LIPSCHITZ TYPE p-SCHATTEN NORM INEQUALITIES FOR THE LOGARITHMIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0, \infty)$ we consider the following *logarithmic integral transform*

$$\mathcal{L}og(w, \mu)(T) = \int_0^\infty w(\lambda) \ln(\lambda + T) \, d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

Assume that A, $B \in \mathcal{B}_p(H)$, $p \geq 1$ and $A \geq m_1 > 0$, $B \geq m_2 > 0$, then

$$\|\mathcal{L}og(w, \mu)(B) - \mathcal{L}og(w, \mu)(A)\|_p \leq \|B - A\|_p \times \begin{cases} \frac{\mathcal{L}og(w,\mu)(m_2) - \mathcal{L}og(w,\mu)(m_1)}{m_2 - m_1} & \text{if } m_1 \neq m_2, \\ \\ \mathcal{D}(w, \mu)(m) & \text{if } m_1 = m_2 = m, \end{cases}$$

where

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0,$$

is the derivative of $\mathcal{L}og(w, \mu)$ as a function of t.

Applications for Hermite-Hadamard type inequalities and some particular examples of interest are also given.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [6] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \leq L \|A - B\|$$

for any A, $B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [30], [31] and Kato in [38], the following inequality holds

$$\||A| - |B|\| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.1}$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator inequalities, Lipschitz type inequalities, Hermite-Hadamard inequalities, p-Schatten norm.

RGMIA Res. Rep. Coll. 24 (2021), Art. 103, 15 pp. Received 24/08/21

for any $A,\, B\in \mathcal{B}(H)$ with $A\neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_2 := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\||A|-|B|\|_2 \leq \sqrt{2}\,\|A-B\|_2 \tag{1.2}$$

for any $A,\, B\in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$. By $A\geq B$ we understand that $A-B\geq 0$

In 1934, K. Löwner [42] had given a definitive characterization of operator monotone functions as follows, see for instance [8, p. 144-145]:

Theorem 1. *A function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty \frac{t\lambda}{t+\lambda}d\mu(\lambda) \tag{1.3}$$

where μ *is a positive measure on* $(0,\infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t^\alpha$ is an operator monotone function for any $\alpha\in[0,1]$, [37]. The function ln is also operator monotone on $(0,\infty)$. For other examples of operator monotone functions, see [33] and [36].

It has been shown in [6] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\|\leq a_1\|A-B\|+a_2\|A-B\|^2+O\left(\|A-B\|^3\right) \tag{1.4}$$

where

$$a_1=\left\|A^{-1}\right\|\|A\| \text{ and } a_2=\left\|A^{-1}\right\|+\left\|A^{-1}\right\|^3\|A\|^2.$$

In [5] the author also obtained the following *Lipschitz type inequality*

$$\|f(A)-f(B)\|\leq f'(a)\|A-B\| \tag{1.5}$$

where f is an *operator monotone function* on $(0,\infty)$ and $A,\, B\geq a>0$.

One of the problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [7], [32] and the references therein.

In order to extend the above results to *p-Schatten norm* and $\mathcal{L}og(w,\mu)$ operator transform, we need the following preparations.

2. Some Preliminary Facts

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{2.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{2.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ then $A^*\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.3}$$

(ii) If $A\in\mathcal{B}_1(H)$ and $T\in\mathcal{B}(H)$, then AT, $TA\in\mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \ \text{ and } \ |\operatorname{tr}(AT)| \le \|A\|_1\|T\|; \tag{2.4}$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\operatorname{tr}\| = 1$;
(iv) If A, $B\in\mathcal{B}_2(H)$ then AB, $BA\in\mathcal{B}_1(H)$ and $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [48, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{2.5}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{2.6}$$

For $p\ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [48, p. 60-64],

$$\|A\|_p = \|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{2.7}$$

$$\|AB\|_p \le \|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{2.8}$$

and

$$\|AB\|_p \le \|A\|_p\|B\|,\ \ \|BA\|_p \le \|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{2.9}$$

This implies that

$$\|CAB\|_p \le \|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{2.10}$$

In terms of p-*Schatten norm* we have the *Hölder inequality* for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\leq)\,\|AB\|_1\leq\|A\|_p\,\|B\|_q\,,\ A\in\mathcal{B}_p(H)\,,\ B\in\mathcal{B}_q(H)\,. \tag{2.11}$$

For the theory of trace functionals and their applications the reader is referred to [46] and [48].

For some classical trace inequalities see [9], [11], and [43], which are continuations of the work of Bellman [3]. For related works the reader can refer to [1], [4], [9], [34], [39], [40], [41], [45], [47] and the recent papers [12]-[28].

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [8, p. 145]

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [8, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{2.12}$$

Observe that for $t>0$, $t\neq 1$, we have

$$\int_0^u\frac{d\lambda}{(\lambda+t)(\lambda+1)}=\frac{\ln t}{t-1}+\frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln t}{t-1}=\int_0^\infty\frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty\frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{2.13}$$

for all $t>0$.

Motivated by these representations, we introduced in [29], for a continuous and positive function $w(\lambda)$, $\lambda>0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\mu(\lambda)\,,\ t>0, \tag{2.14}$$

where μ is a positive measure on $(0,\infty)$ and the integral (2.14) exists for all $t>0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t):=\int_0^\infty\frac{w(\lambda)}{\lambda+t}d\lambda,\ t>0. \tag{2.15}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda)=\lambda^{r-1}$, $r\in(0,1]$, then

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\mathcal{D}(w_r)(t)\,,\ t>0. \tag{2.16}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda)=(\lambda+1)^{-1}$, $t>0$, then we have the representation

$$\ln t=(t-1)\,\mathcal{D}(w_{\ln})(t)\,,\ t>0. \tag{2.17}$$

Assume that $T>0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w,\mu)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\mu(\lambda), \tag{2.18}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T):=\int_0^\infty w(\lambda)(\lambda+T)^{-1}d\lambda, \tag{2.19}$$

for $T>0$.

We define the *logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda>0$ by

$$\mathcal{L}og(w,\mu)(t):=\int_0^\infty w(\lambda)\ln(\lambda+t)\,d\mu(\lambda), \tag{2.20}$$

where μ is a positive measure on $(0,\infty)$ and the integral (2.14) exists for all $t>0$.Also, when μ is the usual Lebesgue measure, then

$$\mathcal{L}og(w)(t):=\int_0^\infty w(\lambda)\ln(\lambda+t)\,d\lambda. \tag{2.21}$$

If we consider the positive kernel $w_{\exp(-a\cdot)}(\lambda):=\exp(-a\lambda)$, $\lambda\geq 0$, then, after some calculations

$$\mathcal{L}og(\exp(-a\cdot))(t)=\int_0^\infty \exp(-a\lambda)\ln(\lambda+t)\,d\lambda=\frac{1}{a}\left[\ln t+E_1(at)\exp(at)\right],$$

for $t>0$,where the *exponential integral* is given by

$$E_1(t):=\int_t^\infty \frac{e^{-u}}{u}du.$$

For $a=1$ we have

$$\mathcal{L}og(\exp(-\cdot))(t)=\int_0^\infty \exp(-\lambda)\ln(\lambda+t)\,d\lambda=\ln t+E_1(t)\exp(t),$$

For $t=0$, we derive

$$\mathcal{L}og(\exp(-\cdot))(0)=\int_0^\infty \exp(-\lambda)\ln(\lambda)\,d\lambda=-\gamma,$$

where γ is Euler–Mascheroni constant.

For $a>0$, by changing the variable $a\lambda=\nu$, then

$$\begin{aligned}\int_0^\infty \exp(-a\lambda)\ln(\lambda)\,d\lambda&=\int_0^\infty \exp(-\nu)\ln\left(\frac{\nu}{a}\right)\frac{1}{a}d\nu\\&=\frac{1}{a}\int_0^\infty\left[\exp(-\nu)\ln\nu-\exp(-\nu)\ln a\right]d\nu\\&=\frac{1}{a}(-\gamma-\ln a)=-\frac{\ln a+\gamma}{a}\end{aligned}$$

and we have for any $a>0$ that

$$\mathcal{L}og(\exp(-a\cdot))(0)=-\frac{\ln a+\gamma}{a}.$$

If we consider the positive kernel $w_{(\cdot+a)^{-2}}(\lambda) := \frac{1}{(\lambda+a)^2}$, $\lambda \geq 0$, $a > 0$, then, after some calculations

$$\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)(t) := \int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+a)^2} d\lambda = \begin{cases} \frac{t\ln t - a\ln a}{a(t-a)}, \text{ if } t \neq a, \\ \\ \frac{\ln a+1}{a}, \text{ if } t = a \end{cases}$$

for $t > 0$.

If $a = 1$, then

$$\mathcal{L}og\left(w_{(\cdot+1)^{-2}}\right)(t) := \int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+1)^2} d\lambda = \begin{cases} \frac{t\ln t}{t-1}, \text{ if } t \neq 1, \\ \\ 1, \text{ if } t = 1 \end{cases}$$

for $t > 0$.

For $t = 0$, we derive

$$\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)(0) := \int_0^\infty \frac{\ln(\lambda)}{(\lambda+a)^2} d\lambda = \frac{\ln a}{a}$$

for $a > 0$.

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{L}og(w,\mu)(T) = \int_0^\infty w(\lambda)\ln(\lambda+T)\,d\mu(\lambda), \tag{2.22}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{L}og(w)(T) = \int_0^\infty w(\lambda)\ln(\lambda+T)\,d\lambda. \tag{2.23}$$

3. Main Results

Start to the following identity for the logarithmic function:

Lemma 1. *For all A, $B > 0$ we have the identity:*

$$\begin{aligned} &\ln B - \ln A \\ &= \int_0^\infty \left(\int_0^1 (s+(1-t)A+tB)^{-1}(B-A)(s+(1-t)A+tB)^{-1}\,dt\right) ds. \end{aligned} \tag{3.1}$$

Proof. We have from (2.17) for A, $B > 0$ that

$$\ln B - \ln A = \int_0^\infty \frac{1}{s+1}\left[(B-1)(s+B)^{-1} - (A-1)(s+A)^{-1}\right] ds. \tag{3.2}$$

Since

$$\begin{aligned} &(B-1)(s+B)^{-1} - (A-1)(s+A)^{-1} \\ &= B(s+B)^{-1} - A(s+A)^{-1} - \left((s+B)^{-1} - (s+A)^{-1}\right) \end{aligned}$$

and

$$\begin{aligned} &B(s+B)^{-1} - A(s+A)^{-1} \\ &= (B+s-s)(s+B)^{-1} - (A+s-s)(s+A)^{-1} \\ &= 1 - s(s+B)^{-1} - 1 + s(s+A)^{-1} = s(s+A)^{-1} - s(s+B)^{-1}, \end{aligned}$$

hence

$$\begin{aligned}&(B-1)(s+B)^{-1}-(A-1)(s+A)^{-1}\\&=s(s+A)^{-1}-s(s+B)^{-1}-\left((s+B)^{-1}-(s+A)^{-1}\right)\\&=(s+1)\left[(s+A)^{-1}-(s+B)^{-1}\right]\end{aligned}$$

and by (3.2) we get

$$\ln B-\ln A=\int_0^\infty\left[(s+A)^{-1}-(s+B)^{-1}\right]ds. \tag{3.3}$$

Let $T,\ S>0$. The function $f(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S):=\lim_{t\to 0}\left[\frac{f(T+tS)-f(T)}{t}\right]=T^{-1}ST^{-1} \tag{3.4}$$

for $T,\ S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{3.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{3.6}$$

Since, by (3.6) we have

$$\begin{aligned}&(s+A)^{-1}-(s+B)^{-1}\\&=\int_0^1(s+(1-t)A+tB)^{-1}(B-A)(s+(1-t)A+tB)^{-1}\,dt,\end{aligned} \tag{3.7}$$

for all $s\geq 0$, hence by (3.3) and (3.7) we get (3.1). □

Our first main result is as follows:

Theorem 3. *Assume that $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$ and $A\geq m_1>0$, $B\geq m_2>0$, then*

$$\begin{aligned}&\|\mathcal{L}og(w,\mu)(B)-\mathcal{L}og(w,\mu)(A)\|_p\\&\leq\|B-A\|_p\times\begin{cases}\frac{\mathcal{L}og(w,\mu)(m_2)-\mathcal{L}og(w,\mu)(m_1)}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ \mathcal{D}(w,\mu)(m) & \text{if } m_1=m_2=m.\end{cases}\end{aligned} \tag{3.8}$$

Proof. For all A, $B>0$ we have

$$\mathcal{L}og\left(w,\mu \right) \left(B\right) -\mathcal{L}og\left(w,\mu \right) \left(A\right) =\int_{0}^{\infty }w\left(\lambda \right) \ln \left(\lambda +B\right) d\mu \left(\lambda \right) -\int_{0}^{\infty }w\left(\lambda \right) \ln \left(\lambda +A\right) d\mu \left(\lambda \right) =\int_{0}^{\infty }w\left(\lambda \right) \left[\ln \left(\lambda +B\right) -\ln \left(\lambda +A\right) \right] d\mu \left(\lambda \right) . \tag{3.9}$$

Since, by (3.1) we get

$$\ln \left(\lambda +B\right) -\ln \left(\lambda +A\right) =\int_{0}^{\infty }\left(\int_{0}^{1}\left(s+\left(1-t\right) \left(\left(\lambda +A\right) \right) +t\left(\lambda +B\right) \right) ^{-1} \times \left(\lambda +B-\left(\lambda +A\right) \right) \left(s+\left(1-t\right) \left(\left(\lambda +A\right) \right) +t\left(\lambda +B\right) \right) ^{-1}dt\right) ds$$

for all $\lambda \geq 0$, then by multiplying with $w\left(\lambda \right)$ and integrating over $\mu \left(\lambda \right)$ we obtain

$$\int_{0}^{\infty }w\left(\lambda \right) \left[\ln \left(\lambda +B\right) -\ln \left(\lambda +A\right) \right] d\mu \left(\lambda \right) =\left(\int_{0}^{\infty }\left(\int_{0}^{1}\left(s+\lambda +\left(1-t\right) A+tB\right) ^{-1}\left(B-A\right) \times \left(s+\lambda +\left(1-t\right) A+tB\right) ^{-1}dt\right) ds\right) d\mu \left(\lambda \right) . \tag{3.10}$$

Finally, by (3.9) and (3.10) we get

$$\mathcal{L}og\left(w,\mu \right) \left(B\right) -\mathcal{L}og\left(w,\mu \right) \left(A\right) =\int_{0}^{\infty }w\left(\lambda \right) \left(\int_{0}^{\infty }\left(\int_{0}^{1}\left(s+\lambda +\left(1-t\right) A+tB\right) ^{-1}\left(B-A\right) \times \left(s+\lambda +\left(1-t\right) A+tB\right) ^{-1}dt\right) ds\right) d\mu \left(\lambda \right) . \tag{3.11}$$

From the identity (3.11) we get by taking the p-*Schatten norm* that

$$\begin{aligned} &\left\Vert \mathcal{L}og\left(w,\mu \right) \left(B\right) -\mathcal{L}og\left(w,\mu \right) \left(A\right) \right\Vert _{p} \\ &\leq \int_{0}^{\infty }w\left(\lambda \right) \left(\int_{0}^{\infty }\left(\int_{0}^{1}\left\Vert \left(s+\lambda +\left(1-t\right) A+tB\right) ^{-1}\left(B-A\right) \right.\right.\right. \\ &\left.\left.\left. \times \left(s+\lambda +\left(1-t\right) A+tB\right) ^{-1}\right\Vert _{p}dt\right) ds\right) d\mu \left(\lambda \right) \\ &\leq \left\Vert B-A\right\Vert _{p} \\ &\times \int_{0}^{\infty }w\left(\lambda \right) \left(\int_{0}^{\infty }\left(\int_{0}^{1}\left\Vert \left(s+\lambda +\left(1-t\right) A+tB\right) ^{-1}\right\Vert ^{2}dt\right) ds\right) d\mu \left(\lambda \right) \end{aligned} \tag{3.12}$$

for all A, $B>0$, A, $B\in \mathcal{B}_{p}\left(H\right)$, where for the last inequality we used the property (2.10).

Assume that $m_{2}>m_{1}$. Then

$$\left(1-t\right) A+tB+\lambda +s\geq \left(1-t\right) m_{1}+tm_{2}+\lambda +s,$$

which implies that

$$\left(\left(1-t\right) A+tB+\lambda +s\right) ^{-1}\leq \left(\left(1-t\right) m_{1}+tm_{2}+\lambda +s\right) ^{-1},$$

and

$$\left\|\left((1-t)A+tB+\lambda+s\right)^{-1}\right\|^{2}\leq\left((1-t)m_{1}+tm_{2}+\lambda+s\right)^{-2} \tag{3.13}$$

for all $t\in[0,1]$ and $\lambda\geq 0$.

Therefore, by integrating (3.13) we derive

$$\begin{aligned}
&\int_{0}^{\infty}w(\lambda)\left(\int_{0}^{\infty}\left(\int_{0}^{1}\left\|\left(s+\lambda+(1-t)A+tB\right)^{-1}\right\|^{2}dt\right)ds\right)d\mu(\lambda)\\
&\leq\int_{0}^{\infty}w(\lambda)\left(\int_{0}^{\infty}\left(\int_{0}^{1}\left((1-t)m_{1}+tm_{2}+\lambda+s\right)^{-2}dt\right)ds\right)d\mu(\lambda)\\
&=\frac{1}{m_{2}-m_{1}}\int_{0}^{\infty}w(\lambda)\left(\int_{0}^{\infty}\left(\int_{0}^{1}\left((1-t)m_{1}+tm_{2}+\lambda+s\right)^{-1}\right.\right.\\
&\quad\left.\left.\times(m_{2}-m_{1})\left((1-t)m_{1}+tm_{2}+\lambda+s\right)^{-1}dt\right)ds\right)d\mu(\lambda)\\
&=\frac{1}{m_{2}-m_{1}}\left[\mathcal{L}og(w,\mu)(m_{2})-\mathcal{L}og(w,\mu)(m_{1})\right]\text{ (by (3.11))}
\end{aligned}$$

and by (3.11) we deduce

$$\begin{aligned}
&\left\|\mathcal{L}og(w,\mu)(B)-\mathcal{L}og(w,\mu)(A)\right\|_{p}\\
&\leq\frac{\|B-A\|_{p}}{m_{2}-m_{1}}\left[\mathcal{L}og(w,\mu)(m_{2})-\mathcal{L}og(w,\mu)(m_{1})\right].
\end{aligned} \tag{3.14}$$

The case $m_2<m_1$ goes in a similar way and we also obtain (3.14).

Let $\epsilon>0$. Then $B+\epsilon\geq m+\epsilon>m$. From (3.14) we get

$$\begin{aligned}
&\left\|\mathcal{L}og(w,\mu)(B+\epsilon)-\mathcal{L}og(w,\mu)(A)\right\|_{p}\\
&\leq\frac{\|B+\epsilon-A\|_{p}}{m+\epsilon-m}\left[\mathcal{L}og(w,\mu)(m+\epsilon)-\mathcal{L}og(w,\mu)(m)\right]
\end{aligned}$$

and by taking the limit over $\epsilon\to 0+$, using the continuity and differentiability of $\mathcal{L}og(w,\mu)$ we deduce

$$\left\|\mathcal{L}og(w,\mu)(B+\epsilon)-\mathcal{L}og(w,\mu)(A)\right\|_{p}\leq\|B-A\|_{p}\left(\mathcal{L}og(w,\mu)\right)'(m).$$

The fact that the derivative of $\mathcal{L}og(w,\mu)(t)$ is $\mathcal{D}(w,\mu)(t)$ follows by the properties of the derivative of a parameter integral, i.e.

$$\frac{d\mathcal{L}og(w,\mu)(t)}{dt}:=\int_{0}^{\infty}w(\lambda)\frac{d\ln(\lambda+t)}{dt}d\mu(\lambda)=\int_{0}^{\infty}w(\lambda)(\lambda+t)^{-1}d\mu(\lambda)$$

for $t>0$, which proves the second part of (3.8). □

Corollary 1. *Assume that A, $B\in\mathcal{B}_1(H)$ and $A\geq m_1>0$, $B\geq m_2>0$, then we have the trace inequality*

$$\begin{aligned}
&\left|\operatorname{tr}\left[\mathcal{L}og(w,\mu)(B)\right]-\operatorname{tr}\left[\mathcal{L}og(w,\mu)(A)\right]\right|\\
&\leq\|B-A\|_{1}\times\begin{cases}\frac{\mathcal{L}og(w,\mu)(m_{2})-\mathcal{L}og(w,\mu)(m_{1})}{m_{2}-m_{1}} & \text{if } m_{1}\neq m_{2},\\[2ex] \mathcal{D}(w,\mu)(m) & \text{if } m_{1}=m_{2}=m.\end{cases}
\end{aligned} \tag{3.15}$$

Proof. From (3.8) we have for $p=1$ that

$$\begin{aligned} & \operatorname{tr}\left|\mathcal{L}og\left(w,\mu\right)\left(B\right)-\mathcal{L}og\left(w,\mu\right)\left(A\right)\right| \\ & \leq \left\|B-A\right\|_{p}\times\begin{cases}\frac{\mathcal{L}og(w,\mu)(m_2)-\mathcal{L}og(w,\mu)(m_1)}{m_2-m_1} \text{ if } m_1\neq m_2,\\ \\ \mathcal{D}\left(w,\mu\right)\left(m\right) \text{ if } m_1=m_2=m.\end{cases}\end{aligned} \tag{3.16}$$

Since

$$\left|\operatorname{tr}\left[\mathcal{L}og\left(w,\mu\right)\left(B\right)\right]-\operatorname{tr}\left[\mathcal{L}og\left(w,\mu\right)\left(A\right)\right]\right|\leq\operatorname{tr}\left|\mathcal{L}og\left(w,\mu\right)\left(B\right)-\mathcal{L}og\left(w,\mu\right)\left(A\right)\right|$$

for $A,\ B\in\mathcal{B}_1\left(H\right)$, hence by (3.16) we derive (3.15). □

4. Inequalities of Hermite-Hadamard Type

Let $\left(X;\left\|\cdot\right\|_X\right)$ and $\left(Y;\left\|\cdot\right\|_Y\right)$ be two Banach spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. For any mapping $F:C\subset X\to Y$ we can consider the associated function $\Phi_{F,x,y,\lambda}:[0,1]\to Y$, where $x,\ y\in C,\ \lambda\in[0,1]$, defined by [19]

$$\begin{aligned}\Phi_{F,x,y,\lambda}\left(t\right) &:= \left(1-\lambda\right)F\left[\left(1-t\right)\left(\left(1-\lambda\right)x+\lambda y\right)+ty\right] \\ &\quad +\lambda F\left[\left(1-t\right)x+t\left(\left(1-\lambda\right)x+\lambda y\right)\right].\end{aligned} \tag{4.1}$$

We observe that for $\lambda=0$ and $\lambda=1$ we have

$$\Phi_{F,x,y,0}\left(t\right)=\Phi_{F,x,y,1}\left(t\right)=F\left[\left(1-t\right)x+ty\right]$$

and

$$\Phi_{F,x,y,\frac{1}{2}}\left(t\right)=\frac{1}{2}\left(F\left[\left(1-t\right)\frac{x+y}{2}+ty\right]+F\left[\left(1-t\right)x+t\frac{x+y}{2}\right]\right)$$

where $x,\ y\in B$.

The following result holds [19]:

Lemma 2. *Let $F:C\subset X\to Y$ be a Lipschitzian mapping with the constant $L>0$ on the convex subset C of X. If $x,\ y\in C$, then we have*

$$\begin{aligned} & \left\|\Phi_{F,x,y,\lambda}\left(t\right)-\int_0^1 F\left[sy+\left(1-s\right)x\right]ds\right\|_Y \\ & \leq 2L\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\left\|x-y\right\|_X\end{aligned} \tag{4.2}$$

for any $t\in[0,1]$ and $\lambda\in[0,1]$.

We define, for the positive operators $A,\ B>0$ and the transform $\mathcal{L}og\left(w,\mu\right)$,

$$\begin{aligned} & \Phi_{\mathcal{L}og(w,\mu),A,B,\lambda}\left(t\right) \\ & :=\left(1-\lambda\right)\mathcal{L}og\left(w,\mu\right)\left[\left(1-t\right)\left(\left(1-\lambda\right)A+\lambda B\right)+tB\right] \\ & +\lambda\mathcal{L}og\left(w,\mu\right)\left[\left(1-t\right)A+t\left(\left(1-\lambda\right)A+\lambda B\right)\right],\end{aligned} \tag{4.3}$$

where $t\in[0,1]$ and $\lambda\in[0,1]$.

By utilising Lemma 2 and Theorem 3 we can state:

Proposition 1. *Assume that* $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$ *and* $A,\ B\geq m>0$, *then*

$$\left\|\Phi_{\mathcal{L}og(w,\mu),A,B,\lambda}(t)-\int_0^1\mathcal{L}og(w,\mu)[sB+(1-s)A]\,ds\right\|_p \tag{4.4}$$
$$\leq 2\mathcal{D}(w,\mu)(m)\|A-B\|_p$$
$$\times\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right],$$

for any $t\in[0,1]$ *and* $\lambda\in[0,1]$.

In particular, we have

$$\left\|\frac{1}{2}\left[\mathcal{L}og(w,\mu)\left(\frac{3A+B}{4}\right)+\mathcal{L}og(w,\mu)\left(\frac{A+3B}{4}\right)\right]\right. \tag{4.5}$$
$$\left.-\int_0^1\mathcal{L}og(w,\mu)[sB+(1-s)A]\,ds\right\|_p$$
$$\leq\frac{1}{8}\mathcal{D}(w,\mu)(m)\|A-B\|_p,$$

$$\left\|\mathcal{L}og(w,\mu)[(1-t)A+tB]-\int_0^1\mathcal{L}og(w,\mu)[sB+(1-s)A]\,ds\right\|_p \tag{4.6}$$
$$\leq\mathcal{D}(w,\mu)(m)\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|A-B\|_p,$$

$$\left\|\mathcal{L}og(w,\mu)\left(\frac{A+B}{2}\right)-\int_0^1\mathcal{L}og(w,\mu)[sB+(1-s)A]\,ds\right\|_p \tag{4.7}$$
$$\leq\frac{1}{4}\mathcal{D}(w,\mu)(m)\|A-B\|_p$$

and

$$\left\|\frac{1}{2}\left[\mathcal{L}og(w,\mu)\left[(1-t)\frac{A+B}{2}+tB\right]\right.\right. \tag{4.8}$$
$$\left.+\mathcal{L}og(w,\mu)\left[(1-t)A+t\frac{A+B}{2}\right]\right]$$
$$\left.-\int_0^1\mathcal{L}og(w,\mu)[sB+(1-s)A]\,ds\right\|_p$$
$$\leq\frac{1}{2}\mathcal{D}(w,\mu)(m)\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|A-B\|_p$$

for any $t\in[0,1]$.

We can also consider another associated function $\Psi_{F,x,y,\lambda}:[0,1]\to Y$, where x, $y\in C$, $\lambda\in[0,1]$, defined by [19]

$$\Psi_{F,x,y,\lambda}(t):=(1-\lambda)F[(1-t)((1-\lambda)x+\lambda y)+ty] \tag{4.9}$$
$$+\lambda F[tx+(1-t)((1-\lambda)x+\lambda y)].$$

We observe that for $\lambda=0$ and $\lambda=1$ we have

$$\Psi_{F,x,y,0}(t)=F[(1-t)x+ty],\ \Psi_{F,x,y,1}(t)=F[tx+(1-t)y]$$

and

$$\Psi_{F,x,y,\frac{1}{2}}(t)=\frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2}+ty\right]+F\left[tx+(1-t)\frac{x+y}{2}\right]\right),$$

where $x,\ y\in B$.

In [19] we also obtained the following result:

Lemma 3. *Let $F:C\subset X\to Y$ be a Lipschitzian mapping with the constant $L>0$ on the convex subset C of X. If $x,\ y\in C$, then we have*

$$\left\|\Psi_{F,x,y,\lambda}(t)-\int_0^1 F[sy+(1-s)x]\,ds\right\|_Y \le 2L\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|x-y\|_X \tag{4.10}$$

for any $t\in[0,1]$ and $\lambda\in[0,1]$.

We define, for the positive operators $A,\ B>0$ and the transform $\mathcal{L}og(w,\mu)$,

$$\begin{aligned}\Psi_{\mathcal{L}og(w,\mu),A,B,\lambda}(t) &:= (1-\lambda)\,\mathcal{L}og(w,\mu)[(1-t)((1-\lambda)A+\lambda B)+tB]\\ &+\lambda\mathcal{L}og(w,\mu)[tA+(1-t)((1-\lambda)A+\lambda B)].\end{aligned} \tag{4.11}$$

By utilising Lemma 3 and Theorem 3 we can also state:

Proposition 2. *Assume that $A,\ B\in\mathcal{B}_p(H)$, $p\ge 1$ and $A,\ B\ge m>0$, then*

$$\begin{aligned}&\left\|\Psi_{\mathcal{L}og(w,\mu),A,B,\lambda}(t)-\int_0^1\mathcal{L}og(w,\mu)[sB+(1-s)A]\,ds\right\|_p\\ &\le 2\mathcal{D}(w,\mu)(m)\|A-B\|_p\\ &\times\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\end{aligned} \tag{4.12}$$

for any $t\in[0,1]$ and $\lambda\in[0,1]$.

In particular,

$$\begin{aligned}&\left\|\frac{1}{2}[\mathcal{L}og(w,\mu)(A)+\mathcal{L}og(w,\mu)(B)]\right.\\ &\left.-\int_0^1\mathcal{L}og(w,\mu)[sB+(1-s)A]\,ds\right\|_p\\ &\le\frac{1}{4}\mathcal{D}(w,\mu)(m)\|A-B\|_p.\end{aligned} \tag{4.13}$$

5. Some Examples

From the introduction we have the following two classes of continuous functions defined on $[0,\infty)$,

$$\begin{aligned}\mathcal{L}og(\exp(-a\cdot))(t) &:= \int_0^\infty \exp(-a\lambda)\ln(\lambda+t)\,d\lambda\\ &=\begin{cases}\frac{1}{a}[\ln t+E_1(at)\exp(at)], & \text{if } t>0,\\ -\frac{\ln a+\gamma}{a}, & \text{if } t=0\end{cases}\end{aligned}$$

and

$$\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)(t) := \int_0^\infty \frac{\ln(\lambda+t)}{(\lambda+a)^2}d\lambda = \begin{cases} \frac{t\ln t - a\ln a}{a(t-a)}, \text{ if } t\neq a, \\ \\ \frac{\ln a+1}{a}, \text{ if } t=a, \\ \\ \frac{\ln a}{a}, \text{ if } t=0 \end{cases}$$

for all $a>0$.

For the kernel $w_{\exp(-a\cdot)}(\lambda) := \exp(-a\lambda)$, $\lambda\geq 0$, we obtain

$$\mathcal{D}\left(w_{\exp(-a\cdot)}\right)(t) := \int_0^\infty \frac{\exp(-a\lambda)}{\lambda+t}d\lambda = E_1(at)\exp(at),$$

for $a,\ t>0$.

For the kernel $w_{(\cdot+a)^{-2}}(\lambda) := \frac{1}{(\lambda+a)^2}$, $\lambda\geq 0$, $a>0$, we get

$$\mathcal{D}\left(w_{(\cdot+a)^{-2}}\right)(t) := \int_0^\infty \frac{1}{(\lambda+t)(\lambda+a)^2}d\lambda = \begin{cases} \frac{t-a-a(\ln t-\ln a)}{a(t-a)^2}, \text{ if } t\neq a \\ \\ \frac{1}{2a^2}, \ t=a \end{cases}$$

for $a,\ t>0$.

Assume that $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$ and $A\geq m_1>0$, $B\geq m_2>0$, then by (3.8) for $\mathcal{L}og(\exp(-a\cdot))$, $a>0$, we get

$$\begin{aligned} &\left\|\ln B + E_1(aB)\exp(aB) - \left[\ln A + E_1(aA)\exp(aA)\right]\right\|_p \\ &\leq \|B-A\|_p \\ &\times \begin{cases} \frac{\ln m_2 + E_1(am_2)\exp(am_2) - \ln m_1 - E_1(am_1)\exp(am_1)}{m_2-m_1} \text{ if } m_1\neq m_2, \\ \\ aE_1(am)\exp(am) \text{ if } m_1=m_2=m. \end{cases} \end{aligned} \tag{5.1}$$

For $a=1$ we derive the simpler inequality

$$\begin{aligned} &\left\|\left[\ln B + E_1(B)\exp(B)\right] - \left[\ln A + E_1(A)\exp(A)\right]\right\|_p \\ &\leq \|B-A\|_p \times \begin{cases} \frac{\ln m_2 + E_1(m_2)\exp(m_2) - \ln m_1 - E_1(m_1)\exp(m_1)}{m_2-m_1} \\ \text{ if } m_1\neq m_2, \\ \\ E_1(m)\exp(m) \text{ if } m_1=m_2=m. \end{cases} \end{aligned} \tag{5.2}$$

Assume that $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$ and $A\geq m_1>a>0$, $B\geq m_2>a>0$, then by (3.8) for $\mathcal{L}og\left(w_{(\cdot+a)^{-2}}\right)$ we get

$$\begin{aligned} &\left\|(B-a)^{-1}(B\ln B - a\ln a) - (A-a)^{-1}(A\ln A - a\ln a)\right\|_p \\ &\leq \|B-A\|_p \\ &\times \begin{cases} (m_2-a)^{-1}(m_2\ln m_2 - a\ln a) - (m_1-a)^{-1}(m_1\ln m_1 - a\ln a) \\ \text{if } m_1\neq m_2 \\ \\ \frac{m-a-a(\ln m-\ln a)}{(m-a)^2} \text{ if } m_1=m_2=m. \end{cases} \end{aligned} \tag{5.3}$$

In particular, if $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$ and $A\geq m_1>1$, $B\geq m_2>1$, then

$$\left\|(B-1)^{-1}B\ln B - (A-1)^{-1}A\ln A\right\|_p \tag{5.4}$$

$$\leq \|B-A\|_p \times \begin{cases} (m_2-1)^{-1} m_2 \ln m_2 - (m_1-1)^{-1} m_1 \ln m_1 \\ \text{if } m_1 \neq m_2 \\ \\ \frac{m-1-\ln m}{(m-1)^2} \text{ if } m_1 = m_2 = m. \end{cases}$$

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