m+\nu M)\leq\frac{1}{2}\nu(1-\nu)D(M-m)^2 \tag{2.16}$$

for any $\nu\in[0,1]$.

Let $x\in[m,M]$ and take $\alpha=\frac{x-m}{M-m}\in[0,1]$ in (2.16) to get

$$\frac{1}{2}d(x-m)(M-x)\leq\frac{M-x}{M-m}f(m)+\frac{x-m}{M-m}f(M)-f(x)\leq\frac{1}{2}D(x-m)(M-x). \tag{2.17}$$

If we use the continuous functional calculus for selfadjoint operators and the inequality (2.17) we get

$$\begin{aligned}&\frac{1}{2}d(T-m1_H)(M1_H-T)\\&\leq f(m)\frac{M1_H-T}{M-m}+f(M)\frac{T-m1_H}{M-m}-f(T)\\&\leq\frac{1}{2}D(T-m1_H)(M1_H-T).\end{aligned} \tag{2.18}$$

So, if we take $T=\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}$ in (2.12), then we get the inequality of interest

$$\begin{aligned}(2.19)\quad &\frac{1}{2}d\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}-m1_H\right)\\ &\times\left(M1_H-\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right)\\ &\le f(m)\frac{M1_H-\frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right]}{M-m}\\ &+f(M)\frac{\frac{1}{3}\left[|A|^2+\mathrm{Re}(B^*A)+|B|^2\right]-m1_H}{M-m}\\ &-f\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right)\\ &\le\frac{1}{2}D\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}-m1_H\right)\\ &\times\left(M1_H-\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right)\\ &\le\frac{1}{8}(M-m)^2D\end{aligned}$$

Now, if we use (2.7) and (2.19) we derive (2.15). □

Corollary 3. *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex on $[0,\infty)$ with $f(0)=0$. If $A,\ B\in\mathcal{B}(H)$ with $|A|^2,\ |B|^2\le M$, where $M>0$, then*

$$\begin{aligned}(2.20)\quad 0&\le\int_0^1 f\left(|(1-t)A+tB|^2\right)dt-f\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right)\\ &\le\frac{f(M)}{M}\left[\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right]-f\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right)\\ &\le\frac{1}{2}D\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right)\left(M1_H-\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}\right)\\ &\le\frac{1}{8}M^2D,\end{aligned}$$

provided that $f''(t)\le D$ for any $t\in(0,M)$.

3. Some Examples

The function $f(t)=t^{p}$, $p\in[1,2]$ is operator convex on $[0,\infty)$ with $f(0)=0$. From (2.8) and (2.14) we derive

$$
\begin{aligned}
(3.1)\qquad 0 &\le \int_0^1 |(1-t)A+tB|^{2p}\,dt-\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^p \\
&\le M^{p-1}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right]-\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^p \\
&\le M^p\left(\frac{2^{p-1}-1}{2^{p-1}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.2)\qquad 0 &\le \int_0^1 |(1-t)A+tB|^{2p}\,dt-\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^p \\
&\le M^{p-1}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right]-\left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^p \\
&\le pM^{p-2}\left(\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]\right) \\
&\times\left(M1_H-\frac{1}{3}\left[|A|^2+\operatorname{Re}(B^*A)+|B|^2\right]\right) \\
&\le \frac{1}{4}pM^p1_H,
\end{aligned}
$$

provided that $A,B\in\mathcal{B}(H)$ with $|A|^2, |B|^2\le M$, where $M>0$.

The function $f(t)=t^{r}$, $r\in(0,1)$ is operator concave on $[0,\infty)$ with $f(0)=0$. From (2.8) we derive

$$
\begin{aligned}
(3.3)\qquad 0 &\le \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^r-\int_0^1 |(1-t)A+tB|^{2r}\,dt \\
&\le \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^r-\frac{1}{M^{1-r}}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right] \\
&\le M^r\left(2^{1-r}-1\right)
\end{aligned}
$$

and, in particular

$$
\begin{aligned}
(3.4)\qquad 0 &\le \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^{1/2}-\int_0^1 |(1-t)A+tB|\,dt \\
&\le \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right)^{1/2}-\frac{1}{\sqrt{M}}\left[\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}\right] \\
&\le \sqrt{M}\left(\sqrt{2}-1\right)
\end{aligned}
$$

provided that $A,B\in\mathcal{B}(H)$ with $|A|^2, |B|^2\le M$, where $M>0$.

Consider the operator concave function $f(t) = \ln(t+1)$. By (2.8) we get

$$0 \leq \ln\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} + 1\right) - \int_0^1 \ln\left(|(1-t)A + tB|^2 + 1\right) dt \tag{3.5}$$
$$\leq \ln\left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} + 1\right)$$
$$- \frac{\ln(M+1)}{M}\left[\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3}\right]$$
$$\leq 2\ln\left(\frac{1}{2}M + 1\right) - \ln(M+1)$$

provided that $A, B \in \mathcal{B}(H)$ with $|A|^2, |B|^2 \leq M$, where $M > 0$.

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