REVERSES OF THE SECOND HERMITE-HADAMARD TYPE INEQUALITY FOR THE SQUARE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$ and $\operatorname{Re}A:=\frac{1}{2}(A^*+A)$. In paper we show among other that, if $A,\ B\in\mathcal{B}(H)$ with $\operatorname{Re}(B^*A)\geq 0$ and $0\leq m\leq |(1-t)A+tB|^2\leq M$ for all $t\in[0,1]$, then

$$0\leq \frac{f\left(|A|^2\right)+f\left(\operatorname{Re}(B^*A)\right)+f\left(|B|^2\right)}{3}-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt$$
$$\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_H.$$

Applications for power and logarithmic functions are also provided.

1. Introduction

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as that arithmetic mean-geometric mean inequality, Hölder and Minkowski inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p}=(p_1,\ldots,p_n)$ where $p_j\geq 0,\ j\in\{1,...,n\}$ with $P_n:=\sum_{j=1}^n p_j>0$ and $\mathbf{x}=(x_1,\ldots,x_n)\in C^n$, then

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right)\leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i), \tag{1.1}$$

is well known in the literature as Jensen's inequality.

In order to extend this inequality for operator convex functions of selfadjoint bounded linear operators on complex Hilbert spaces we need the following preliminary facts.

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1-\lambda)A+\lambda B)\leq(\geq)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

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A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [14] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

We also have the following Jensen type inequality for operator convex functions $f : I \to \mathbb{R}$.

Let A_j be selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq I$, $j \in \{1, ..., n\}$. If $p_j \geq 0$, $j \in \{1, ..., n\}$ with $P_n > 0$ and f is an operator convex function on I then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i A_j\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(A_j), \tag{1.2}$$

in the operator order.

This is a well known result and can be proved easily by mathematical induction over $n \geq 2$. The details are left to the reader.

For recent results related to the Jensen inequality for selfadjoint operators in Hilbert spaces see the papers [1]-[6], [15]-[22], [23] and the monograph [7].

In the recent paper [13] we obtained the following Hermite-Hadamard type inequalities for operator convex functions $f : [0, \infty) \to \mathbb{R}$ and A, $B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) \geq 0$,

$$\begin{aligned}
0 &\leq \frac{1}{3}\left[\frac{f\left(|A|^2\right) + 2f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{4} - f\left(\left|\frac{A+B}{2}\right|^2\right)\right] \\
&\leq \frac{f\left(|A|^2\right) + f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{3} - \int_0^1 f\left(|(1-t)A + tB|^2\right) dt \\
&\leq \frac{7}{3}\left[\frac{f\left(|A|^2\right) + 2f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{4} - f\left(\left|\frac{A+B}{2}\right|^2\right)\right].
\end{aligned}$$

For other similar inequalities for operator convex functions see [1] and [24]-[28].

In paper we show among other that, if A, $B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) \geq 0$ and $0 \leq m \leq |(1-t)A + tB|^2 \leq M$ for all $t \in [0, 1]$, then

$$\begin{aligned}
0 &\leq \frac{f\left(|A|^2\right) + f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{3} - \int_0^1 f\left(|(1-t)A + tB|^2\right) dt \\
&\leq 2\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] 1_H.
\end{aligned}$$

Applications for power and logarithmic functions are also provided.

2. Some Preliminary Facts

We consider the functional

$$J_n(\mathbf{p};\mathbf{A},f,[0,\infty)) := \sum_{j=1}^{n} p_j f(A_j) - P_n f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \tag{2.1}$$

where $\mathbf{p} = (p_1,...,p_n)$, $p_j \geq 0$ with $j \in \{1,...,n\}$ and $P_n > 0$, $\mathbf{A} = (A_1,...,A_n)$ is an n-tuple of selfadjoint operators with $A_j \geq 0$ for $j \in \{1,...,n\}$ and $f : [0,\infty) \to \mathbb{R}$ is a operator convex function defined on the interval $[0,\infty)$.

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $\mathbf{p} = (p_1,...,p_n)$, $p_j \geq 0$ with $j \in \{1,...,n\}$ and $P_n > 0$. For $\mathbf{p},\mathbf{q} \in \mathcal{P}_n^+$ we denote $\mathbf{p} \geq \mathbf{q}$ if $p_j \geq q_j$ for any $j \in \{1,...,n\}$.

We have the following operator version of the scalar result obtained in [3]:

Lemma 1. *Assume that $f : [0,\infty) \to \mathbb{R}$ is an operator convex function and $\mathbf{A} = (A_1,...,A_n)$ an n-tuple of selfadjoint operators with $A_j \geq 0$, then for any $\mathbf{p},\mathbf{q} \in \mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,[0,\infty)) \geq J_n(\mathbf{p};\mathbf{A},f,[0,\infty)) + J_n(\mathbf{q};\mathbf{A},f,[0,\infty)) \geq 0, \tag{2.2}$$

i.e., $J_n(\cdot;\mathbf{A},f,[0,\infty))$ is a super-additive functional in the operator order.

Moreover, if $\mathbf{p},\mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$J_n(\mathbf{p};\mathbf{A},f,[0,\infty)) \geq J_n(\mathbf{q};\mathbf{A},f,[0,\infty)) \geq 0, \tag{2.3}$$

i.e., $J_n(\cdot;\mathbf{A},f,[0,\infty))$ is a monotonic functional in the operator order.

Proof. We have

$$\begin{aligned} & J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,[0,\infty)) \\ &= \sum_{j=1}^{n}(p_j+q_j) f(A_j) - (P_n+Q_n) f\left(\frac{1}{P_n+Q_n}\sum_{j=1}^{n}(p_j+q_j)A_j\right) \\ &= \sum_{j=1}^{n}(p_j+q_j) f(A_j) \\ &- (P_n+Q_n) f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^{n} p_jA_j\right) + Q_n\left(\frac{1}{Q_n}\sum_{j=1}^{n} q_jA_j\right)}{P_n+Q_n}\right). \end{aligned} \tag{2.4}$$

Now, consider the operators

$$A := \frac{1}{P_n}\sum_{j=1}^{n} p_jA_j \geq 0 \text{ and } B := \frac{1}{Q_n}\sum_{j=1}^{n} q_jA_j \geq 0.$$

Applying the inequality (OC) for A and B given above and $\lambda = \frac{Q_n}{P_n+Q_n}$ we have

$$\begin{aligned} & f\left(\frac{P_n\left(\frac{1}{P_n}\sum_{j=1}^{n} p_jA_j\right) + Q_n\left(\frac{1}{Q_n}\sum_{j=1}^{n} q_jA_j\right)}{P_n+Q_n}\right) \\ &\leq \frac{P_n}{P_n+Q_n} f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_jA_j\right) + \frac{Q_n}{P_n+Q_n} f\left(\frac{1}{Q_n}\sum_{j=1}^{n} q_jA_j\right) \end{aligned} \tag{2.5}$$

in the operator order.

Making use of (2.4) and (2.5) we have

$$
\begin{aligned}
(2.6)\qquad & J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,[0,\infty)) \\
& \geq \sum_{j=1}^{n}(p_j+q_j)f(A_j)-(P_n+Q_n) \\
& \times\left[\frac{P_n}{P_n+Q_n}f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)+\frac{Q_n}{P_n+Q_n}f\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right)\right] \\
& =\sum_{j=1}^{n}p_jf(A_j)-P_nf\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right) \\
& +\sum_{j=1}^{n}q_jf(A_j)-Q_nf\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_jA_j\right) \\
& =J_n(\mathbf{p};\mathbf{A},f,[0,\infty))+J_n(\mathbf{q};\mathbf{A},f,[0,\infty))
\end{aligned}
$$

in the operator order, and the inequality (2.2) is proved.

Now, let $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$. Then by the super-additivity property (2.2) we have

$$
\begin{aligned}
(2.7)\qquad J_n(\mathbf{p};\mathbf{A},f,[0,\infty)) & =J_n((\mathbf{p}-\mathbf{q})+\mathbf{q};\mathbf{A},f,[0,\infty)) \\
& \geq J_n((\mathbf{p}-\mathbf{q});\mathbf{A},f,[0,\infty))+J_n(\mathbf{q};\mathbf{A},f,[0,\infty)) \\
& \geq J_n(\mathbf{q};\mathbf{A},f,[0,\infty))
\end{aligned}
$$

in the operator order, and the monotonicity property (2.3) is proved. □

Corollary 1. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator convex and the n-tuple of selfadjoint operators $(A_1,...,A_n)$ satisfies the condition $A_j\geq 0$ for any $j\in\{1,...,n\}$. If $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and there exists the positive constants m,M such that*

$$
(2.8)\qquad m\mathbf{q}\leq\mathbf{p}\leq M\mathbf{q}
$$

then

$$
(2.9)\qquad mJ_n(\mathbf{q};\mathbf{A},f,[0,\infty))\leq J_n(\mathbf{p};\mathbf{A},f,[0,\infty))\leq MJ_n(\mathbf{q};\mathbf{A},f,[0,\infty))
$$

in the operator order.

Proof. Observe that for $\alpha>0$ we have $J_n(\alpha\mathbf{p};\mathbf{A},f,[0,\infty))=\alpha J_n(\mathbf{p};\mathbf{A},f,[0,\infty))$.

Utilising the monotonicity property (2.3) we have

$$
J_n(m\mathbf{q};\mathbf{A},f,[0,\infty))\leq J_n(\mathbf{p};\mathbf{A},f,[0,\infty))\leq J_n(M\mathbf{q};\mathbf{A},f,[0,\infty))
$$

which imply the desired result (2.9). □

Remark 1. *We observe that if all $q_j>0$ then we have the inequality*

$$
\begin{aligned}
(2.10)\qquad \min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,[0,\infty)) & \leq J_n(\mathbf{p};\mathbf{A},f,[0,\infty)) \\
& \leq\max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,[0,\infty))
\end{aligned}
$$

in the operator order.

In particular, if $\mathbf{q}$ *is the uniform distribution, i.e.,* $q_j = \frac{1}{n}, j \in \{1, ..., n\}$, *then we have the inequalities*

$$n \min_{j\in\{1,...,n\}} \{p_j\} J_n (\mathbf{A}, f, [0, \infty)) \leq J_n (\mathbf{p}; \mathbf{A}, f, [0, \infty)) \leq n \max_{j\in\{1,...,n\}} \{p_j\} J_n (\mathbf{A}, f, [0, \infty)) \tag{2.11}$$

where

$$J_n (\mathbf{A}, f, [0, \infty)) := \frac{1}{n} \sum_{j=1}^{n} f (A_j) - f \left(\frac{1}{n} \sum_{j=1}^{n} A_j \right). \tag{2.12}$$

For $n = 2$ *and by choosing* $p_1 = \alpha, p_2 = 1 - \alpha$ *with* $\alpha \in [0, 1]$, *we get from (2.11) the inequality*

$$\begin{aligned} &2 \min \{\alpha, 1 - \alpha\} \left[\frac{f (A) + f (B)}{2} - f \left(\frac{A + B}{2} \right) \right] \\ &\leq (1 - \alpha) f (A) + \alpha f (B) - f ((1 - \alpha) A + \alpha B) \\ &\leq 2 \max \{\alpha, 1 - \alpha\} \left[\frac{f (A) + f (B)}{2} - f \left(\frac{A + B}{2} \right) \right], \end{aligned} \tag{2.13}$$

in the operator order, where $f : [0, \infty) \to \mathbb{R}$ *is an operator convex function and* A *and* B *are two bounded selfadjoint operators on the complex Hilbert space* H *with* $A, B \geq 0$.

The following result also holds:

Theorem 1. *If the function* $f : [0, \infty) \to \mathbb{R}$ *is operator convex and if the* n*-tuple of selfadjoint operators* $(A_1, ..., A_n)$ *has the property that*

$$0 \leq m \leq \frac{1}{P_n} \sum_{j=1}^{n} p_j A \leq M \tag{2.14}$$

for $p_j \geq 0$ *with* $j \in \{1, ..., n\}$ *and* $P_n := \sum_{j=1}^{n} p_j > 0$, *then we have*

$$\begin{aligned} 0 &\leq \frac{1}{P_n} \sum_{j=1}^{n} p_j f (A_j) - f \left(\frac{1}{P_n} \sum_{j=1}^{n} p_j A_j \right) \\ &\leq \frac{2}{M - m} \left[\frac{f (m) + f (M)}{2} - f \left(\frac{m + M}{2} \right) \right] \\ &\quad \times \left(\frac{1}{2} (M - m) 1_H + \left| \frac{1}{P_n} \sum_{j=1}^{n} p_j A_j - \frac{m + M}{2} 1_H \right| \right) \\ &\leq 2 \left[\frac{f (m) + f (M)}{2} - f \left(\frac{m + M}{2} \right) \right] 1_H \end{aligned} \tag{2.15}$$

in the operator order.

Proof. Since the function $f : [0, \infty) \to \mathbb{R}$ is operator convex, then it is convex and we have the inequality

$$f (t) = f \left(\frac{(M - t) m + (t - m) M}{M - m} \right) \leq \frac{(M - t) f (m) + (t - m) f (M)}{M - m}$$

for any $t \in [m, M]$.

Utilising the *continuous functional calculus* for a selfadjoint operator $A \geq 0$, we have in the operator order

$$f(A_j) \leq \frac{f(m)(M1_H - A_j) + f(M)(A_j - m1_H)}{M - m} \tag{2.16}$$

for any $j \in \{1, ..., n\}$.

If we multiply the inequality (2.16) by p_j and sum over j from 1 to n we get

$$\begin{aligned} &\frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) \\ &\leq \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M - m} \end{aligned} \tag{2.17}$$

in the operator order.

Therefore we have

$$\begin{aligned} 0 &\leq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \\ &\leq \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M - m} \\ &\quad - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \end{aligned} \tag{2.18}$$

in the operator order, which is a reverse of Jensen's inequality that is of interest in itself.

Now, from the scalar version of (2.13) we have

$$\begin{aligned} 0 &\leq (1-t) f(m) + t f(M) - f((1-t) m + tM) \\ &\leq 2\max\{t, 1-t\}\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &= 2\left(\frac{1}{2} + \left|t - \frac{1}{2}\right|\right)\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \end{aligned} \tag{2.19}$$

for any $t \in [m, M]$, where $f : [m, M] \to \mathbb{R}$ is a continuous convex function on $[m, M]$.

Utilising the *continuous functional calculus* for a selfadjoint operator T with $0 \leq T \leq 1_H$ we have from (2.19) that

$$\begin{aligned} 0 &\leq f(m)(1_H - T) + f(M) T - f((1_H - T) m + TM) \\ &\leq 2\left[\frac{f(m) + f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2} + \left|T - \frac{1}{2}1_H\right|\right) \end{aligned} \tag{2.20}$$

in the operator order.

Writing the inequality (2.20) for the operator

$$T = \frac{\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H}{M - m}$$

which, by (2.14), satisfies the conditions

$$0 \leq T = \frac{\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H}{M-m} \leq 1_H$$

we have

$$\begin{aligned}
(2.21)\qquad & \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M-m} \\
& - f\left[\frac{m\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + M\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M-m}\right] \\
& = \frac{f(m)\left(M1_H - \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) + f(M)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - m1_H\right)}{M-m} \\
& - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \\
& \leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
& \times \left(\frac{1}{2}(M-m)1_H + \left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}1_H\right|\right)
\end{aligned}$$

in the operator order.

The last part is obvious since

$$\left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}1_H\right| \leq \frac{1}{2}(M-m)1_H.$$

□

Corollary 2. *Assume that* $A, B \geq 0$ *and* $f : [0,\infty) \to \mathbb{R}$ *is operator convex with* $0 \leq m \leq (1-t)A + tB \leq M$ *for* $t \in [0,1]$, *then*

$$\begin{aligned}
(2.22)\qquad & 0 \leq (1-t)f(A) + tf(B) - f((1-t)A + tB) \\
& \leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\
& \times \left(\frac{1}{2}(M-m)1_H + \left|(1-t)A + tB - \frac{m+M}{2}1_H\right|\right) \\
& \leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]1_H.
\end{aligned}$$

We have the following result for general convex functions, see [11]:

Lemma 2. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on the interval* I, $a, b \in \mathring{I}$, *the interior of* I, *with* $a < b$ *and* $\nu \in [0,1]$. *Then*

$$\begin{aligned}
(2.23)\qquad & \nu(1-\nu)(b-a)\left[f'_+((1-\nu)a + \nu b) - f'_-((1-\nu)a + \nu b)\right] \\
& \leq (1-\nu)f(a) + \nu f(b) - f((1-\nu)a + \nu b) \\
& \leq \nu(1-\nu)(b-a)\left[f'_-(b) - f'_+(a)\right].
\end{aligned}$$

In particular, we have

$$\frac{1}{4}(b-a)\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right]\leq\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\leq\frac{1}{4}(b-a)\left[f'_{-}(b)-f'_{+}(a)\right]. \tag{2.24}$$

The constant $\frac{1}{4}$ *is best possible in both inequalities from (2.24).*

Corollary 3. *If the function* $f:I\subset\mathbb{R}\to\mathbb{R}$ *is a differentiable convex function on* $\mathring{I}$*, then for any* $a,b\in\mathring{I}$ *and* $\nu\in[0,1]$ *we have*

$$\begin{aligned}0&\leq(1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b)\\&\leq\nu(1-\nu)(b-a)\left[f'(b)-f'(a)\right].\end{aligned} \tag{2.25}$$

We also have:

Theorem 2. *If the function* $f:[0,\infty)\to\mathbb{R}$ *is operator convex and if the n-tuple of selfadjoint operators* $(A_1,...,A_n)$ *has the property (2.14) for* $p_j\geq0$ *with* $j\in\{1,...,n\}$ *and* $P_n:=\sum_{j=1}^{n}p_j>0$*, then we have*

$$\begin{aligned}0&\leq\frac{1}{P_n}\sum_{j=1}^{n}p_jf(A_j)-f\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)\\&\leq\frac{\left[f'_{-}(M)-f'_{+}(m)\right]}{M-m}\frac{1}{P_n}\sum_{j=1}^{n}p_j(A_j-m1_H)\frac{1}{P_n}\sum_{j=1}^{n}p_j(M1_H-A_j)\\&\leq\frac{1}{4}(M-m)\left[f'_{-}(M)-f'_{+}(m)\right]1_H.\end{aligned} \tag{2.26}$$

Proof. Using the *continuous functional calculus* for a selfadjoint operator T with $0\leq T\leq1_H$ we have from (2.23) that

$$\begin{aligned}0&\leq f(m)(1-T)+f(M)T-f(m(1-T)+MT)\\&\leq(M-m)\left[f'_{-}(M)-f'_{+}(m)\right]T(1-T).\end{aligned} \tag{2.27}$$

Writing the inequality (2.27) for the operator

$$0\leq T=\frac{\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-m1_H}{M-m}\leq1_H$$

we have

$$\begin{aligned}
&\frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H\right)}{M-m} \\
&-f\left[\frac{m\left(M1_H-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+M\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H\right)}{M-m}\right] \\
&=\frac{f(m)\left(M1_H-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)+f(M)\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H\right)}{M-m} \\
&-f\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right) \\
&\leq (M-m)\left[f'_-(M)-f'_+(m)\right] \\
&\times\frac{\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H}{M-m}\left(1-\frac{\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m1_H}{M-m}\right) \\
&=\frac{\left[f'_-(M)-f'_+(m)\right]}{M-m}\frac{1}{P_n}\sum_{j=1}^n p_j\left(A_j-m1_H\right)\frac{1}{P_n}\sum_{j=1}^n p_j\left(M1_H-A_j\right).
\end{aligned} \tag{2.28}$$

By making use of (2.18) and (2.28) we derive the first inequality in (2.26).

We also have

$$\begin{aligned}
&\frac{1}{P_n}\sum_{j=1}^n p_j\left(A_j-m1_H\right)\frac{1}{P_n}\sum_{j=1}^n p_j\left(M1_H-A_j\right) \\
&\leq\frac{1}{4}\left[\frac{1}{P_n}\sum_{j=1}^n p_j\left(A_j-m1_H\right)+\frac{1}{P_n}\sum_{j=1}^n p_j\left(M1_H-A_j\right)\right]^2 \\
&=\frac{1}{4}(M-m)^2 1_H,
\end{aligned}$$

which proves the last part of (2.26). □

Corollary 4. *Assume that $A,B\geq 0$ and $f:[0,\infty)\to\mathbb{R}$ is operator convex with $0\leq m\leq(1-t)A+tB\leq M$ for $t\in[0,1]$, then*

$$\begin{aligned}
0&\leq(1-t)f(A)+tf(B)-f((1-t)A+tB) \\
&\leq\frac{\left[f'_-(M)-f'_+(m)\right]}{M-m}\left((1-t)A+tB-m1_H\right)\left(M1_H-(1-t)A-tB\right) \\
&\leq\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_H.
\end{aligned} \tag{2.29}$$

We also have the following result [10]:

Lemma 3. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that*

$$d\leq f''(t)\leq D \text{ for any } t\in\mathring{I}, \tag{2.30}$$

then

$$\frac{1}{2}\nu(1-\nu)d(b-a)^2 \leq (1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b) \leq \frac{1}{2}\nu(1-\nu)D(b-a)^2 \tag{2.31}$$

for any $a, b \in \mathring{I}$ *and* $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8}(b-a)^2 d \leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)^2 D, \tag{2.32}$$

for any $a, b \in \mathring{I}$.

The constant $\frac{1}{8}$ *is best possible in both inequalities in (2.32).*

We also have:

Theorem 3. *If the function* $f:[0,\infty)\to\mathbb{R}$ *is operator convex with* $f''(x) \leq D$ *for all* $x \in (m,M)$ *and if the* n*-tuple of selfadjoint operators* $(A_1,...,A_n)$ *has the property (2.14) for* $p_j \geq 0$ *with* $j \in \{1,...,n\}$ *and* $P_n := \sum_{j=1}^n p_j > 0$*, then we have*

$$\begin{aligned} 0 &\leq \frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right) \\ &\leq \frac{1}{2}D\frac{1}{P_n}\sum_{j=1}^n p_j(A_j - m1_H)\frac{1}{P_n}\sum_{j=1}^n p_j(M1_H - A_j) \leq \frac{1}{8}(M-m)^2 D. \end{aligned} \tag{2.33}$$

Proof. Using the *continuous functional calculus* for a selfadjoint operator T with $0 \leq T \leq 1_H$ we have from (2.31) that

$$\begin{aligned} 0 &\leq f(m)(1-T)+f(M)T - f(m(1-T)+MT) \\ &\leq \frac{1}{2}T(1-T)(M-m)^2 D \end{aligned} \tag{2.34}$$

Writing the inequality (2.34) for the operator

$$0 \leq T = \frac{\frac{1}{P_n}\sum_{j=1}^n p_j A_j - m1_H}{M-m} \leq 1_H,$$

we derive (2.33). □

Corollary 5. *Assume that* $A, B \geq 0$ *and* $f:[0,\infty)\to\mathbb{R}$ *is operator convex with* $0 \leq m \leq (1-t)A+tB \leq M$ *for* $t \in [0,1]$, *then*

$$\begin{aligned} 0 &\leq (1-t)f(A)+tf(B)-f((1-t)A+tB) \\ &\leq \frac{1}{2}D((1-t)A+tB-m1_H)(M1_H-(1-t)A-tB) \\ &\leq \frac{1}{8}D(M-m)^2 1_H. \end{aligned} \tag{2.35}$$

3. Quadratic Hermite-Hadamard Inequalities

We have the following reverses of Hermite-Hadamard type inequalities:

Theorem 4. *Let $f:[0,\infty)\to\mathbb{R}$ be operator convex on $[0,\infty)$. If $A,\ B\in\mathcal{B}(H)$ with $\operatorname{Re}(B^{*}A)\geq 0$ and $0\leq m\leq |(1-t)A+tB|^{2}\leq M$ for all $t\in[0,1]$, then*

$$
\begin{aligned}
(3.1)\qquad 0 &\leq (1-t)^{2} f\left(|A|^{2}\right)+2t(1-t) f(\operatorname{Re}(B^{*}A))+t^{2} f\left(|B|^{2}\right) \\
&\quad - f\left(|(1-t)A+tB|^{2}\right) \\
&\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad \times\left(\frac{1}{2}(M-m)1_{H}+\left||(1-t)A+tB|^{2}-\frac{m+M}{2}1_{H}\right|\right) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_{H}.
\end{aligned}
$$

Also, we have the integral inequalities

$$
\begin{aligned}
(3.2)\qquad 0 &\leq \frac{f\left(|A|^{2}\right)+f(\operatorname{Re}(B^{*}A))+f\left(|B|^{2}\right)}{3}-\int_{0}^{1} f\left(|(1-t)A+tB|^{2}\right)dt \\
&\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\quad \times\left(\frac{1}{2}(M-m)1_{H}+\int_{0}^{1}\left||(1-t)A+tB|^{2}-\frac{m+M}{2}1_{H}\right|dt\right) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]1_{H}.
\end{aligned}
$$

Proof. Observe that, by the properties of modulus, we have for $t\in[0,1]$

$$
\begin{aligned}
&|(1-t)A+tB|^{2} \\
&=((1-t)A+tB)^{*}(1-t)A+tB \\
&=((1-t)A^{*}+tB^{*})(1-t)A+tB \\
&=(1-t)^{2}A^{*}A+t(1-t)B^{*}A+(1-t)tA^{*}B+t^{2}B^{*}B \\
&=(1-t)^{2}|A|^{2}+2t(1-t)\operatorname{Re}(B^{*}A)+t^{2}|B|^{2}.
\end{aligned}
$$

Now, consider $p_{1}:=(1-t)^{2}$, $p_{2}:=2t(1-t)$ and $p_{3}:=t^{2}$. We observe that p_{1}, p_{2}, $p_{3}\geq 0$ and

$$p_{1}+p_{2}+p_{3}=(1-t)^{2}+2t(1-t)+t^{2}=(1-t+t)^{2}=1.$$

Also, put $A_{1}:=|A|^{2}$, $A_{2}:=\operatorname{Re}(B^{*}A)$ and $A_{3}:=|B|^{2}$. Then

$$p_{1}A_{1}+p_{2}A_{3}+p_{3}A_{3}=|(1-t)A+tB|^{2}$$

and

$$m1_{H}\leq p_{1}A_{1}+p_{2}A_{3}+p_{3}A_{3}\leq M1_{H}.$$

If we use (2.11) for $n=3$ we get

$$\begin{aligned} 0 &\leq (1-t)^2 f\left(|A|^2\right) + 2t(1-t) f(\operatorname{Re}(B^*A)) + t^2 f\left(|B|^2\right) \\ &- f\left(|(1-t)A+tB|^2\right) \\ &\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\times \left(\frac{1}{2}(M-m)1_H + \left||(1-t)A+tB|^2 - \frac{m+M}{2}1_H\right|\right) \\ &\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]1_H, \end{aligned}$$

which proves (3.1).

If we take the integral and observe that

$$\int_0^1 (1-t)^2 dt = 2\int_0^1 t(1-t)\,dt = \int_0^1 t^2 dt = \frac{1}{3},$$

then we also obtain (3.2). □

Theorem 5. *With the assumptions of Theorem 4 we have*

$$\begin{aligned} 0 &\leq (1-t)^2 f\left(|A|^2\right) + 2t(1-t) f(\operatorname{Re}(B^*A)) + t^2 f\left(|B|^2\right) \\ &- f\left(|(1-t)A+tB|^2\right) \\ &\leq \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m} \\ &\times \left(|(1-t)A+tB|^2 - m1_H\right)\left(M1_H - |(1-t)A+tB|^2\right) \\ &\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]1_H \end{aligned} \tag{3.3}$$

for all $t\in[0,1]$ and the integral inequality

$$\begin{aligned} 0 &\leq \frac{f\left(|A|^2\right) + f(\operatorname{Re}(B^*A)) + f\left(|B|^2\right)}{3} - \int_0^1 f\left(|(1-t)A+tB|^2\right)dt \\ &\leq \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m} \\ &\times \left[\frac{1}{4}(M-m)^2 1_H - \int_0^1 \left(|(1-t)A+tB|^2 - \frac{m+M}{2}1_H\right)^2 dt\right] \\ &\leq \frac{\left[f'_-(M) - f'_+(m)\right]}{M-m} \\ &\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2}1_H\right)^2\right] \\ &\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]1_H. \end{aligned} \tag{3.4}$$

Proof. The inequality (3.3) follows by (2.26) by taking $p_1 := (1-t)^2$, $p_2 := 2t(1-t)$ and $p_3 := t^2$ and $A_1 := |A|^2$, $A_2 := \operatorname{Re}(B^*A)$ and $A_3 := |B|^2$.

Now, observe that, by using the elementary identity

$$(X - m1_H)(M1_H - X) = \frac{1}{4}(M-m)^2 1_H - \left(X - \frac{m+M}{2}1_H\right)^2$$

we get

$$\begin{aligned}&\int_0^1 \left(|(1-t)A + tB|^2 - m1_H\right)\left(M1_H - |(1-t)A + tB|^2\right) dt \\ &= \frac{1}{4}(M-m)^2 1_H - \int_0^1 \left(|(1-t)A + tB|^2 - \frac{m+M}{2}1_H\right)^2 dt.\end{aligned}$$

By the operator convexity of the square function and Jensen's inequality we have

$$\begin{aligned}&\int_0^1 \left(|(1-t)A + tB|^2 - \frac{m+M}{2}1_H\right)^2 \\ &\geq \left(\int_0^1 |(1-t)A + tB|^2 - \frac{m+M}{2}1_H\right)^2 \\ &= \left(\int_0^1 \left[(1-t)^2 |A|^2 + 2t(1-t)\operatorname{Re}(B^*A) + t^2 |B|^2\right] dt - \frac{m+M}{2}1_H\right)^2 \\ &= \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2}1_H\right)^2,\end{aligned}$$

which gives that

$$\begin{aligned}&\frac{1}{4}(M-m)^2 1_H - \int_0^1 \left(|(1-t)A + tB|^2 - \frac{m+M}{2}1_H\right)^2 \\ &\leq \frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2}1_H\right)^2 \\ &\leq \frac{1}{4}(M-m)^2 1_H,\end{aligned}$$

which proves the last part of (3.4). □

Theorem 6. *Assume that the function $f : [0, \infty) \to \mathbb{R}$ is operator convex with $f''(x) \leq D$ for all $x \in (m, M)$. If A, $B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) \geq 0$ and $0 \leq m \leq |(1-t)A + tB|^2 \leq M$ for all $t \in [0,1]$, then*

$$\begin{aligned}0 &\leq (1-t)^2 f\left(|A|^2\right) + 2t(1-t) f(\operatorname{Re}(B^*A)) + t^2 f\left(|B|^2\right) \\ &\quad - f\left(|(1-t)A + tB|^2\right) \\ &\leq \frac{1}{2}D\left(|(1-t)A + tB|^2 - m1_H\right)\left(M1_H - |(1-t)A + tB|^2\right) \\ &\leq \frac{1}{8}(M-m)^2 D1_H\end{aligned} \tag{3.5}$$

for all $t\in[0,1]$ and the integral inequality

$$0\le\frac{f\left(|A|^2\right)+f\left(\operatorname{Re}\left(B^*A\right)\right)+f\left(|B|^2\right)}{3}-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt \tag{3.6}$$
$$\le\frac{1}{2}D\left[\frac{1}{4}(M-m)^2 1_H-\int_0^1\left(|(1-t)A+tB|^2-\frac{m+M}{2}1_H\right)^2dt\right]$$
$$\le\frac{1}{2}D\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{|A|^2+\operatorname{Re}\left(B^*A\right)+|B|^2}{3}-\frac{m+M}{2}1_H\right)^2\right]$$
$$\le\frac{1}{8}(M-m)^2 D1_H.$$

The proof follows in a similar way from Theorem 3.

Remark 2. *We observe that, if $0\le m1_H\le|A|^2,\ \operatorname{Re}\left(B^*A\right),\ |B|^2\le M1_H$ then for all $t\in[0,1]$*

$$\begin{aligned}0&\le(1-t)^2 m+2t(1-t)m+t^2m\\&\le(1-t)^2|A|^2+2t(1-t)\operatorname{Re}\left(B^*A\right)+t^2|B|^2\\&\le(1-t)^2M+2t(1-t)\operatorname{Re}M+t^2M\end{aligned}$$

that is equivalent to $0\le m1_H\le|(1-t)A+tB|^2\le M1_H$, which gives a sufficient condition for the assumption in the above theorems to hold.

The following Cauchy-Bunyakowsky-Schwarz inequality holds

$$\sum_{k=1}^n w_k|z_k|^2\sum_{k=1}^n w_k|A_k|^2\ge\left|\sum_{k=1}^n w_kz_kA_k\right|^2, \tag{3.7}$$

where $z_k\in\mathbb{C},\ A_k\in\mathcal{B}(H),\ w_k\ge0$ for $k\in\{1,...,n\}$ and $\sum_{k=1}^n w_k=1$.

Let $A,\ B\in\mathcal{B}(H)$ and $\alpha\in[0,1]$. Then by (3.7), we get

$$\begin{aligned}|(1-\alpha)A+\alpha B|^2&=\left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A+\alpha^{1/2}\alpha^{1/2}B\right|^2\\&\le\left[\left((1-\alpha)^{1/2}\right)^2+\left(\alpha^{1/2}\right)^2\right]\left[\left|(1-\alpha)^{1/2}A\right|^2+\left|\alpha^{1/2}B\right|^2\right]\\&=(1-\alpha+\alpha)\left[(1-\alpha)|A|^2+\alpha|B|^2\right]\\&=(1-\alpha)|A|^2+\alpha|B|^2.\end{aligned}$$

So, if $|A|^2\le M_1$ and $|B|^2\le M_2$, then $|(1-\alpha)A+\alpha B|^2\le(1-\alpha)M_1+\alpha M_2$. So, if we take $m=0$ above and $M=(1-\alpha)M_1+\alpha M_2$ in the theorems above, we can get several reverse inequalities as well, of course, for functions for which the upper bounds are finite on $(0,(1-\alpha)M_1+\alpha M_2)$.

4. Some Examples

By writing the inequalities from Theorems 4-6 for the operator convex function $f(t)=t^r$ for $r\in[1,2]$ we get for $A,\ B\in\mathcal{B}(H)$ with $\operatorname{Re}\left(B^*A\right)\ge0$ and $0\le m\le$

$|(1-t)A+tB|^2 \le M$ for all $t\in[0,1]$, the following integral inequalities

$$0 \le \frac{|A|^{2r}+[\mathrm{Re}(B^*A)]^r+|B|^{2r}}{3}-\int_0^1 |(1-t)A+tB|^{2r}dt \le 2\left[\frac{m^r+M^r}{2}-\left(\frac{m+M}{2}\right)^r\right]1_H, \tag{4.1}$$

$$\begin{aligned} 0 &\le \frac{|A|^{2r}+[\mathrm{Re}(B^*A)]^r+|B|^{2r}}{3}-\int_0^1 |(1-t)A+tB|^{2r}dt \\ &\le r\frac{M^{r-1}-m^{r-1}}{M-m} \\ &\times\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}-\frac{m+M}{2}1_H\right)^2\right] \\ &\le \frac{1}{4}r(M-m)\left(M^{r-1}-m^{r-1}\right)1_H \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} 0 &\le \frac{f\left(|A|^2\right)+f(\mathrm{Re}(B^*A))+f\left(|B|^2\right)}{3}-\int_0^1 f\left(|(1-t)A+tB|^2\right)dt \\ &\le \frac{1}{2m^{2-r}}r(r-1) \\ &\times\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}-\frac{m+M}{2}1_H\right)^2\right] \\ &\le \frac{1}{8m^{2-r}}r(r-1)(M-m)^2 1_H. \end{aligned} \tag{4.3}$$

By writing the inequalities from Theorems 4-6 for the operator concave function $f(t)=t^p$ for $p\in(0,1]$ we get for $A,\ B\in\mathcal{B}(H)$ with $\mathrm{Re}(B^*A)>0$ and $0<m\le |(1-t)A+tB|^2\le M$ for all $t\in[0,1]$, the following integral inequalities

$$0 \le \int_0^1 |(1-t)A+tB|^{2p}dt-\frac{|A|^{2p}+[\mathrm{Re}(B^*A)]^p+|B|^{2p}}{3} \le 2\left[\left(\frac{m+M}{2}\right)^p-\frac{m^p+M^p}{2}\right]1_H. \tag{4.4}$$

$$\begin{aligned} 0 &\le \int_0^1 |(1-t)A+tB|^{2p}dt-\frac{|A|^{2p}+[\mathrm{Re}(B^*A)]^p+|B|^{2p}}{3} \\ &\le \frac{p\left(M^{1-p}-m^{1-p}\right)}{m^{1-p}M^{1-p}(M-m)} \\ &\times\left[\frac{1}{4}(M-m)^2 1_H-\left(\frac{|A|^2+\mathrm{Re}(B^*A)+|B|^2}{3}-\frac{m+M}{2}1_H\right)^2\right] \\ &\le \frac{1}{4}p\frac{M^{1-p}-m^{1-p}}{m^{1-p}M^{1-p}}(M-m). \end{aligned} \tag{4.5}$$

and

$$0 \leq \int_0^1 |(1-t)A + tB|^{2p} dt - \frac{|A|^{2p} + [\operatorname{Re}(B^*A)]^p + |B|^{2p}}{3} \tag{4.6}$$
$$\leq \frac{1}{2m^{2-p}} p(1-p)$$
$$\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2} 1_H\right)^2\right]$$
$$\leq \frac{1}{8m^{2-p}} p(1-p)(M-m)^2 1_H.$$

For $p = 1/2$ we derive

$$0 \leq \int_0^1 |(1-t)A + tB| dt - \frac{|A| + [\operatorname{Re}(B^*A)]^{1/2} + |B|}{3} \tag{4.7}$$
$$\leq 2\left[\sqrt{\frac{m+M}{2}} - \frac{\sqrt{m} + \sqrt{M}}{2}\right] 1_H.$$

$$0 \leq \int_0^1 |(1-t)A + tB| dt - \frac{|A| + [\operatorname{Re}(B^*A)]^{1/2} + |B|}{3} \tag{4.8}$$
$$\leq \frac{p\left(\sqrt{M} - \sqrt{m}\right)}{\sqrt{mM}(M-m)}$$
$$\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2} 1_H\right)^2\right]$$
$$\leq \frac{1}{4} \frac{p\left(\sqrt{M} - \sqrt{m}\right)}{\sqrt{mM}(M-m)} (M-m).$$

and

$$0 \leq \int_0^1 |(1-t)A + tB| dt - \frac{|A| + [\operatorname{Re}(B^*A)]^{1/2} + |B|}{3} \tag{4.9}$$
$$\leq \frac{1}{8m^{3/2}}$$
$$\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2} 1_H\right)^2\right]$$
$$\leq \frac{1}{32m^{3/2}}(M-m)^2 1_H.$$

By writing the inequalities from Theorems 4-6 for the operator convex function $f(t) = t^{-q}$ on $(0, \infty)$ with $q \in (0, 1]$ we get for $A, B \in \mathcal{B}(H)$ with $\operatorname{Re}(B^*A) > 0$ and

$0 < m \leq |(1-t)A + tB|^2 \leq M$ for all $t \in [0,1]$, the following integral inequalities

$$0 \leq \frac{|A|^{-2q} + [\operatorname{Re}(B^*A)]^{-q} + |B|^{-2q}}{3} - \int_0^1 |(1-t)A + tB|^{-2q} dt \leq 2\left[\frac{m^q + M^q}{2m^q M^q} - \left(\frac{2}{m+M}\right)^q\right] 1_H, \tag{4.10}$$

$$\begin{aligned} 0 &\leq \frac{|A|^{-2q} + [\operatorname{Re}(B^*A)]^{-q} + |B|^{-2q}}{3} - \int_0^1 |(1-t)A + tB|^{-2q} dt \\ &\leq q\frac{M^q - m^q}{M^q m^q (M-m)} \\ &\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2}1_H\right)^2\right] \\ &\leq \frac{1}{4}q(M-m)\frac{M^q - m^q}{M^q m^q}1_H. \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} 0 &\leq \frac{|A|^{-2q} + [\operatorname{Re}(B^*A)]^{-q} + |B|^{-2q}}{3} - \int_0^1 |(1-t)A + tB|^{-2q} dt \\ &\leq \frac{1}{2m^{q+2}}q(q+1) \\ &\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2}1_H\right)^2\right] \\ &\leq \frac{1}{8m^{q+2}}q(q+1)(M-m)^2 1_H. \end{aligned} \tag{4.12}$$

For $q = 1$ we obtain

$$0 \leq \frac{|A|^{-2} + [\operatorname{Re}(B^*A)]^{-1} + |B|^{-2}}{3} - \int_0^1 |(1-t)A + tB|^{-2} dt \leq \frac{M-m}{mM(m+M)}1_H, \tag{4.13}$$

$$\begin{aligned} 0 &\leq \frac{|A|^{-2} + [\operatorname{Re}(B^*A)]^{-1} + |B|^{-2}}{3} - \int_0^1 |(1-t)A + tB|^{-2} dt \\ &\leq \frac{1}{Mm}\left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \operatorname{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2}1_H\right)^2\right] \\ &\leq \frac{1}{4}\frac{(M-m)^2}{Mm}1_H. \end{aligned} \tag{4.14}$$

and

$$0 \leq \frac{|A|^{-2} + [\mathrm{Re}(B^*A)]^{-1} + |B|^{-2}}{3} - \int_0^1 |(1-t)A + tB|^{-2} dt \tag{4.15}$$
$$\leq \frac{1}{m^3}\left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \mathrm{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2} 1_H\right)^2\right]$$
$$\leq \frac{1}{4m^3}(M-m)^2 1_H.$$

For $q = 1/2$ we derive

$$0 \leq \frac{|A|^{-1} + [\mathrm{Re}(B^*A)]^{-1/2} + |B|^{-1}}{3} - \int_0^1 |(1-t)A + tB|^{-1} dt \tag{4.16}$$
$$\leq 2\left[\frac{m^{1/2} + M^{1/2}}{2m^{1/2}M^{1/2}} - \left(\frac{2}{m+M}\right)^{1/2}\right] 1_H,$$

$$0 \leq \frac{|A|^{-1} + [\mathrm{Re}(B^*A)]^{-1/2} + |B|^{-1}}{3} - \int_0^1 |(1-t)A + tB|^{-1} dt \tag{4.17}$$
$$\leq \frac{M^{1/2} - m^{1/2}}{2M^{1/2}m^{1/2}(M-m)}$$
$$\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \mathrm{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2} 1_H\right)^2\right]$$
$$\leq \frac{1}{8}(M-m)\frac{M^{1/2} - m^{1/2}}{M^{1/2}m^{1/2}} 1_H.$$

and

$$0 \leq \frac{|A|^{-1} + [\mathrm{Re}(B^*A)]^{-1/2} + |B|^{-1}}{3} - \int_0^1 |(1-t)A + tB|^{-1} dt \tag{4.18}$$
$$\leq \frac{3}{8m^{3/2}} q(q+1)$$
$$\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2 + \mathrm{Re}(B^*A) + |B|^2}{3} - \frac{m+M}{2} 1_H\right)^2\right]$$
$$\leq \frac{3}{32m^{3/2}}(M-m)^2 1_H.$$

By writing the inequalities from Theorems 4-6 for the operator convex function $f(t) = -\ln t$ on $(0, \infty)$ with $q \in (0, 1]$ we get for $A, B \in \mathcal{B}(H)$ with $\mathrm{Re}(B^*A) > 0$ and $0 < m \leq |(1-t)A + tB|^2 \leq M$ for all $t \in [0, 1]$, the following integral inequalities

$$0 \leq \int_0^1 \ln\left(|(1-t)A + tB|^2\right) dt - \frac{\ln\left(|A|^2\right) + \ln(\mathrm{Re}(B^*A)) + \ln\left(|B|^2\right)}{3} \tag{4.19}$$
$$\leq \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^2 1_H,$$

$$
\begin{aligned}
(4.20)\quad 0 &\leq \int_0^1 \ln\left(\left|(1-t)A+tB\right|^2\right)dt - \frac{\ln\left(|A|^2\right)+\ln\left(\operatorname{Re}(B^*A)\right)+\ln\left(|B|^2\right)}{3} \\
&\leq \frac{1}{mM} \\
&\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}-\frac{m+M}{2}1_H\right)^2\right] \\
&\leq \frac{1}{4mM}(M-m)^2 1_H.
\end{aligned}
$$

and

$$
\begin{aligned}
(4.21)\quad 0 &\leq \int_0^1 \ln\left(\left|(1-t)A+tB\right|^2\right)dt - \frac{\ln\left(|A|^2\right)+\ln\left(\operatorname{Re}(B^*A)\right)+\ln\left(|B|^2\right)}{3} \\
&\leq \frac{1}{2m^2} \\
&\times \left[\frac{1}{4}(M-m)^2 1_H - \left(\frac{|A|^2+\operatorname{Re}(B^*A)+|B|^2}{3}-\frac{m+M}{2}1_H\right)^2\right] \\
&\leq \frac{1}{8m^2}(M-m)^2 D1_H.
\end{aligned}
$$

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