LIPSCHITZ TYPE p-SCHATTEN NORM INEQUALITIES FOR THE MONOTONIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS

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Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following *monotonic integral transform*

$$\mathcal{M}(w,\mu)(T) := \int_0^\infty w(\lambda) T(\lambda+T)^{-1} d\mu(\lambda),$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

Assume that $A, B \in \mathcal{B}_p(H)$, $p \ge 1$ and $A \ge m_1 > 0$, $B \ge m_2 > 0$, then

$$\|\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)\|_p \le \|B-A\|_p \times \begin{cases} \frac{\mathcal{M}(w,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)}{m_2-m_1} & \text{if } m_1 \ne m_2, \\ \\ \mathcal{M}'(w,\mu)(m) & \text{if } m_1 = m_2 = m, \end{cases}$$

where $\mathcal{M}'(w,\mu)(t)$ is the derivative of $\mathcal{M}(w,\mu)$ as a function of t.

Similar inequalities for operator monotonic and operator convex functions with some particular examples of interest are also given.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [6] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \le L \|A - B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [28], [29] and Kato in [36], the following inequality holds

$$\||A| - |B|\| \le \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.1}$$

for any $A, B \in \mathcal{B}(H)$ with $A \ne B$.

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If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_2 := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\||A| - |B|\|_2 \le \sqrt{2}\, \|A - B\|_2 \tag{1.2}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x \rangle \ge 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible. A real valued continuous function f on $(0, \infty)$ is said to be operator monotone if $f(A) \ge f(B)$ holds for any $A \ge B > 0$. By $A \ge B$ we understand that $A - B \ge 0$

In 1934, K. Löwner [40] had given a definitive characterization of operator monotone functions as follows, see for instance [8, p. 144-145]:

Theorem 1. *A function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} d\mu(\lambda) \tag{1.3}$$

where μ is a positive measure on $(0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1 + \lambda} d\mu(\lambda) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, [35]. The function ln is also operator monotone on $(0, \infty)$. For other examples of operator monotone functions, see [31] and [34].

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* on I if

$$f((1 - \lambda) A + \lambda B) \le (\ge) (1 - \lambda) f(A) + \lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda \in [0, 1]$ and for every selfadjoint operator A and B on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

We have the following representation of operator convex functions [8, p. 147]:

Theorem 2. *A function $f : [0, \infty) \to \mathbb{R}$ is operator convex in $[0, \infty)$ with $f'_+(0) \in \mathbb{R}$ if and only if it has the representation*

$$f(t) = f(0) + f'_+(0)\, t + ct^2 + \int_0^\infty \frac{t^2\lambda}{t + \lambda} d\mu(\lambda), \tag{1.4}$$

where $c \ge 0$ and a positive measure μ on $(0, \infty)$ such that (1.4) holds.

It has been shown in [6] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \le a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right) \tag{1.5}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [5] the author also obtained the following *Lipschitz type inequality*

$$\|f(A)-f(B)\| \leq f'(a)\|A-B\| \tag{1.6}$$

where f is an *operator monotone function* on $(0,\infty)$ and $A,\ B \geq a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [7], [30] and the references therein.

In order to extend the above results to *p-Schatten norm* and $\mathcal{D}(w,\mu)$ operator transform, we need the following preparations.

2. Some Preliminary Facts

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{2.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{2.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.3}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1\|T\|; \tag{2.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\leq p<\infty$ if the *p-Schatten norm* is finite [46, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{2.5}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{2.6}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [46, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{2.7}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{2.8}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \ \|BA\|_p \leq \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{2.9}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \ A \in \mathcal{B}_p(H), \ B, \ C \in \mathcal{B}(H). \tag{2.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}_q(H). \tag{2.11}$$

For the theory of trace functionals and their applications the reader is referred to [44] and [46].

For some classical trace inequalities see [9], [11], and [41], which are continuations of the work of Bellman [3]. For related works the reader can refer to [1], [4], [9], [32], [37], [38], [39], [43], [45] and the recent papers [12]-[27].

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [8, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda.$$

Motivated by these representations, we introduce, for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{2.12}$$

where μ is a positive measure on $(0, \infty)$ and the integral (2.12) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda, \ t > 0. \tag{2.13}$$

Now, assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator

$$\mathcal{D}(w, \mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{2.14}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{2.15}$$

for $T > 0$.

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t), \ t > 0. \tag{2.16}$$

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t), \ t > 0. \tag{2.17}$$

For $t > 0$ we have

$$\begin{aligned} \mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t) &= \int_0^\infty w(\lambda)\, t\, (t+\lambda)^{-1}\, d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\,(t+\lambda-\lambda)\,(t+\lambda)^{-1}\, d\mu(\lambda) \\ &= \int_0^\infty w(\lambda)\left[1-\lambda\,(t+\lambda)^{-1}\right] d\mu(\lambda). \end{aligned} \tag{2.18}$$

If $\int_0^\infty w(\lambda)\, d\mu(\lambda) < \infty$, then

$$\mathcal{M}(w, \mu)(t) = \int_0^\infty w(\lambda)\, d\mu(\lambda) - \mathcal{D}(\ell w, \mu)(t), \tag{2.19}$$

where $\ell(t) = t$, $t > 0$.

Consider the kernel $e_{-a}(\lambda) := \exp(-a\lambda)$, $\lambda \geq 0$ and $a > 0$. Then, after some calculations, we get

$$\mathcal{D}(e_{-a})(t) = \int_0^\infty \frac{\exp(-a\lambda)}{t+\lambda} d\lambda = E_1(at)\exp(at), \ t \geq 0$$

and

$$\int_0^\infty w(\lambda)\, d\lambda = \int_0^\infty \exp(-a\lambda)\, d\lambda = \frac{1}{a},$$

where the *exponential integral* is defined by

$$E_1(t) := \int_t^\infty \frac{e^{-u}}{u} du.$$

This gives that

$$\mathcal{M}(e_{-a})(t) = t\mathcal{D}(w, \mu)(t) = tE_1(at)\exp(at), \ t \geq 0.$$

By integration we also have

$$\mathcal{D}(\ell e_{-a}, \mu)(t) = \int_0^\infty \frac{\lambda \exp(-a\lambda)}{t+\lambda} d\lambda = \frac{1}{a} - tE_1(at)\exp(at)$$

for $t > 0$.

One observes that

$$\mathcal{M}(e_{-a})(t) = \int_0^\infty w(\lambda)\, d\lambda - \mathcal{D}(\ell e_{-a}, \mu)(t), \ t > 0$$

and the equality (2.19) is verified in this case.

If we take $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then $\int_0^\infty w_r(\lambda)\, d\lambda = \infty$ and the equality (2.19) does not hold in this case.

For all $T > 0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w, \mu)(T) = T\mathcal{D}(w, \mu)(T) = \int_0^\infty w(\lambda)\left[1-\lambda\,(T+\lambda)^{-1}\right] d\mu(\lambda). \tag{2.20}$$

This gives the representation

$$T^{r}=\frac{\sin\left(r\pi\right)}{\pi}\mathcal{M}(w_{r},\mu)\left(T\right),$$

where $w_{r}\left(\lambda\right)=\lambda^{r-1}$, $r\in\left(0,1\right]$ and μ is the usual Lebesgue norm.

3. Main Results

Our first main result is as follows:

Theorem 4. *Assume that* $A,\ B\in\mathcal{B}_{p}\left(H\right),\ p\geq1$ *and* $A\geq m_{1}>0,\ B\geq m_{2}>0,$ *then*

$$\left\Vert \mathcal{M}\left(w,\mu\right)\left(B\right)-\mathcal{M}\left(w,\mu\right)\left(A\right)\right\Vert _{p} \leq\left\Vert B-A\right\Vert _{p}\times\begin{cases}\frac{\mathcal{M}(w,\mu)(m_{2})-\mathcal{M}(w,\mu)(m_{1})}{m_{2}-m_{1}} & \text{if } m_{1}\neq m_{2},\\ \\ \mathcal{M}'\left(w,\mu\right)\left(m\right) & \text{if } m_{1}=m_{2}=m,\end{cases} \tag{3.1}$$

where $\mathcal{M}'\left(w,\mu\right)\left(t\right)$ *is the derivative of* $\mathcal{M}\left(w,\mu\right)$ *as a function of* t*.*

Proof. From (2.20) we have for all $A,\ B\geq0$ that

$$\begin{aligned}&\mathcal{M}(w,\mu)\left(B\right)-\mathcal{M}(w,\mu)\left(A\right)\\ &=\int_{0}^{\infty}w\left(\lambda\right)\left[1-\lambda\left(B+\lambda\right)^{-1}\right]d\mu\left(\lambda\right)-\int_{0}^{\infty}w\left(\lambda\right)\left[1-\lambda\left(A+\lambda\right)^{-1}\right]d\mu\left(\lambda\right)\\ &=\int_{0}^{\infty}\lambda w\left(\lambda\right)\left[\left(A+\lambda\right)^{-1}-\left(B+\lambda\right)^{-1}\right]d\mu\left(\lambda\right).\end{aligned} \tag{3.2}$$

Let $T,\ S>0$. The function $f\left(t\right)=-t^{-1}$ is operator monotone on $\left(0,\infty\right)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_{T}\left(S\right):=\lim_{t\rightarrow0}\left[\frac{f\left(T+tS\right)-f\left(T\right)}{t}\right]=T^{-1}ST^{-1} \tag{3.3}$$

for $T,\ S>0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $\left[C,D\right]$: $\left\{\left(1-t\right)C+tD,\ t\in\left[0,1\right]\right\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $\left[0,1\right]$ by

$$f_{C,D}\left(t\right):=f\left(\left(1-t\right)C+tD\right),\ t\in\left[0,1\right].$$

Then we have, by the properties of the Bochner integral, that

$$f\left(D\right)-f\left(C\right)=\int_{0}^{1}\frac{d}{dt}\left(f_{C,D}\left(t\right)\right)dt=\int_{0}^{1}\nabla f_{\left(1-t\right)C+tD}\left(D-C\right)dt. \tag{3.4}$$

If we write this equality for the function $f\left(t\right)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_{0}^{1}\left(\left(1-t\right)C+tD\right)^{-1}\left(D-C\right)\left(\left(1-t\right)C+tD\right)^{-1}dt. \tag{3.5}$$

Now, if we take in (3.5) $C = \lambda + A$, $D = \lambda + B$, then

$$(\lambda + A)^{-1} - (\lambda + B)^{-1} = \int_0^1 ((1-t)(\lambda + A) + t(\lambda + B))^{-1} (B - A) \times ((1-t)(\lambda + A) + t(\lambda + B))^{-1} dt = \int_0^1 (\lambda + (1-t)A + tB)^{-1} (B - A)(\lambda + (1-t)A + tB)^{-1} dt. \tag{3.6}$$

By employing (3.2) and (3.6), we derive

$$\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A) = \int_0^\infty \left(\int_0^1 (\lambda + (1-t)A + tB)^{-1} (B - A)(\lambda + (1-t)A + tB)^{-1} dt \right) \times \lambda w(\lambda)\, d\mu(\lambda), \tag{3.7}$$

for all A, $B \geq 0$.

From the identity (3.7) we get by taking the *p-Schatten norm* that

$$\begin{aligned} \|\mathcal{M}(w,\mu)(B) - \mathcal{M}(w,\mu)(A)\|_p &\leq \int_0^\infty \left\| \int_0^1 (\lambda + (1-t)A + tB)^{-1} (B - A)(\lambda + (1-t)A + tB)^{-1} dt \right\|_p \times \lambda w(\lambda)\, d\mu(\lambda) \\ &\leq \int_0^\infty \left(\int_0^1 \left\| (\lambda + (1-t)A + tB)^{-1} (B - A)(\lambda + (1-t)A + tB)^{-1} \right\|_p dt \right) \times \lambda w(\lambda)\, d\mu(\lambda) \\ &\leq \|B - A\|_p \int_0^\infty \lambda w(\lambda) \left(\int_0^1 \left\| (\lambda + (1-t)A + tB)^{-1} \right\|^2 dt \right) d\mu(\lambda) \end{aligned} \tag{3.8}$$

for all A, $B > 0$, A, $B \in \mathcal{B}_p(H)$, where for the last inequality we used the property (2.10).

Assume that $m_2 > m_1$. Then

$$(1-t)A + tB + \lambda \geq (1-t)m_1 + tm_2 + \lambda,$$

which implies that

$$((1-t)A + tB + \lambda)^{-1} \leq ((1-t)m_1 + tm_2 + \lambda)^{-1},$$

and

$$\left\| ((1-t)A + tB + \lambda)^{-1} \right\|^2 \leq ((1-t)m_1 + tm_2 + \lambda)^{-2} \tag{3.9}$$

for all $t \in [0,1]$ and $\lambda \geq 0$.

Therefore, by integrating (3.9) we derive

$$\int_0^\infty \lambda w(\lambda)\left(\int_0^1 \left\|((1-t)A+tB+\lambda)^{-1}\right\|^2 dt\right) dw(\lambda)$$
$$\le \int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)m_1+tm_2+\lambda)^{-2} dt\right) dw(\lambda)$$
$$= \frac{1}{m_2-m_1}\int_0^\infty \lambda w(\lambda)\left(\int_0^1 ((1-t)m_1+tm_2+\lambda)^{-1}\right.$$
$$\left.\times (m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1} dt\right) dw(\lambda)$$
$$= \frac{1}{m_2-m_1}\left[\mathcal{M}(w,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)\right] \text{ (by (3.7))}$$

and by (3.8) we deduce

$$\|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)(A)\|_p \le \frac{\|B-A\|_p}{m_2-m_1}\left[\mathcal{M}(w,\mu)(m_2)-\mathcal{M}(w,\mu)(m_2)\right]. \tag{3.10}$$

The case $m_2 < m_1$ goes in a similar way and we also obtain (3.10).

Let $\epsilon > 0$. Then $B+\epsilon \ge m+\epsilon > m$. From (3.10) we get

$$\|\mathcal{M}(w,\mu)(B+\epsilon)-\mathcal{M}(w,\mu)(A)\|_p \le \frac{\|B+\epsilon-A\|_p}{m+\epsilon-m}\left[\mathcal{M}(w,\mu)(m+\epsilon)-\mathcal{M}(w,\mu)(m)\right]$$

and by taking the limit over $\epsilon \to 0+$, using the continuity and differentiability of $\mathcal{M}(w,\mu)$ we deduce the second part of (3.1). □

Remark 1. *Assume that* $A, B \in \mathcal{B}_1(H)$, *and* $A \ge m_1 > 0$, $B \ge m_2 > 0$, *then we have the trace inequality*

$$|\operatorname{tr}[\mathcal{M}(w,\mu)(B)]-\operatorname{tr}[\mathcal{M}(w,\mu)(A)]|_p \le \|B-A\|_1 \times \begin{cases} \frac{\mathcal{M}(w,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)}{m_2-m_1} & \text{if } m_1 \ne m_2, \\ \mathcal{M}'(w,\mu)(m) & \text{if } m_1=m_2=m. \end{cases} \tag{3.11}$$

Corollary 1. *Assume that* $f : [0,\infty) \to \mathbb{R}$ *is an operator monotone function. If* $A, B \in \mathcal{B}_p(H)$, $p \ge 1$ *and* $A \ge m_1 > 0$, $B \ge m_2 > 0$, *then*

$$\|f(B)-f(A)\|_p \le \|B-A\|_p \times \begin{cases} \frac{f(m_2)-f(m_1)}{m_2-m_1} & \text{if } m_1 \ne m_2, \\ f'(m) & \text{if } m_1=m_2=m. \end{cases} \tag{3.12}$$

We also have the trace inequality

$$|\operatorname{tr}[f(B)]-\operatorname{tr}[f(A)]| \le \|B-A\|_1 \times \begin{cases} \frac{f(m_2)-f(m_1)}{m_2-m_1} & \text{if } m_1 \ne m_2, \\ f'(m) & \text{if } m_1=m_2=m, \end{cases} \tag{3.13}$$

provided that $A, B \in \mathcal{B}_1(H)$ *with* $A \ge m_1 > 0$, $B \ge m_2 > 0$.

Proof. From (1.3) we have for $T > 0$ that

$$f(T) - f(0) - bT = \mathcal{M}(\ell, \mu)(T),$$

for some positive measure μ, where $\ell(\lambda) = \lambda$, $\lambda \geq 0$. Therefore

$$\mathcal{M}(\ell, \mu)(B) - \mathcal{M}(w, \mu)(A) = f(B) - f(A) - b(B - A),$$

$$\mathcal{M}(\ell, \mu)(m_2) - \mathcal{M}(w, \mu)(m_1) = f(m_2) - f(m_1) - b(m_2 - m_1)$$

and

$$\mathcal{M}'(\ell, \mu)(m) = f'(m) - b.$$

By (3.1) we obtain

$$\begin{aligned} &\|f(B) - f(A) - b(B - A)\|_p \\ &\leq \|B - A\|_p \times \begin{cases} \left(\frac{f(m_2) - f(m_1)}{m_2 - m_1} - b\right) \text{ if } m_1 \neq m_2, \\ \\ (f'(m) - bm) \text{ if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

which is an inequality of interest in itself.

By the properties of the norm, we have

$$\begin{aligned} &\|f(B) - f(A)\|_p - b\|B - A\|_p \\ &\leq \|f(B) - f(A) - b(B - A)\|_p \\ &\leq \|B - A\|_p \times \begin{cases} \left(\frac{f(m_2) - f(m_1)}{m_2 - m_1} - b\right) \text{ if } m_1 \neq m_2, \\ \\ (f'(m) - b) \text{ if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

which implies (3.12). □

Remark 2. *If A, $B \in \mathcal{B}_p(H)$, $p \geq 1$ and $A \geq m_1 > 0$, $B \geq m_2 > 0$, then for $r \in (0, 1]$ we have, by (3.12), the power inequalities*

$$\|B^r - A^r\|_p \leq \|B - A\|_p \times \begin{cases} \frac{m_2^r - m_1^r}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ rm^{r-1} \text{ if } m_1 = m_2 = m. \end{cases} \tag{3.14}$$

Let $\epsilon > 0$ and write the inequality (3.12) for $f_\epsilon(t) := \ln(t + \epsilon)$, which is operator monotone in $[0, \infty)$, to get

$$\begin{aligned} &\|\ln(B + \epsilon) - \ln(A + \epsilon)\|_p \\ &\leq \|B - A\|_p \times \begin{cases} \frac{\ln(m_2 + \epsilon) - \ln(m_1 + \epsilon)}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ \frac{1}{m + \epsilon} \text{ if } m_1 = m_2 = m. \end{cases} \end{aligned}$$

By letting $\epsilon \to 0+$ we obtain

$$\|\ln B - \ln A\|_p \leq \|B - A\|_p \times \begin{cases} \frac{\ln m_2 - \ln m_1}{m_2 - m_1} \text{ if } m_1 \neq m_2, \\ \\ \frac{1}{m} \text{ if } m_1 = m_2 = m \end{cases} \tag{3.15}$$

for A, $B \in \mathcal{B}_p(H)$, $p \geq 1$ and $A \geq m_1 > 0$, $B \geq m_2 > 0$.

Corollary 2. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$. If A, $B\in\mathcal{B}_p(H)$, $p\geq 1$ and $A\geq m_1>0$, $B\geq m_2>0$, then*

$$\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\right\|_p \leq \|B-A\|_p\times\begin{cases}\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \frac{f'(m)m-f(m)+f(0)}{m^2} & \text{if } m_1=m_2=m.\end{cases} \tag{3.16}$$

In particular, if $f(0)=0$, then we get the simpler inequality

$$\left\|f(B)B^{-1}-f(A)A^{-1}\right\|_p \leq \|B-A\|_p\times\begin{cases}\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \frac{f'(m)m-f(m)}{m^2} & \text{if } m_1=m_2=m.\end{cases} \tag{3.17}$$

We also have the trace inequality

$$\left|\operatorname{tr}\left[f(B)B^{-1}\right]-\operatorname{tr}\left[f(A)A^{-1}\right]\right| \leq \|B-A\|_1\times\begin{cases}\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \frac{f'(m)m-f(m)}{m^2} & \text{if } m_1=m_2=m,\end{cases} \tag{3.18}$$

provided that A, $B\in\mathcal{B}_1(H)$ with $A\geq m_1>0$, $B\geq m_2>0$.

Proof. From (1.4) we have for $T>0$ that

$$(f(T)-f(0))T^{-1}-f'_+(0)-cT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ. Therefore

$$\begin{aligned}&\mathcal{M}(\ell,\mu)(B)-\mathcal{M}(w,\mu)(A)\\&=f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A),\end{aligned}$$

$$\begin{aligned}&\mathcal{M}(\ell,\mu)(m_2)-\mathcal{M}(w,\mu)(m_1)\\&=f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)-c(m_2-m_1)\end{aligned}$$

and

$$\mathcal{M}(\ell,\mu)(m)=\frac{f'(m)m-f(m)+f(0)}{m^2}-c.$$

Then by (3.1) we get

$$\begin{aligned}&\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\right\|_p\\&\leq\|B-A\|_p\\&\times\begin{cases}\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1}-c & \text{if } m_1\neq m_2,\\ \left(\frac{f'(m)m-f(m)+f(0)}{m^2}-c\right) & \text{if } m_1=m_2=m,\end{cases}\end{aligned}$$

which is an inequality of interest in itself, if $c\geq 0$ is known.

By the properties of the norm, we have

$$\begin{aligned}
&\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)\right\|_p-c\|B-A\|_p\\
&\le\left\|f(B)B^{-1}-f(A)A^{-1}-f(0)\left(B^{-1}-A^{-1}\right)-c(B-A)\right\|_p\\
&\le\|B-A\|_p\begin{cases}\left(\frac{f(m_2)m_2^{-1}-f(m_1)m_1^{-1}-f(0)\left(m_2^{-1}-m_1^{-1}\right)}{m_2-m_1}-c\right) & \text{if } m_1\neq m_2,\\ \\ \left(\frac{f'(m)m-f(m)+f(0)}{m^2}-c\right) & \text{if } m_1=m_2=m,\end{cases}
\end{aligned}$$

which implies (3.16). □

Remark 3. *By applying this inequality to the operator convex function* $f(t)=-\ln(t+1)$, *then we can state the following result:*

$$\begin{aligned}
(3.19)\qquad &\left\|B^{-1}\ln(B+1)-A^{-1}\ln(A+1)\right\|_p\\
&\le\|B-A\|_p\times\begin{cases}\frac{m_1^{-1}\ln(m_1+1)-m_2^{-1}\ln(m_2+1)}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ \frac{\ln(m+1)-m(m+1)^{-1}}{m^2} & \text{if } m_1=m_2=m,\end{cases}
\end{aligned}$$

provided $A,\ B\in\mathcal{B}_p(H)$, $p\ge 1$ *and* $A\ge m_1>0,\ B\ge m_2>0.$

4. Midpoint Inequalities

We have the following midpoint type inequalities:

Proposition 1. *If* $A,\ B\in\mathcal{B}_p(H)$, $p\ge 1$ *and* $A,\ B\ge m0>0$, *then we get the midpoint inequality*

$$\begin{aligned}
(4.1)\qquad &\left\|\int_0^1\mathcal{M}(w,\mu)((1-t)A+tB)\,dt-\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_p\\
&\le\frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\|_p.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
(4.2)\qquad &\left|\int_0^1\operatorname{tr}\left[\mathcal{M}(w,\mu)((1-t)A+tB)\right]dt-\operatorname{tr}\left[\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right]\right|\\
&\le\frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\|_1,
\end{aligned}$$

provided $A,\ B\in\mathcal{B}_1(H)$ *and* $A\ge m_1>0,\ B\ge m_2>0$

Proof. Since $A,\ B\ge m$, hence $\frac{A+B}{2}\ge m>0$ and $(1-t)A+tB\ge m>0$ for all $t\in[0,1]$ and by (3.1)

$$\begin{aligned}
(4.3)\qquad &\left\|\mathcal{M}(w,\mu)((1-t)A+tB)-\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_p\\
&\le\mathcal{M}'(w,\mu)(m)\left\|(1-t)A+tB-\frac{A+B}{2}\right\|_p\\
&=\mathcal{M}'(w,\mu)(m)\left|t-\frac{1}{2}\right|\|B-A\|_p
\end{aligned}$$

for all $t\in[0,1]$.

Taking the integral in (4.3), we get

$$\begin{aligned}&\left\|\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt-\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_p\\&\le\int_0^1\left\|\mathcal{M}(w,\mu)((1-t)A+tB)-\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_p dt\\&\le\mathcal{M}'(w,\mu)(m)\|B-A\|_p\int_0^1\left|t-\frac{1}{2}\right|dt\\&=\frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\|_p\end{aligned}$$

and the inequality (4.1) is proved.

For $p=1$ in (4.3) we get

$$\begin{aligned}&\left\|\mathcal{M}(w,\mu)((1-t)A+tB)-\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_1\\&\le\mathcal{M}'(w,\mu)(m)\left|t-\frac{1}{2}\right|\|B-A\|_1,\end{aligned}$$

namely

$$\begin{aligned}&\operatorname{tr}\left|\mathcal{M}(w,\mu)((1-t)A+tB)-\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right|\\&\le\mathcal{M}'(w,\mu)(m)\left|t-\frac{1}{2}\right|\|B-A\|_1,\end{aligned}$$

for $t\in[0,1]$.

If we take the integral and use the properties of modulus, we have

$$\begin{aligned}&\left|\int_0^1\operatorname{tr}[\mathcal{M}(w,\mu)((1-t)A+tB)]\,dt-\operatorname{tr}\left[\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right]\right|\\&\le\int_0^1\operatorname{tr}\left|\mathcal{M}(w,\mu)((1-t)A+tB)-\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right|dt\\&\le\mathcal{M}'(w,\mu)(m)\int_0^1\left|t-\frac{1}{2}\right|dt\,\|B-A\|_1\\&=\frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\|_1,\end{aligned}$$

which proves (4.2). □

Corollary 3. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. If A, $B\in\mathcal{B}_p(H)$, $p\ge1$, with $A\ge m>0$, then we have the midpoint inequality*

$$\left\|\int_0^1 f((1-t)A+tB)\,dt-f\left(\frac{A+B}{2}\right)\right\|_p\le\frac{1}{4}f'(m)\|B-A\|_p. \tag{4.4}$$

In particular, we have

$$\left|\int_0^1\operatorname{tr}[f((1-t)A+tB)]\,dt-\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\right]\right|\le\frac{1}{4}f'(m)\|B-A\|_1, \tag{4.5}$$

provided that A, $B\in\mathcal{B}_p(H)$, $p\ge1$, with A, $B\ge m>0$.

Proof. From (1.3) we have for $T>0$ that

$$f(T)-f(0)-bT=\mathcal{M}(\ell,\mu)(T),$$

for some positive measure μ, where $\ell(\lambda)=\lambda$, $\lambda\geq 0$.

Therefore

$$\int_0^1 \mathcal{M}(\ell,\mu)((1-t)A+tB)\,dt = \int_0^1 f((1-t)A+tB)\,dt - f(0) - b\left(\frac{A+B}{2}\right),$$

$$\mathcal{M}(\ell,\mu)\left(\frac{A+B}{2}\right) = f\left(\frac{A+B}{2}\right) - f(0) - b\left(\frac{A+B}{2}\right)$$

and

$$\mathcal{M}'(\ell,\mu)(m)=f'(m)-b.$$

From (4.1) we derive

$$\left\|\int_0^1 \mathcal{M}(w,\mu)((1-t)A+tB)\,dt-\mathcal{M}(w,\mu)\left(\frac{A+B}{2}\right)\right\|_p \leq \frac{1}{4}\left[f'(m)-b\right]\|B-A\|_p \leq \frac{1}{4}f'(m)\|B-A\|_p, \tag{4.6}$$

which is an inequality of interest if the nonnegative parameter b is known. □

Remark 4. *If $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$, with $A,\ B\geq m>0$, then we have the midpoint inequality for power function with exponent $r\in(0,1]$*

$$\left\|\int_0^1 ((1-t)A+tB)^r\,dt-\left(\frac{A+B}{2}\right)^r\right\|_p \leq \frac{1}{4}rm^{r-1}\|B-A\|_p \tag{4.7}$$

The following inequalities for logarithm also holds

$$\left\|\int_0^1 \ln((1-t)A+tB)\,dt-\ln\left(\frac{A+B}{2}\right)\right\|_p \leq \frac{1}{4m}\|B-A\|_p. \tag{4.8}$$

If $A,\ B\in\mathcal{B}_1(H)$ with $A\geq m>0$, then we have the trace inequalities

$$\left|\int_0^1 \operatorname{tr}\left[((1-t)A+tB)^r\right]dt-\operatorname{tr}\left[\left(\frac{A+B}{2}\right)^r\right]\right| \leq \frac{1}{4}rm^{r-1}\|B-A\|_1 \tag{4.9}$$

and

$$\left|\int_0^1 \operatorname{tr}\left[\ln((1-t)A+tB)\right]dt-\operatorname{tr}\left[\ln\left(\frac{A+B}{2}\right)\right]\right| \leq \frac{1}{4m}\|B-A\|. \tag{4.10}$$

Corollary 4. *Assume that $f:[0,\infty)\to\mathbb{R}$ is operator convex in $[0,\infty)$. If A, $B\in\mathcal{B}_p(H)$, $p\geq 1$, with $A,\ B\geq m>0$, then*

$$\begin{aligned}&\left\|\int_0^1 f((1-t)A+tB)((1-t)A+tB)^{-1}\,dt-f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}\right.\\&\left.-f(0)\left(\int_0^1((1-t)A+tB)^{-1}\,dt-\left(\frac{A+B}{2}\right)^{-1}\right)\right\|_p\\&\leq\frac{f'(m)m-f(m)+f(0)}{4m^2}\|B-A\|_p.\end{aligned} \tag{4.11}$$

We also have the trace inequality

$$\begin{aligned}(4.12)\quad & \left|\int_0^1 \operatorname{tr}\left[f\left((1-t)A+tB\right)\left((1-t)A+tB\right)^{-1}\right]dt\right. \\ & -\operatorname{tr}\left[f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}\right] \\ & \left.-f(0)\left(\int_0^1 \operatorname{tr}\left[\left((1-t)A+tB\right)^{-1}\right]dt-\operatorname{tr}\left[\left(\frac{A+B}{2}\right)^{-1}\right]\right)\right| \\ & \leq \frac{f'(m)m-f(m)+f(0)}{4m^2}\left\|B-A\right\|_1,\end{aligned}$$

if $A,\ B\in\mathcal{B}_1(H)$, *with* $A,\ B\geq m>0$.

Proof. From (1.4) we have for $T>0$ that

$$\mathcal{M}(\ell,\mu)(T)=\left(f(T)-f(0)\right)T^{-1}-f'_+(0)-cT,$$

for some positive measure μ. Therefore

$$\begin{aligned}&\int_0^1 \mathcal{M}(\ell,\mu)\left((1-t)A+tB\right)dt \\ &=\int_0^1 f\left((1-t)A+tB\right)\left((1-t)A+tB\right)^{-1}dt-f(0)\int_0^1\left((1-t)A+tB\right)^{-1}dt \\ &-f'_+(0)-c\left(\frac{A+B}{2}\right),\end{aligned}$$

$$\begin{aligned}\mathcal{M}(\ell,\mu)\left(\frac{A+B}{2}\right)&=f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}-f(0)\left(\frac{A+B}{2}\right)^{-1} \\ &-f'_+(0)-c\left(\frac{A+B}{2}\right),\end{aligned}$$

and

$$\mathcal{M}(\ell,\mu)(m)=\frac{f'(m)m-f(m)+f(0)}{m^2}-c.$$

By utilising (4.1) we get

$$\begin{aligned}&\left\|\int_0^1 f\left((1-t)A+tB\right)\left((1-t)A+tB\right)^{-1}dt-f\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^{-1}\right. \\ &\left.-f(0)\left(\int_0^1\left((1-t)A+tB\right)^{-1}dt-\left(\frac{A+B}{2}\right)^{-1}\right)\right\|_p \\ &\leq\frac{1}{4}\left(\frac{f'(m)m-f(m)+f(0)}{m^2}-c\right)\left\|B-A\right\|_p \\ &\leq\frac{f'(m)m-f(m)+f(0)}{4m^2}\left\|B-A\right\|_p,\end{aligned}$$

which is an inequality of interest if $c\geq 0$ is known. □

Remark 5. *If in these inequalities we take the operator convex function* $f(t) = -\ln(t+1)$, *then we get*

$$\begin{aligned}(4.13)\qquad & \left\| \int_0^1 \ln((1-t)A + tB + 1)((1-t)A + tB)^{-1} dt \right. \\ & \left. - \ln\left(\frac{A+B}{2} + 1\right)\left(\frac{A+B}{2}\right)^{-1} \right\|_p \\ & \leq \frac{\ln(m+1) - m(m+1)^{-1}}{m^2} \|B - A\|_p \end{aligned}$$

if $A, B \in \mathcal{B}_p(H)$, $p \geq 1$, *with* $A, B \geq m > 0$ *and*

$$\begin{aligned}(4.14)\qquad & \left| \int_0^1 \operatorname{tr}\left[\ln((1-t)A + tB + 1)((1-t)A + tB)^{-1}\right] dt \right. \\ & \left. - \operatorname{tr}\left[\ln\left(\frac{A+B}{2} + 1\right)\left(\frac{A+B}{2}\right)^{-1}\right] \right| \\ & \leq \frac{\ln(m+1) - m(m+1)^{-1}}{m^2} \|B - A\|_1 \end{aligned}$$

if $A, B \in \mathcal{B}_1(H)$, *with* $A, B \geq m > 0$.

5. Trapezoid Inequalities

We have the following trapezoid type inequalities:

Proposition 2. *If* $A, B \in \mathcal{B}_p(H)$, $p \geq 1$ *and* $A, B \geq m > 0$, *then we get the midpoint inequality the trapezoid inequality*

$$\begin{aligned}(5.1)\qquad & \left\| \frac{\mathcal{M}(w,\mu)(A) + \mathcal{M}(w,\mu)(B)}{2} - \int_0^1 \mathcal{M}(w,\mu)((1-t)A + tB) dt \right\|_p \\ & \leq \frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B - A\|_p . \end{aligned}$$

In particular,

$$\begin{aligned}(5.2)\qquad & \left| \frac{\operatorname{tr}[\mathcal{M}(w,\mu)(A)] + \operatorname{tr}[\mathcal{M}(w,\mu)(B)]}{2} \right. \\ & \left. - \int_0^1 \operatorname{tr}[\mathcal{M}(w,\mu)((1-t)A + tB)] dt \right| \\ & \leq \frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B - A\|_1 \end{aligned}$$

provided that $A, B \in \mathcal{B}_1(H)$ *with* $A, B \geq m > 0$.

Proof. Since $A, B \geq m$, hence $(1-s)A + s\frac{A+B}{2}$, $s\frac{A+B}{2} + (1-s)B \geq m > 0$ for all $s \in [0,1]$ and by (3.1) we get

$$\begin{aligned}(5.3)\qquad & \left\| \mathcal{M}(w,\mu)(A) - \mathcal{M}(w,\mu)\left((1-s)A + s\frac{A+B}{2}\right) \right\|_p \\ & \leq \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B - A\|_p s \end{aligned}$$

and

$$\left\|\mathcal{M}(w,\mu)(B)-\mathcal{M}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right\|_p \tag{5.4}$$
$$\leq \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|_p s.$$

From (5.3) and (5.4) we derive by addition, division by 2 and triangle inequality that

$$\left\|\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2}\right.$$
$$\left.-\frac{1}{2}\left[\mathcal{M}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)+\mathcal{M}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right]\right\|_p$$
$$\leq \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|_p s$$

for all $s\in[0,1]$.

By taking the integral and using its properties, we derive

$$\left\|\frac{\mathcal{M}(w,\mu)(A)+\mathcal{M}(w,\mu)(B)}{2}\right. \tag{5.5}$$
$$-\frac{1}{2}\left[\int_0^1\mathcal{M}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)\right.$$
$$\left.\left.+\mathcal{M}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)ds\right]\right\|_p$$
$$\leq \frac{1}{2}\mathcal{M}'(w,\mu)(m)\|B-A\|_p\int_0^1 sds=\frac{1}{4}\mathcal{M}'(w,\mu)(m)\|B-A\|_p.$$

Now, using the change of variable $t=2s$ we have

$$\frac{1}{2}\int_0^1\mathcal{M}(w,\mu)\left((1-t)A+t\frac{A+B}{2}\right)dt$$
$$=\int_0^{1/2}\mathcal{M}(w,\mu)((1-s)A+sB)ds$$

and by the change of variable $t=1-v$ we have

$$\frac{1}{2}\int_0^1\mathcal{M}(w,\mu)\left(t\frac{A+B}{2}+(1-t)A\right)dt$$
$$=\frac{1}{2}\int_0^1\mathcal{M}(w,\mu)\left((1-v)\frac{A+B}{2}+vB\right)dv.$$

Moreover, if we make the change of variable $v=2s-1$ we also have

$$\frac{1}{2}\int_0^1\mathcal{M}(w,\mu)\left((1-v)\frac{A+B}{2}+vB\right)dv$$
$$=\int_{1/2}^1\mathcal{M}(w,\mu)((1-s)A+sB)ds.$$

Therefore

$$\frac{1}{2}\int_0^1\left[\mathcal{M}(w,\mu)\left((1-s)A+s\frac{A+B}{2}\right)+\mathcal{M}(w,\mu)\left(s\frac{A+B}{2}+(1-s)B\right)\right]ds$$
$$=\int_0^{1/2}\mathcal{M}(w,\mu)((1-s)A+sB)\,dt+\int_{1/2}^1\mathcal{M}(w,\mu)((1-s)A+sB)\,ds$$
$$=\int_0^1\mathcal{M}(w,\mu)((1-s)A+sB)\,ds$$

and by (5.5) we deduce the desired result (5.1). □

Corollary 5. *Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$. If $A,\ B\in\mathcal{B}_p(H),\ p\geq 1$ and $A,\ B\geq m>0$, then we get the midpoint inequality the trapezoid inequality*

$$\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\|_p\leq\frac{1}{4}f'(m)\|B-A\|_p. \tag{5.6}$$

In particular,

$$\left|\frac{\operatorname{tr}[f(A)]+\operatorname{tr}[f(B)]}{2}-\int_0^1\operatorname{tr}[f((1-t)A+tB)]\,dt\right|\leq\frac{1}{4}f'(m)\|B-A\|_1, \tag{5.7}$$

provided that $A,\ B\in\mathcal{B}_1(H)$ with $A,\ B\geq m>0$.

Proof. Since

$$\mathcal{M}(\ell,\mu)(A)=f(A)-f(0)-bA,\ \text{and}\ \mathcal{M}(\ell,\mu)(B)=f(B)-f(0)-bB,$$

then by (5.1) we get

$$\left\|\frac{f(A)+f(B)}{2}-\int_0^1 f((1-t)A+tB)\,dt\right\|_p$$
$$\leq\frac{1}{4}[f'(m)-b]\|B-A\|_p\leq\frac{1}{4}f'(m)\|B-A\|_p,$$

which is of interest if $b\geq 0$ is known. □

Remark 6. *If $A,\ B\in\mathcal{B}_p(H),\ p\geq 1$, with $A,\ B\geq m>0$, then we have the midpoint inequality for power function with exponent $r\in(0,1]$*

$$\left\|\frac{A^r+B^r}{2}-\int_0^1((1-t)A+tB)^r\,dt\right\|_p\leq\frac{1}{4}rm^{r-1}\|B-A\|_p \tag{5.8}$$

The following inequalities for logarithm also holds

$$\left\|\frac{\ln A+\ln B}{2}-\int_0^1\ln((1-t)A+tB)\,dt\right\|_p\leq\frac{1}{4m}\|B-A\|_p. \tag{5.9}$$

If $A,\ B\in\mathcal{B}_1(H)$ with $A\geq m>0$, then we have the trace inequalities

$$\left|\frac{\operatorname{tr}(A^r)+\operatorname{tr}(B^r)}{2}-\int_0^1\operatorname{tr}[((1-t)A+tB)^r]\,dt\right|\leq\frac{1}{4}rm^{r-1}\|B-A\|_1 \tag{5.10}$$

and

$$\left|\frac{\operatorname{tr}(\ln A)+\operatorname{tr}(\ln B)}{2}-\int_0^1\operatorname{tr}[\ln((1-t)A+tB)]\,dt\right|\leq\frac{1}{4m}\|B-A\|_1. \tag{5.11}$$

Finally, we can state:

Corollary 6. *Assume that* $f:[0,\infty)\rightarrow\mathbb{R}$ *is operator convex in* $[0,\infty)$*. If* A, $B\in\mathcal{B}_p(H)$, $p\geq 1$, *with* A, $B\geq m>0$, *then*

$$\begin{aligned}(5.12)\quad &\left\Vert \frac{f(A)A^{-1}+f(B)B^{-1}}{2}-\int_0^1 f((1-t)A+tB)((1-t)A+tB)^{-1}dt\right.\\ &\left.-f(0)\left(\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right)\right\Vert_p\\ &\leq\frac{f'(m)m-f(m)+f(0)}{4m^2}\Vert B-A\Vert_p.\end{aligned}$$

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