LIPSCHITZ TYPE p-SCHATTEN NORM INEQUALITIES FOR $\mathcal{D}(w,\mu)$ INTEGRAL TRANSFORM OF POSITIVE OPERATORS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *integral transform*

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

Assume that A, $B \in \mathcal{B}_p(H)$, $p \ge 1$ and $A \ge m_1 > 0$, $B \ge m_2 > 0$, then

$$\begin{aligned}&\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A)\|_p \\ &\le \|B-A\|_p \times \begin{cases} \frac{\mathcal{D}(w,\mu)(m_1) - \mathcal{D}(w,\mu)(m_2)}{m_2 - m_1} & \text{if } m_1 \ne m_2, \\ \\ (-\mathcal{D}'(w,\mu)(m)) & \text{if } m_1 = m_2 = m, \end{cases}\end{aligned}$$

where $\mathcal{D}'(w,\mu)(t)$ is the derivative of $\mathcal{D}(w,\mu)$ as a function of t.

If $f : [0,\infty) \to \mathbb{R}$ is an operator monotone function with $f(0) = 0$, then

$$\begin{aligned}&\left\|f(A)A^{-1} - f(B)B^{-1}\right\|_p \\ &\le \|B-A\|_p \begin{cases} \frac{f(m_1)m_1^{-1} - f(m_2)m_2^{-1}}{m_2 - m_1} & \text{if } m_1 \ne m_2, \\ \\ \frac{f(m) - f'(m)m}{m^2} & \text{if } m_1 = m_2 = m. \end{cases}\end{aligned}$$

Similar inequalities for operator convex functions and some particular examples of interest are also given.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [6] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \le L\|A - B\|$$

for any A, $B \in \mathcal{B}(H)$.

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However, as shown by Farforovskaya in [28], [29] and Kato in [36], the following inequality holds

$$\| |A| - |B| \| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log \left(\frac{\|A\| + \|B\|}{\|A - B\|} \right) \right) \tag{1.1}$$

for any $A, B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_2 := (\operatorname{tr} C^* C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\| |A| - |B| \|_2 \leq \sqrt{2} \|A - B\|_2 \tag{1.2}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$. By $A \geq B$ we understand that $A - B \geq 0$

In 1934, K. Löwner [40] had given a definitive characterization of operator monotone functions as follows, see for instance [8, p. 144-145]:

Theorem 1. *A function* $f : [0, \infty) \to \mathbb{R}$ *is operator monotone in* $[0, \infty)$ *if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda) \tag{1.3}$$

where μ *is a positive measure on* $(0, \infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, [35]. The function ln is also operator monotone on $(0, \infty)$. For other examples of operator monotone functions, see [31] and [34].

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1 - \lambda) A + \lambda B) \leq (\geq) (1 - \lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [8, p. 147]:

Theorem 2. *A function* $f : [0, \infty) \to \mathbb{R}$ *is operator convex in* $[0, \infty)$ *with* $f'_+(0) \in \mathbb{R}$ *if and only if it has the representation*

$$f(t) = f(0) + f'_+(0) t + ct^2 + \int_0^\infty \frac{t^2 \lambda}{t + \lambda} d\mu(\lambda), \tag{1.4}$$

where $c \geq 0$ *and a positive measure* μ *on* $(0, \infty)$ *such that (1.4) holds.*

It has been shown in [6] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\| \leq a_1 \|A-B\| + a_2 \|A-B\|^2 + O\left(\|A-B\|^3\right) \tag{1.5}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2 .$$

In [5] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A-B\| \tag{1.6}$$

where f is an *operator monotone function* on $(0,\infty)$ and $A,\ B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [7], [30] and the references therein.

We have the following integral representation for the power function when $t>0$, $r \in (0,1]$, see for instance [8, p. 145]

$$t^r = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{t\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.7}$$

Observe that for $t>0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{1.8}$$

for all $t>0$.

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\mu(\lambda),\ t>0, \tag{1.9}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.9) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t} d\lambda,\ t>0. \tag{1.10}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t),\ t>0. \tag{1.11}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, we have the representation

$$\ln t = (t-1)\,\mathcal{D}(w_{\ln})(t), \ t > 0. \tag{1.12}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.13}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.14}$$

for $T > 0$.

In order to extend the above results to *p-Schatten norm* and $\mathcal{D}(w, \mu)$ operator transform, we need the following preparations.

Let $(H; \langle ., . \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.15}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle, \tag{1.16}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.16) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.17}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \textit{and} \ |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.18}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;*

(iv) If A, $B \in \mathcal{B}_2(H)$ then AB, $BA \in \mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite [46, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.19}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\| . \tag{1.20}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [46, p. 60-64], for $p \geq 1$

$$\|A\|_p = \|A^*\|_p , \; A \in \mathcal{B}_p(H) \tag{1.21}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p , \; A, B \in \mathcal{B}_p(H) \tag{1.22}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\| , \; \|BA\|_p \leq \|B\| \|A\|_p , \; A \in \mathcal{B}_p(H) , \; B \in \mathcal{B}(H) . \tag{1.23}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\| , \; A \in \mathcal{B}_p(H) , \; B, \; C \in \mathcal{B}(H) . \tag{1.24}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q , \; A \in \mathcal{B}_p(H) , \; B \in \mathcal{B}_q(H) . \tag{1.25}$$

For the theory of trace functionals and their applications the reader is referred to [44] and [46].

For some classical trace inequalities see [9], [11], and [41], which are continuations of the work of Bellman [3]. For related works the reader can refer to [1], [4], [9], [32], [37], [38], [39], [43], [45] and the recent papers [12]-[27].

2. Main Results

We have the following Lipschitz type inequality for the p-Schatten norm $\|\cdot\|_p$ for $p \geq 1$.

Theorem 4. *Assume that A, $B \in \mathcal{B}_p(H)$, $p \geq 1$ and $A \geq m_1 > 0$, $B \geq m_2 > 0$, then*

$$\begin{aligned} &\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A)\|_p \\ &\leq \|B - A\|_p \times \begin{cases} \frac{\mathcal{D}(w,\mu)(m_1) - \mathcal{D}(w,\mu)(m_2)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ (-\mathcal{D}'(w,\mu)(m)) & \text{if } m_1 = m_2 = m, \end{cases} \end{aligned} \tag{2.1}$$

where $\mathcal{D}'(w,\mu)(t)$ is the derivative of $\mathcal{D}(w,\mu)$ as a function of t.

Proof. Observe that, for all A, $B > 0$

$$\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A) = \int_0^\infty w(\lambda) \left[(\lambda + B)^{-1} - (\lambda + A)^{-1}\right] d\mu(\lambda) . \tag{2.2}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotone on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \to 0} \left[\frac{f(T + tS) - f(T)}{t}\right] = T^{-1} S T^{-1} \tag{2.3}$$

for T, $S > 0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C, D] :=$

$\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.4}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1 ((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.5}$$

Now, if we take in (2.5) $C=\lambda+B$, $D=\lambda+A$, then

$$\begin{aligned}(\lambda+B)^{-1}-(\lambda+A)^{-1} &= \int_0^1 ((1-t)(\lambda+B)+t(\lambda+A))^{-1}(A-B) \\ &\times ((1-t)(\lambda+B)+t(\lambda+A))^{-1}\,dt \\ &= \int_0^1 (\lambda+(1-t)B+tA)^{-1}(A-B)(\lambda+(1-t)B+tA)^{-1}\,dt \\ &= \int_0^1 (\lambda+(1-t)A+tB)^{-1}(A-B)(\lambda+(1-t)A+tB)^{-1}\,dt\end{aligned} \tag{2.6}$$

and by (2.2) we derive the following identity of interest:

$$\begin{aligned}&\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A) \\ &= -\int_0^\infty w(\lambda) \\ &\times \left(\int_0^1 (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,dt\right)d\mu(\lambda).\end{aligned} \tag{2.7}$$

From the identity (2.7) we get by taking the *p-Schatten norm* that

$$\begin{aligned}&\|\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)(A)\|_p \le \int_0^\infty w(\lambda) \\ &\times \left\|\int_0^1 (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,dt\right\|_p d\mu(\lambda) \\ &\le \int_0^\infty w(\lambda) \\ &\times \left(\int_0^1 \left\|(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\right\|_p dt\right)d\mu(\lambda) \\ &\le \|B-A\|_p \int_0^\infty w(\lambda)\left(\int_0^1 \left\|(\lambda+(1-t)A+tB)^{-1}\right\|^2 dt\right)d\mu(\lambda)\end{aligned} \tag{2.8}$$

for all A, $B>0$, A, $B\in\mathcal{B}_p(H)$, where for the last inequality we used the property (1.24).

Assume that $m_2 > m_1$. Then

$$(1-t)A + tB + \lambda \geq (1-t)m_1 + tm_2 + \lambda,$$

which implies that

$$((1-t)A + tB + \lambda)^{-1} \leq ((1-t)m_1 + tm_2 + \lambda)^{-1},$$

and

$$\left\| ((1-t)A + tB + \lambda)^{-1} \right\|^2 \leq ((1-t)m_1 + tm_2 + \lambda)^{-2} \tag{2.9}$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

Therefore, by integrating (2.9) we derive

$$\begin{aligned}
&\int_0^\infty w(\lambda) \left(\int_0^1 \left\| ((1-t)A + tB + \lambda)^{-1} \right\|^2 dt \right) dw(\lambda) \\
&\leq \int_0^\infty w(\lambda) \left(\int_0^1 ((1-t)m_1 + tm_2 + \lambda)^{-2} dt \right) dw(\lambda) \\
&= \frac{1}{m_2 - m_1} \int_0^\infty w(\lambda) \left(\int_0^1 ((1-t)m_1 + tm_2 + \lambda)^{-1} \right. \\
&\left. \times (m_2 - m_1)((1-t)m_1 + tm_2 + \lambda)^{-1} dt \right) dw(\lambda) \\
&= \frac{1}{m_2 - m_1} [\mathcal{D}(w,\mu)(m_1) - \mathcal{D}(w,\mu)(m_2)] \text{ (by (2.7))}
\end{aligned}$$

and by (2.8) we deduce

$$\begin{aligned}
&\|\mathcal{D}(w,\mu)(B) - \mathcal{D}(w,\mu)(A)\|_p \\
&\leq \frac{\|B-A\|_p}{m_2 - m_1} [\mathcal{D}(w,\mu)(m_1) - \mathcal{D}(w,\mu)(m_2)].
\end{aligned} \tag{2.10}$$

The case $m_2 < m_1$ goes in a similar way and we also obtain (2.10).

Let $\epsilon > 0$. Then $B + \epsilon \geq m + \epsilon > m$. From (2.10) we get

$$\begin{aligned}
&\|\mathcal{D}(w,\mu)(B+\epsilon) - \mathcal{D}(w,\mu)(A)\|_p \\
&\leq \frac{\|B-A\|_p}{m+\epsilon-m} [\mathcal{D}(w,\mu)(m) - \mathcal{D}(w,\mu)(m+\epsilon)] \\
&= \|B-A\|_p \left[\frac{\mathcal{D}(w,\mu)(m) - \mathcal{D}(w,\mu)(m+\epsilon)}{\epsilon} \right]
\end{aligned}$$

and by taking the limit over $\epsilon \to 0+$, using the continuity and differentiability of $\mathcal{D}(w,\mu)$ we deduce the second part of (2.1). □

Remark 1. *Assume that* $A, B \in \mathcal{B}_1(H)$, *and* $A \geq m_1 > 0$, $B \geq m_2 > 0$, *then we have the trace inequality*

$$\begin{aligned}
&|\operatorname{tr}[\mathcal{D}(w,\mu)(B)] - \operatorname{tr}[\mathcal{D}(w,\mu)(A)]| \\
&\leq \|B-A\|_1 \times \begin{cases} \frac{\mathcal{D}(w,\mu)(m_1) - \mathcal{D}(w,\mu)(m_2)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ (-\mathcal{D}'(w,\mu)(m)) & \text{if } m_1 = m_2 = m. \end{cases}
\end{aligned} \tag{2.11}$$

Corollary 1. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator monotone function. If $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$ and $A\geq m_1>0$, $B\geq m_2>0$, then*

$$\left\|f(A)A^{-1}-f(B)B^{-1}-f(0)\left(A^{-1}-B^{-1}\right)\right\|_p \leq \|B-A\|_p\times\begin{cases}\left(\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1}-\frac{f(0)}{m_2m_1}\right) & \text{if } m_1\neq m_2,\\ \frac{f(m)-f(0)-f'(m)m}{m^2} & \text{if } m_1=m_2=m.\end{cases} \tag{2.12}$$

If $f(0)=0$, then we have the simpler inequalities

$$\left\|f(A)A^{-1}-f(B)B^{-1}\right\|_p \leq \|B-A\|_p\times\begin{cases}\frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \frac{f(m)-f'(m)m}{m^2} & \text{if } m_1=m_2=m.\end{cases} \tag{2.13}$$

Proof. From (1.3) we have that

$$\frac{f(t)-f(0)}{t}-b=\mathcal{D}(\ell,\mu)(t),$$

where $\ell(\lambda)=\lambda$, $\lambda>0$.

Then

$$\frac{1}{m_2-m_1}\left[\mathcal{D}(w,\mu)(m_1)-\mathcal{D}(w,\mu)(m_2)\right] = \frac{f(m_1)m_1^{-1}-f(m_2)m_2^{-1}}{m_2-m_1}-\frac{f(0)}{m_2m_1}$$

and

$$-\mathcal{D}'(w,\mu)(m)=\frac{f(m)-f(0)-f'(m)m}{m^2}.$$

By making use of (2.1) we derive (2.12). □

Remark 2. *Assume that $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$ and $A\geq m_1>0$, $B\geq m_2>0$ and $r\in(0,1]$. Then by (2.13) we have the power inequalities*

$$\left\|A^{r-1}-B^{r-1}\right\|_p\leq\|B-A\|_p\times\begin{cases}\frac{m_1^{r-1}-m_2^{r-1}}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \frac{1-r}{m^{2-r}} & \text{if } m_1=m_2=m.\end{cases} \tag{2.14}$$

If we take $f(t)=\ln(t+1)$, then we get by (2.13) that

$$\left\|A^{-1}\ln(A+1)-B^{-1}\ln(B+1)\right\|_p \leq \|B-A\|_p\times\begin{cases}\frac{m_1^{-1}\ln(m_1+1)-m_2^{-1}\ln(m_2+1)}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \frac{(m+1)\ln(m+1)-m}{m^2(m+1)} & \text{if } m_1=m_2=m.\end{cases} \tag{2.15}$$

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function. If A, $B\in\mathcal{B}_p(H)$, $p\geq 1$, and $A\geq m_1>0$, $B\geq m_2>0$, then*

$$\left\|f(B)B^{-2}-f(A)A^{-2}-f'_+(0)\left(B^{-1}-A^{-1}\right)-f(0)\left(B^{-2}-A^{-2}\right)\right\|_p \tag{2.16}$$
$$\leq\|B-A\|_p \times\begin{cases}\frac{f(m_1)m_1^{-2}-f(m_2)m_2^{-2}}{m_2-m_1}-\frac{f'_+(0)}{m_1m_2}-f(0)\frac{m_1+m_2}{m_1^2m_1^2} & \text{if } m_1\neq m_2,\\ 2\frac{f(m)-f(0)}{m^3}-\frac{f'(m)+f'_+(0)}{m^2} & \text{if } m_1=m_2=m.\end{cases}$$

If $f(0)=0$, then

$$\left\|f(B)B^{-2}-f(A)A^{-2}-f'_+(0)\left(B^{-1}-A^{-1}\right)\right\| \tag{2.17}$$
$$\leq\|B-A\|\begin{cases}\frac{f(m_1)m_1^{-2}-f(m_2)m_2^{-2}}{m_2-m_1}-\frac{f'_+(0)}{m_1m_2} & \text{if } m_1\neq m_2,\\ \frac{2}{m^2}\left[\frac{f(m)}{m}-\frac{f'(m)+f'_+(0)}{2}\right] & \text{if } m_1=m_2=m.\end{cases}$$

Proof. From (1.5) we have that

$$\frac{f(t)-f(0)-f'_+(0)t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for $t>0$.

Then

$$\frac{\mathcal{D}(w,\mu)(m_1)-\mathcal{D}(w,\mu)(m_2)}{m_2-m_1}$$
$$=\frac{1}{m_2-m_1}\left[\frac{f(m_1)-f(0)-f'_+(0)m_1}{m_1^2}-\frac{f(m_2)-f(0)-f'_+(0)m_2}{m_2^2}\right]$$
$$=\frac{f(m_1)m_1^{-2}-f(m_2)m_2^{-2}}{m_2-m_1}-\frac{f'_+(0)}{m_1m_2}-f(0)\frac{m_1+m_2}{m_1^2m_1^2}.$$

Since

$$\left(\frac{f(t)-f(0)-f'_+(0)t}{t^2}\right)'=\frac{\left(f'(t)-f'_+(0)\right)t^2-2t\left(f(t)-f(0)-f'_+(0)t\right)}{t^4}$$
$$=\frac{f'(t)+f'_+(0)}{t^2}-2\frac{f(t)-f(0)}{t^3},$$

hence

$$-\mathcal{D}'(w,\mu)(m)=2\frac{f(m)-f(0)}{m^3}-\frac{f'(m)+f'_+(0)}{m^2}$$

and by (2.1) we obtain (2.16). □

Remark 3. *If we take $f(t)=-\ln(t+1)$ in (2.17), then for A, $B\in\mathcal{B}_p(H)$, $p\geq 1$, with $A\geq m_1>0$, $B\geq m_2>0$ we get*

$$\left\|B^{-2}\ln(B+1)-A^{-2}\ln(A+1)-B^{-1}+A^{-1}\right\|_p \tag{2.18}$$
$$\leq\|B-A\|_p\times\begin{cases}\frac{m_2^{-2}\ln(m_2+1)-m_1^{-2}\ln(m_1+1)}{m_2-m_1}+\frac{1}{m_1m_2} & \text{if } m_1\neq m_2,\\ \frac{m+2}{m^2(m+1)}-\frac{2\ln(m+1)}{m^3} & \text{if } m_1=m_2=m.\end{cases}$$

3. Applications for Midpoint and Trapezoid Inequalities

We have the following midpoint type inequalities:

Proposition 1. *For all* $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$,*with* $A,\ B\geq m>0$ *we have the midpoint inequality*

$$\left\|\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_p \leq -\frac{1}{4}\mathcal{D}'(w,\mu)(m)\|B-A\|_p. \tag{3.1}$$

Proof. Since $A,\ B\geq m$, hence $\frac{A+B}{2}\geq m>0$ and $(1-t)A+tB\geq m>0$ for all $t\in[0,1]$ and by (2.1)

$$\begin{aligned}&\left\|\mathcal{D}(w,\mu)((1-t)A+tB)-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_p\\ &\leq(-\mathcal{D}'(w,\mu)(m))\left\|(1-t)A+tB-\frac{A+B}{2}\right\|_p\\ &=-\mathcal{D}'(w,\mu)(m)\left|t-\frac{1}{2}\right|\|B-A\|_p\end{aligned} \tag{3.2}$$

for all $t\in[0,1]$.

Taking the integral in (3.2), we get

$$\begin{aligned}&\left\|\int_0^1\mathcal{D}(w,\mu)((1-t)A+tB)\,dt-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_p\\ &\leq\int_0^1\left\|\mathcal{D}(w,\mu)((1-t)A+tB)-\mathcal{D}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_p dt\\ &\leq-\mathcal{D}'(w,\mu)(m)\|B-A\|_p\int_0^1\left|t-\frac{1}{2}\right|dt=-\frac{1}{4}\mathcal{D}'(w,\mu)(m)\|B-A\|_p\end{aligned}$$

and the inequality (3.1) is proved. □

The case of operator monotone functions is as follows:

Corollary 3. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function. If* $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$, *with* $A,\ B\geq m>0$, *then*

$$\begin{aligned}&\left\|\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)\,dt-\left(\frac{A+B}{2}\right)^{-1}f\left(\frac{A+B}{2}\right)\right.\\ &\left.-f(0)\left(\int_0^1((1-t)A+tB)^{-1}dt-\left(\frac{A+B}{2}\right)^{-1}\right)\right\|_p\\ &\leq\frac{1}{4}\|B-A\|_p\frac{f(m)-f(0)-f'(m)m}{m^2}.\end{aligned} \tag{3.3}$$

If $f(0)=0$, *then*

$$\left\|\int_0^1 ((1-t)A+tB)^{-1} f((1-t)A+tB)\,dt - \left(\frac{A+B}{2}\right)^{-1} f\left(\frac{A+B}{2}\right)\right\|_p \le \frac{1}{4}\|B-A\|_p \frac{f(m)-f'(m)m}{m^2}. \tag{3.4}$$

Proof. From (1.3) we have that

$$\frac{f(t)-f(0)}{t} - b = \mathcal{D}(\ell,\mu)(t),$$

where $\ell(\lambda)=\lambda$, $\lambda>0$.

Then

$$\int_0^1 \mathcal{D}(\ell,\mu)((1-t)A+tB)\,dt = \int_0^1 ((1-t)A+tB)^{-1} f((1-t)A+tB)\,dt - f(0)\int_0^1 ((1-t)A+tB)^{-1}\,dt - b,$$

and

$$\mathcal{D}(\ell,\mu)\left(\frac{A+B}{2}\right) = \left(\frac{A+B}{2}\right)^{-1} f\left(\frac{A+B}{2}\right) - f(0)\left(\frac{A+B}{2}\right)^{-1} - b$$

and by (3.1) we get (3.3). □

From inequality (3.4) we get the following power inequality

$$\left\|\int_0^1 ((1-t)A+tB)^{r-1}\,dt - \left(\frac{A+B}{2}\right)^{r-1}\right\|_p \le \frac{1-r}{4m^{2-r}}\|B-A\|_p, \tag{3.5}$$

where $r\in(0,1]$ and $A,\,B\in\mathcal{B}_p(H)$, $p\ge 1$, with $A,\,B\ge m>0$.

The following logarithmic inequality also holds

$$\left\|\int_0^1 ((1-t)A+tB)^{-1}\ln((1-t)A+tB+1)\,dt - \left(\frac{A+B}{2}\right)^{-1}\ln\left(\frac{A+B}{2}+1\right)\right\|_p \le \frac{(m+1)\ln(m+1)-m}{4m^2(m+1)}\|B-A\|_p, \tag{3.6}$$

where $A,\,B\in\mathcal{B}_p(H)$, $p\ge 1$, with $A,\,B\ge m>0$.

Corollary 4. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function. If A, $B\in\mathcal{B}_p(H)$, $p\geq 1$, with A, $B\geq m>0$, then*

$$\begin{aligned}(3.7)\quad &\left\|\int_0^1((1-t)A+tB)^{-2}f((1-t)A+tB)\,dt-\left(\frac{A+B}{2}\right)^{-2}f\left(\frac{A+B}{2}\right)\right.\\&-f(0)\left(\int_0^1((1-t)A+tB)^{-2}dt-\left(\frac{A+B}{2}\right)^{-2}\right)\\&\left.-f'_+(0)\left(\int_0^1((1-t)A+tB)^{-1}dt-\left(\frac{A+B}{2}\right)^{-1}\right)\right\|_p\\&\leq\frac{1}{2m^2}\|B-A\|_p\left(\frac{f(m)-f(0)}{m}-\frac{f'(m)+f'_+(0)}{2}\right).\end{aligned}$$

If $f(0)=0$, then

$$\begin{aligned}(3.8)\quad &\left\|\int_0^1((1-t)A+tB)^{-2}f((1-t)A+tB)\,dt-\left(\frac{A+B}{2}\right)^{-2}f\left(\frac{A+B}{2}\right)\right.\\&\left.-f'_+(0)\left(\int_0^1((1-t)A+tB)^{-1}dt-\left(\frac{A+B}{2}\right)^{-1}\right)\right\|_p\\&\leq\frac{1}{2m^2}\|B-A\|_p\left(\frac{f(m)}{m}-\frac{f'(m)+f'_+(0)}{2}\right).\end{aligned}$$

Proof. From (1.5) we have that

$$\frac{f(t)-f(0)-f'_+(0)\,t}{t^2}-c=\mathcal{D}(\ell,\mu)(t),$$

for $t>0$. We have

$$\begin{aligned}\int_0^1\mathcal{D}(\ell,\mu)((1-t)A+tB)\,dt&=\int_0^1((1-t)A+tB)^{-2}f((1-t)A+tB)\,dt\\&-f(0)\int_0^1((1-t)A+tB)^{-2}dt\\&-f'_+(0)\int_0^1((1-t)A+tB)^{-1}dt-c\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}(\ell,\mu)\left(\frac{A+B}{2}\right)&=\left(\frac{A+B}{2}\right)^{-2}f\left(\frac{A+B}{2}\right)-f(0)\left(\frac{A+B}{2}\right)^{-2}\\&-f'_+(0)\left(\frac{A+B}{2}\right)^{-1}-c\end{aligned}$$

and by (3.1) we get (3.7). □

From (3.8) we get the logarithmic inequality

$$\begin{aligned}(3.9)\qquad &\left\| \int_0^1 ((1-t)A+tB)^{-2} \ln((1-t)A+tB+1)\,dt \right.\\ &-\left(\frac{A+B}{2}\right)^{-2} \ln\left(\frac{A+B}{2}+1\right)\\ &\left.-\int_0^1 ((1-t)A+tB)^{-1}\,dt+\left(\frac{A+B}{2}\right)^{-1}\right\|_p\\ &\le \frac{1}{4m^2}\|B-A\|_p\left(\frac{m+2}{m+1}-\frac{2\ln(m+1)}{m}\right)\end{aligned}$$

for $A,\ B\in\mathcal{B}_p(H)$, $p\ge 1$, with $A,\ B\ge m>0$.

We have the following midpoint type inequalities:

Proposition 2. *For all $A,\ B\in\mathcal{B}_p(H)$, $p\ge 1$, with $A,\ B\ge m>0$ we have the trapezoid inequality*

$$\begin{aligned}(3.10)\qquad &\left\|\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}-\int_0^1 \mathcal{D}(w,\mu)((1-t)A+tB)\,dt\right\|_p\\ &\le -\frac{1}{4}\mathcal{D}'(w,\mu)(m)\|B-A\|_p\,.\end{aligned}$$

Proof. Since $A,\ B\ge m$, hence $(1-s)A+s\frac{A+B}{2}$, $s\frac{A+B}{2}+(1-s)B\ge m>0$ for all $s\in[0,1]$ and by (2.5) we get

$$\begin{aligned}(3.11)\qquad &\left\|\mathcal{D}(w,\mu)(A)-\mathcal{D}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)\right\|_p\\ &\le \frac{1}{2}(-\mathcal{D}'(w,\mu)(m))\|B-A\|_p\,s\end{aligned}$$

and

$$\begin{aligned}(3.12)\qquad &\left\|\mathcal{D}(w,\mu)(B)-\mathcal{D}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right\|_p\\ &\le \frac{1}{2}(-\mathcal{D}'(w,\mu)(m))\|B-A\|_p\,s.\end{aligned}$$

From (3.11) and (3.12) we derive by addition, division by 2 and triangle inequality that

$$\begin{aligned}&\left\|\frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2}\right.\\ &\left.-\frac{1}{2}\left[\mathcal{D}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)+\mathcal{D}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right]\right\|_p\\ &\le \frac{1}{2}(-\mathcal{D}'(w,\mu)(m))\|B-A\|_p\,s\end{aligned}$$

for all $s\in[0,1]$.

By taking the integral and using its properties, we derive

$$\begin{aligned}(3.13)\quad & \Bigg\| \frac{\mathcal{D}(w,\mu)(A)+\mathcal{D}(w,\mu)(B)}{2} \\ & -\frac{1}{2}\left[\int_0^1 \mathcal{D}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)\right. \\ & \left.+\mathcal{D}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)ds\right]\Bigg\|_p \\ & \le \frac{1}{2}\left(-\mathcal{D}'(w,\mu)(m)\right)\|B-A\|_p\int_0^1 sds=\frac{1}{4}\left(-\mathcal{D}'(w,\mu)(m)\right)\|B-A\|_p .\end{aligned}$$

Now, using the change of variable $t=2s$ we have

$$\frac{1}{2}\int_0^1 \mathcal{D}(w,\mu)\left((1-t)A+t\frac{A+B}{2}\right)dt=\int_0^{1/2}\mathcal{D}(w,\mu)((1-s)A+sB)\,ds$$

and by the change of variable $t=1-v$ we have

$$\begin{aligned}&\frac{1}{2}\int_0^1 \mathcal{D}(w,\mu)\left(t\frac{A+B}{2}+(1-t)A\right)dt \\ &=\frac{1}{2}\int_0^1 \mathcal{D}(w,\mu)\left((1-v)\frac{A+B}{2}+vB\right)dv.\end{aligned}$$

Moreover, if we make the change of variable $v=2s-1$ we also have

$$\frac{1}{2}\int_0^1 \mathcal{D}(w,\mu)\left((1-v)\frac{A+B}{2}+vB\right)dv=\int_{1/2}^1 \mathcal{D}(w,\mu)((1-s)A+sB)\,ds.$$

Therefore

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\mathcal{D}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)+\mathcal{D}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right]ds \\ &=\int_0^{1/2}\mathcal{D}(w,\mu)((1-s)A+sB)\,dt+\int_{1/2}^1\mathcal{D}(w,\mu)((1-s)A+sB)\,ds \\ &=\int_0^1 \mathcal{D}(w,\mu)((1-s)A+sB)\,ds\end{aligned}$$

and by (3.13) we deduce the desired result (3.10). □

Corollary 5. *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is an operator monotone function with* $f(0)=0$. *If* $A,\ B\in\mathcal{B}_p(H)$, $p\ge 1$, *with* $A,\ B\ge m>0$, *then*

$$\begin{aligned}(3.14)\quad &\left\|\frac{f(A)A^{-1}+f(B)B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}f((1-t)A+tB)\,dt\right\|_p \\ &\le \frac{f(m)-f'(m)m}{4m^2}\|B-A\|_p .\end{aligned}$$

Assume that $A,\ B\in\mathcal{B}_p(H)$, $p\ge 1$, with $A,\ B\ge m>0$, then by Corollary 5 we obtain the following power inequalities

$$(3.15)\quad \left\|\frac{A^{r-1}+B^{r-1}}{2}-\int_0^1((1-t)A+tB)^{r-1}dt\right\|_p\le\frac{1-r}{4m^{2-r}}\|B-A\|_p ,$$

where $r\in(0,1]$.

We can also state the logarithmic inequality

$$\begin{aligned}
&\left\| \frac{A^{-1}\ln(A+1)+B^{-1}\ln(B+1)}{2} \right. \\
&\left. -\int_0^1 ((1-t)A+tB)^{-1}\ln((1-t)A+tB+1)\,dt \right\|_p \\
&\leq \frac{(m+1)\ln(m+1)-m}{4m^2(m+1)}\|B-A\|_p,
\end{aligned} \tag{3.16}$$

provided that $A,\, B\in\mathcal{B}_p(H)$, $p\geq 1$, with $A,\, B\geq m>0$.

Corollary 6. *Assume that $f:[0,\infty)\to\mathbb{R}$ is an operator convex function with $f(0)=0$. If $A,\, B\in\mathcal{B}_p(H)$, $p\geq 1$, with $A,\, B\geq m>0$, then*

$$\begin{aligned}
&\left\| \frac{f(A)A^{-2}+f(B)B^{-2}}{2} - \int_0^1 ((1-t)A+tB)^{-2} f((1-t)A+tB)\,dt \right. \\
&\left. -f'_+(0)\left(\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right)\right\|_p \\
&\leq \frac{1}{2m^2}\|B-A\|_p\left(\frac{f(m)}{m}-\frac{f'(m)+f'_+(0)}{2}\right).
\end{aligned} \tag{3.17}$$

Assume that $A,\, B\in\mathcal{B}_p(H)$, $p\geq 1$, with $A,\, B\geq m>0$, then by Corollary 6 we obtain the following logarithmic inequalities

$$\begin{aligned}
&\left\| \frac{A^{-2}\ln(A+1)+B^{-2}\ln(B+1)}{2} \right. \\
&-\int_0^1 ((1-t)A+tB)^{-2}\ln((1-t)A+tB+1)\,dt \\
&\left. -\left(\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right)\right\|_p \\
&\leq \frac{1}{4m^2}\|B-A\|_p\left(\frac{m+2}{m+1}-\frac{2\ln(m+1)}{m}\right).
\end{aligned} \tag{3.18}$$

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