SOME LIPSCHITZ TYPE p-SCHATTEN NORM INEQUALITIES FOR THE $\mathcal{D}$-LOGARITHMIC INTEGRAL TRANSFORM OF POSITIVE OPERATORS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and μ a positive measure on $(0,\infty)$ we consider the following $\mathcal{D}$-*logarithmic integral transform*

$$\mathcal{DL}og(w,\mu)(t) := \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\mu(\lambda)$$

where the integral is assumed to exist for T a positive operator on a complex Hilbert space H. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

Assume that A, $B \in \mathcal{B}_p(H)$, $p \ge 1$ and $A \ge m_1 > 0$, $B \ge m_2 > 0$, then

$$\|\mathcal{DL}og(w,\mu)(B) - \mathcal{DL}og(w,\mu)(A)\|_p \le \|B-A\|_p \times \begin{cases} \frac{\mathcal{DL}og(w,\mu)(m_2)-\mathcal{DL}og(w,\mu)(m_1)}{m_2-m_1} \text{ if } m_1 \ne m_2, \\ \\ \mathcal{D}(w,\mu)(m) \text{ if } m_1 = m_2 = m, \end{cases}$$

where

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t}d\mu(\lambda),\ t>0,$$

is the derivative of $\mathcal{DL}og(w,\mu)$ as a function of t.

Applications for Hermite-Hadamard type inequalities and some particular examples of interest are also given.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [6] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \le L\|A-B\|$$

for any A, $B \in \mathcal{B}(H)$.

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However, as shown by Farforovskaya in [30], [31] and Kato in [38], the following inequality holds

$$\||A|-|B|\| \leq \frac{2}{\pi}\|A-B\|\left(2+\log\left(\frac{\|A\|+\|B\|}{\|A-B\|}\right)\right)$$

for any A, $B\in\mathcal{B}(H)$ with $A\neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS}:=(\operatorname{tr}C^{*}C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\||A|-|B|\|_{HS}\leq\sqrt{2}\,\|A-B\|_{HS}$$

for any A, $B\in\mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [6] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\|\leq a_1\|A-B\|+a_2\|A-B\|^2+O\left(\|A-B\|^3\right)$$

where

$$a_1=\left\|A^{-1}\right\|\|A\| \text{ and } a_2=\left\|A^{-1}\right\|+\left\|A^{-1}\right\|^3\|A\|^2.$$

An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$. In [5] the author also obtained the following *Lipschitz type inequality*

$$\|f(A)-f(B)\|\leq f'(a)\|A-B\|$$

where f is an *operator monotone function* on $(0,\infty)$ and A, $B\geq a>0$.

One of the problems in perturbation theory is to find bounds for $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [7], [32] and the references therein.

We have the following representation of operator monotone functions [42], see for instance [8, p. 144-145]:

Theorem 1. *A function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t)=f(0)+bt+\int_0^\infty\frac{t\lambda}{t+\lambda}d\mu(\lambda), \tag{1.1}$$

where $b\geq 0$ and a positive measure μ on $(0,\infty)$ such that

$$\int_0^\infty\frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty.$$

For some recent results related to operator monotone functions we refer to [33], [35] [30] and the references therein.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [8, p. 145]

$$s^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+s}d\lambda. \tag{1.2}$$

Observe that for $s>0$, $s\neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+s)(\lambda+1)}=\frac{\ln s}{s-1}+\frac{1}{1-s}\ln\left(\frac{u+s}{u+1}\right)$$

for all $u>0$.

By taking the limit over $u\to\infty$ in this equality, we derive

$$\frac{\ln s}{s-1}=\int_0^\infty \frac{d\lambda}{(\lambda+s)(\lambda+1)}, \tag{1.3}$$

which gives the representation for the logarithm

$$\ln s=(s-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+s)} \tag{1.4}$$

If we integrate (1.2) over s from 0 to $t>0$, we get by Fubini's theorem

$$\begin{aligned}\frac{t^r}{r}&=\frac{\sin(r\pi)}{\pi}\int_0^\infty\left(\int_0^t\left(\frac{1}{\lambda+s}\right)ds\right)\lambda^{r-1}d\lambda\\&=\frac{\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda\end{aligned}$$

giving the identity of interest

$$t^r=\frac{r\sin(r\pi)}{\pi}\int_0^\infty \lambda^{r-1}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda,\ t>0 \text{ and } r\in(0,1].$$

Recall the *dilogarithmic function* dilog : $[0,\infty)\to\mathbb{R}$ defined by

$$\operatorname{dilog}(t):=\int_1^t\frac{\ln s}{1-s}ds,\ t\geq 0.$$

Some particular values of interest are

$$\operatorname{dilog}(1)=0,\ \operatorname{dilog}(0)=\int_1^0\frac{\ln s}{1-s}ds=\int_0^1\frac{\ln s}{s-1}ds=\frac{1}{6}\pi^2,$$

and

$$\operatorname{dilog}\left(\frac{1}{2}\right)=\frac{1}{12}\pi^2-\frac{1}{2}(\ln 2)^2.$$

If we integrate the identity (1.3) over s from 0 to $t>0$, we get by Fubini's theorem

$$\int_0^t\frac{\ln s}{s-1}ds=\int_0^\infty\left(\int_0^t\frac{1}{\lambda+s}ds\right)\frac{1}{(\lambda+1)}d\lambda=\int_0^\infty\frac{1}{(\lambda+1)}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda$$

and since

$$\begin{aligned}\int_0^t\frac{\ln s}{s-1}ds&=\int_0^1\frac{\ln s}{s-1}ds+\int_1^t\frac{\ln s}{s-1}ds=\frac{1}{6}\pi^2-\int_1^t\frac{\ln s}{1-s}ds\\&=\frac{1}{6}\pi^2-\operatorname{dilog}(t)\end{aligned}$$

then we get the identity of interest

$$\frac{1}{6}\pi^2-\operatorname{dilog}(t)=\int_0^\infty\frac{1}{(\lambda+1)}\ln\left(\frac{t+\lambda}{\lambda}\right)d\lambda,\ t>0.$$

Motivated by the above representations, we define the *$\mathcal{D}$-logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda > 0$ by

$$\mathcal{DL}og(w,\mu)(t) := \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\mu(\lambda), \tag{1.5}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.9) exists for all $t > 0$. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{DL}og(w)(t) := \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\lambda. \tag{1.6}$$

Obviously,

$$\begin{aligned}\mathcal{DL}og(w,\mu)(t) &= \int_0^\infty w(\lambda)\ln\left(1+\frac{t}{\lambda}\right)d\mu(\lambda)\\ &= \int_0^\infty w(\lambda)\left[\ln(\lambda+t)-\ln(\lambda)\right]d\mu(\lambda)\end{aligned}$$

and one can use either of these representations when is needed.

If we use the *$\mathcal{D}$-logarithmic transform* for the kernel $w_{\ell^{r-1}}(\lambda) := \frac{r\sin(r\pi)}{\pi}\lambda^{r-1}$, $r \in (0,1]$ we have

$$\mathcal{DL}og(w_{\ell^{r-1}})(t) = t^r,\ t \geq 0$$

while for the kernel $w_{(\ell+1)^{-1}}(\lambda) := \frac{1}{\lambda+1}$ we have

$$\mathcal{DL}og\left(w_{(\ell+1)^{-1}}\right)(t) = \frac{1}{6}\pi^2 - \operatorname{dilog}(t),\ t \geq 0. \tag{1.7}$$

In the recent paper [29] we introduced the following *integral transform*

$$\mathcal{D}(w,\mu)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s}d\mu(\lambda),\ s > 0, \tag{1.8}$$

for a continuous and positive function $w(\lambda)$, $\lambda > 0$,where μ is a positive measure on $(0,\infty)$ and the integral (3.3) exists for all $s > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(s) := \int_0^\infty \frac{w(\lambda)}{\lambda+s}d\lambda,\ s > 0. \tag{1.9}$$

Several examples of integral transforms $\mathcal{D}(w,\mu)$ have also been given in [29].

If we integrate the identity (1.3) over s from 0 to $t > 0$, we get by Fubini's theorem

$$\begin{aligned}\int_0^t \mathcal{D}(w,\mu)(s)\,ds &:= \int_0^\infty \left(\int_0^t \frac{1}{\lambda+s}ds\right)w(\lambda)\,d\mu(\lambda)\\ &= \int_0^\infty w(\lambda)\ln\left(\frac{\lambda+t}{\lambda}\right)d\mu(\lambda)\end{aligned} \tag{1.10}$$

for $t > 0$, which provides the equality of interest

$$\mathcal{DL}og(w,\mu)(t) = \int_0^t \mathcal{D}(w,\mu)(s)\,ds,\ t > 0, \tag{1.11}$$

provided that the integral on the right side exists for all $t > 0$.

2. Some Preliminary Facts on p-Schatten Norm

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A| e_i, e_i\rangle < \infty. \tag{2.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{2.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.3}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* AT, $TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1\|T\|; \tag{2.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [48, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{2.5}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{2.6}$$

For $p\ge 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [48, p. 60-64],

$$\|A\|_p = \|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{2.7}$$

$$\|AB\|_p \le \|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{2.8}$$

and

$$\|AB\|_p \le \|A\|_p\|B\|,\ \|BA\|_p \le \|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{2.9}$$

This implies that

$$\|CAB\|_p \le \|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{2.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \; A \in \mathcal{B}_p(H), \; B \in \mathcal{B}_q(H). \tag{2.11}$$

For the theory of trace functionals and their applications the reader is referred to [46] and [48].

For some classical trace inequalities see [9], [11], and [43], which are continuations of the work of Bellman [3]. For related works the reader can refer to [1], [4], [9], [34], [39], [40], [41], [45], [47] and the recent papers [12]-[28].

3. Main Results

Start to the following identity for the logarithmic function:

Lemma 1. *For all A, $B > 0$ we have the identity:*

$$\begin{aligned} &\ln B - \ln A \\ &= \int_0^\infty \left(\int_0^1 (s + (1-t)A + tB)^{-1} (B - A) (s + (1-t)A + tB)^{-1} dt \right) ds. \end{aligned} \tag{3.1}$$

Proof. We have from (1.4) for A, $B > 0$ that

$$\ln B - \ln A = \int_0^\infty \frac{1}{s+1} \left[(B-1)(s+B)^{-1} - (A-1)(s+A)^{-1} \right] ds. \tag{3.2}$$

Since

$$\begin{aligned} &(B-1)(s+B)^{-1} - (A-1)(s+A)^{-1} \\ &= B(s+B)^{-1} - A(s+A)^{-1} - \left((s+B)^{-1} - (s+A)^{-1} \right) \end{aligned}$$

and

$$\begin{aligned} &B(s+B)^{-1} - A(s+A)^{-1} \\ &= (B+s-s)(s+B)^{-1} - (A+s-s)(s+A)^{-1} \\ &= 1 - s(s+B)^{-1} - 1 + s(s+A)^{-1} = s(s+A)^{-1} - s(s+B)^{-1}, \end{aligned}$$

hence

$$\begin{aligned} &(B-1)(s+B)^{-1} - (A-1)(s+A)^{-1} \\ &= s(s+A)^{-1} - s(s+B)^{-1} - \left((s+B)^{-1} - (s+A)^{-1} \right) \\ &= (s+1) \left[(s+A)^{-1} - (s+B)^{-1} \right] \end{aligned}$$

and by (3.2) we get

$$\ln B - \ln A = \int_0^\infty \left[(s+A)^{-1} - (s+B)^{-1} \right] ds. \tag{3.3}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotone on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \to 0} \left[\frac{f(T+tS) - f(T)}{t} \right] = T^{-1} S T^{-1} \tag{3.4}$$

for T, $S > 0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{3.5}$$

If we write this equality for the function $f(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{3.6}$$

Since, by (3.6) we have

$$\begin{aligned}&(s+A)^{-1}-(s+B)^{-1}\\&=\int_0^1(s+(1-t)A+tB)^{-1}(B-A)(s+(1-t)A+tB)^{-1}\,dt,\end{aligned} \tag{3.7}$$

for all $s\geq 0$, hence by (3.3) and (3.7) we get (3.1). □

Lemma 2. *For all A, $B>0$ we have the identity:*

$$\begin{aligned}&\mathcal{D}\mathcal{L}og(w,\mu)(B)-\mathcal{D}\mathcal{L}og(w,\mu)(A)\\&=\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\\&\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}\,dt\right)ds\right)d\mu(\lambda).\end{aligned} \tag{3.8}$$

Proof. For all A, $B>0$ we have

$$\begin{aligned}&\mathcal{D}\mathcal{L}og(w,\mu)(B)-\mathcal{D}\mathcal{L}og(w,\mu)(A)\\&=\int_0^\infty w(\lambda)[\ln(\lambda+B)-\ln\lambda]\,d\mu(\lambda)-\int_0^\infty w(\lambda)[\ln(\lambda+A)-\ln\lambda]\,d\mu(\lambda)\\&=\int_0^\infty w(\lambda)[\ln(\lambda+B)-\ln(\lambda+A)]\,d\mu(\lambda).\end{aligned} \tag{3.9}$$

Since, by (3.1) we get

$$\begin{aligned}&\ln(\lambda+B)-\ln(\lambda+A)\\&=\int_0^\infty\left(\int_0^1(s+(1-t)((\lambda+A))+t(\lambda+B))^{-1}\right.\\&\left.\times(\lambda+B-(\lambda+A))(s+(1-t)((\lambda+A))+t(\lambda+B))^{-1}\,dt\right)ds\end{aligned}$$

for all $\lambda \geq 0$, then by multiplying with $w(\lambda)$ and integrating over $\mu(\lambda)$ we obtain

$$\int_0^\infty w(\lambda)[\ln(\lambda+B)-\ln(\lambda+A)]\,d\mu(\lambda) \tag{3.10}$$
$$=\left(\int_0^\infty\left(\int_0^1 (s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.$$
$$\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}dt\right)ds\right)d\mu(\lambda).$$

Finally, by (3.9) and (3.10) we get (3.8). □

We have the following Lipschitz type inequality for the p-Schatten norm:

Theorem 3. *Assume that* $A,\ B\in\mathcal{B}_p(H)$, $p\geq 1$ *and* $A\geq m_1>0$, $B\geq m_2>0$, *then*

$$\|\mathcal{DL}og(w,\mu)(B)-\mathcal{DL}og(w,\mu)(A)\|_p \tag{3.11}$$
$$\leq \|B-A\|_p\times\begin{cases}\frac{\mathcal{DL}og(w,\mu)(m_2)-\mathcal{DL}og(w,\mu)(m_1)}{m_2-m_1} & \text{if } m_1\neq m_2,\\ \\ \mathcal{D}(w,\mu)(m) & \text{if } m_1=m_2=m.\end{cases}$$

Proof. By taking the p-Schatten norm in (3.8) we get by

$$\|\mathcal{DL}og(w,\mu)(B)-\mathcal{DL}og(w,\mu)(A)\|_p \tag{3.12}$$
$$\leq\int_0^\infty w(\lambda)\left(\int_0^\infty\left\|\left(\int_0^1(s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\right.$$
$$\left.\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}dt\right)\right\|_p ds\right)d\mu(\lambda)$$
$$\leq\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\right.$$
$$\left.\left.\left.\times(s+\lambda+(1-t)A+tB)^{-1}\right\|_p dt\right)ds\right)d\mu(\lambda)$$
$$\leq\|B-A\|_p\int_0^\infty w(\lambda)$$
$$\times\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2dt\right)ds\right)d\mu(\lambda),$$

for all $A,\ B>0$, $A,\ B\in\mathcal{B}_p(H)$, where for the last inequality we used the property (2.10).

Assume that $m_2>m_1$. Then

$$s+\lambda+(1-t)A+tB\geq(1-t)m_1+tm_2+s+\lambda,$$

for $t\in[0,1]$ and $s,\ \lambda\geq 0$.

This implies that

$$(s+\lambda+(1-t)A+tB)^{-1}\leq((1-t)m_1+tm_2+s+\lambda)^{-1}$$

and

$$\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2\leq((1-t)m_1+tm_2+s+\lambda)^{-2}$$

for $t\in[0,1]$ and $s,\ \lambda\geq 0$.

Therefore

$$\begin{aligned}(3.13)\qquad &\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2 dt\right)ds\right)d\mu(\lambda)\\ &\le\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1((1-t)m_1+tm_2+s+\lambda)^{-2}dt\right)ds\right)d\mu(\lambda)\\ &=\frac{1}{m_2-m_1}\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1((1-t)m_1+tm_2+s+\lambda)^{-1}\right.\right.\\ &\times\left.\left.(m_2-m_1)((1-t)m_1+tm_2+s+\lambda)^{-1}dt\right)ds\right)d\mu(\lambda).\end{aligned}$$

If in the identity (3.8) we take $A=m_1$, $B=m_2$, then we get

$$\begin{aligned}&\mathcal{DL}og(w,\mu)(m_2)-\mathcal{DL}og(w,\mu)(m_1)\\ &=\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1(s+\lambda+(1-t)m_1+tm_2)^{-1}(m_2-m_1)\right.\right.\\ &\times\left.\left.(s+\lambda+(1-t)m_1+tm_2)^{-1}dt\right)ds\right)d\mu(\lambda).\end{aligned}$$

and by (3.13) we get

$$\begin{aligned}(3.14)\qquad &\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2 dt\right)ds\right)d\mu(\lambda)\\ &\le\frac{1}{m_2-m_1}\left[\mathcal{DL}og(w,\mu)(m_2)-\mathcal{DL}og(w,\mu)(m_1)\right].\end{aligned}$$

The case $m_2<m_1$ goes in a similar way and we also obtain (3.14).

Assume that $m_2=m_1=m$. Let $\epsilon>0$, then $B+\epsilon\ge m+\epsilon>m$. From (3.14) we get

$$\begin{aligned}&\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+t(B+\epsilon))^{-1}\right\|^2 dt\right)ds\right)d\mu(\lambda)\\ &\le\frac{1}{m+\epsilon-m}\left[\mathcal{DL}og(w,\mu)(m+\epsilon)-\mathcal{DL}og(w,\mu)(m)\right]\end{aligned}$$

and by taking the limit over $\epsilon\to 0+$, using the continuity and differentiability of $\mathcal{DL}og$ we deduce

$$\begin{aligned}&\int_0^\infty w(\lambda)\left(\int_0^\infty\left(\int_0^1\left\|(s+\lambda+(1-t)A+tB)^{-1}\right\|^2 dt\right)ds\right)d\mu(\lambda)\\ &\le(\mathcal{DL}og(w,\mu))'(m)=\mathcal{D}(w,\mu)(m),\end{aligned}$$

which proves the second part of (3.11). □

We have:

Lemma 3. *Assume that function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation (1.1), where $b\ge 0$ and μ is a positive measure on $[0,\infty)$. Then*

$$\mathcal{DL}og(\ell,\mu)(t)=F_f(t)-bt \tag{3.15}$$

provided the function

$$F_f(t):=\int_0^t\frac{f(s)-f(0)}{s}ds \tag{3.16}$$

is defined for all $t\in(0,\infty)$.

Proof. From (1.1) we have

$$\frac{f(s)-f(0)}{s}-b=\int_0^\infty\frac{\lambda}{s+\lambda}d\mu(\lambda)=\mathcal{D}(\ell,\mu)(s) \tag{3.17}$$

where $\ell(\lambda)=\lambda$, $\lambda\geq0$.

By taking the integral over s on $(0,t)$, we have

$$\int_0^t\frac{f(s)-f(0)}{s}ds-bt=\int_0^t\mathcal{D}(\ell,\mu)(s)\,ds=\mathcal{D}\mathcal{L}og(\ell,\mu)(t)$$

for $t>0$, and the proposition is proved. □

Corollary 1. *With the assumptions of Lemma 3 and if* $A,B\in\mathcal{B}_p(H)$, $p\geq1$ *with* $A\geq m_1>0$, $B\geq m_2>0$, *then*

$$\|F_f(B)-F_f(A)\|_p\leq\|B-A\|_p\times\begin{cases}\frac{F_f(m_2)-F_f(m_1)}{m_2-m_1} & \text{if } m_1\neq m_2,\\[2ex] \frac{f(m)-f(0)}{m} & \text{if } m_1=m_2=m.\end{cases} \tag{3.18}$$

Proof. The inequality

$$\begin{aligned}&\|F_f(B)-F_f(A)-b(B-A)\|_p \\ &\leq\|B-A\|_p\times\begin{cases}\left(\frac{F_f(m_2)-F_f(m_1)}{m_2-m_1}-b\right) & \text{if } m_1\neq m_2,\\[2ex] \left(\frac{f(m)-f(0)}{m}-b\right) & \text{if } m_1=m_2=m.\end{cases}\end{aligned} \tag{3.19}$$

follows by (3.11) for $\mathcal{D}\mathcal{L}og(\ell,\mu)(t)=F_f(t)-bt$, $t>0$.

By the triangle inequality and (3.19) we have

$$\begin{aligned}&\|F_f(B)-F_f(A)\|_p-b\|B-A\|_p\\ &\leq\|F_f(B)-F_f(A)-b(B-A)\|_p\\ &\leq\|B-A\|_p\times\begin{cases}\left(\frac{F_f(m_2)-F_f(m_1)}{m_2-m_1}-b\right) & \text{if } m_1\neq m_2,\\[2ex] \left(\frac{f(m)-f(0)}{m}-b\right) & \text{if } m_1=m_2=m.\end{cases}\end{aligned}$$

and we derive (3.18). □

Remark 1. *Assume that* $A,B\in\mathcal{B}_p(H)$, $p\geq1$ *and* $A\geq m_1>0$, $B\geq m_2>0$. *Consider the kernel* $w_{\ell^{r-1}}(\lambda):=\frac{r\sin(r\pi)}{\pi}\lambda^{r-1}$, $r\in(0,1]$. *Then we have*

$$\mathcal{D}\mathcal{L}og(w_{\ell^{r-1}})(t)=t^r,\ t\geq0$$

and by (3.11),

$$\|B^r-A^r\|_p\leq\|B-A\|_p\times\begin{cases}\frac{m_2^r-m_1^r}{m_2-m_1} & \text{if } m_1\neq m_2,\\[2ex] rm^r & \text{if } m_1=m_2=m.\end{cases} \tag{3.20}$$

For the kernel $w_{(\ell+1)^{-1}}(\lambda):=\frac{1}{\lambda+1}$ *we have*

$$\mathcal{D}\mathcal{L}og\left(w_{(\ell+1)^{-1}}\right)(t)=\frac{1}{6}\pi^2-\operatorname{dilog}(t),\ t\geq0$$

and by (3.11),

$$\left\|\operatorname{dilog}(B)-\operatorname{dilog}(A)\right\|_p \le \|B-A\|_p \times \begin{cases} \frac{\operatorname{dilog}(m_1)-\operatorname{dilog}(m_2)}{m_2-m_1} & \text{if } m_1 \ne m_2, \\ \\ u(m) & \text{if } m_1 = m_2 = m, \end{cases} \tag{3.21}$$

where

$$u(t) := \begin{cases} \frac{\ln t}{t-1}, \ t \ne 1, \ t > 0, \\ \\ 1, \ t = 1. \end{cases}$$

If we take $f(t) = \ln(t+a)$, *for* $a,\ t > 0$, *then we have*

$$F_{\ln(t+a)}(t) := \int_0^t \frac{\ln(s+a)-\ln(a)}{s} ds = \int_0^t \frac{1}{s} \ln\left(\frac{s}{a}+1\right) ds.$$

If we change the variable $u = \frac{s}{a}$, *then we get*

$$\int_0^t \frac{1}{s} \ln\left(\frac{s}{a}+1\right) ds = \int_0^{t/a} \frac{1}{ua} \ln(u+1)\, a du = \int_0^{t/a} \frac{1}{u} \ln(u+1)\, du$$
$$= -\operatorname{dilog}\left(\frac{t}{a}+1\right),$$

which gives

$$F_{\ln(t+a)}(t) = -\operatorname{dilog}\left(\frac{t}{a}+1\right), \ t > 0.$$

By (3.18) we then get

$$\begin{aligned} &\left\|\operatorname{dilog}\left(\frac{1}{a}B+1\right)-\operatorname{dilog}\left(\frac{1}{a}A+1\right)\right\|_p \\ &\le \|B-A\|_p \times \begin{cases} \frac{\operatorname{dilog}\left(\frac{m_1}{a}+1\right)-\operatorname{dilog}\left(\frac{m_2}{a}+1\right)}{m_2-m_1} & \text{if } m_1 \ne m_2, \\ \\ \frac{\ln(m+a)-\ln a}{m} & \text{if } m_1 = m_2 = m, \end{cases} \end{aligned} \tag{3.22}$$

for $A,\ B \in \mathcal{B}_p(H)$, $p \ge 1$ *and* $A \ge m_1 > 0,\ B \ge m_2 > 0$.

4. Applications for Hermite-Hadamard Type Inequalities

Let $(X; \|\cdot\|_X)$ and $(Y; \|\cdot\|_Y)$ be two Banach spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. For any mapping $F : C \subset X \to Y$ we can consider the associated function $\Phi_{F,x,y,\lambda} : [0,1] \to Y$, where $x,\ y \in C$, $\lambda \in [0,1]$, defined by [19]

$$\begin{aligned} \Phi_{F,x,y,\lambda}(t) &:= (1-\lambda) F[(1-t)((1-\lambda)x+\lambda y)+ty] \\ &\quad + \lambda F[(1-t)x + t((1-\lambda)x+\lambda y)]. \end{aligned} \tag{4.1}$$

We observe that for $\lambda = 0$ and $\lambda = 1$ we have

$$\Phi_{F,x,y,0}(t) = \Phi_{F,x,y,1}(t) = F[(1-t)x+ty]$$

and

$$\Phi_{F,x,y,\frac{1}{2}}(t) = \frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2}+ty\right]+F\left[(1-t)x+t\frac{x+y}{2}\right]\right)$$

where $x,\ y \in B$.

The following result holds [19]:

Lemma 4. *Let $F : C \subset X \to Y$ be a Lipschitzian mapping with the constant $L > 0$ on the convex subset C of X. If $x,\ y \in C$, then we have*

$$\left\| \Phi_{F,x,y,\lambda}(t) - \int_0^1 F[sy + (1-s)x]\,ds \right\|_Y \le 2L\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|x - y\|_X \tag{4.2}$$

for any $t \in [0,1]$ and $\lambda \in [0,1]$.

We define, for the positive operators $A,\ B > 0$ and the transform $\mathcal{L}og(w,\mu)$,

$$\begin{aligned} &\Phi_{\mathcal{L}og(w,\mu),A,B,\lambda}(t) \\ &:= (1-\lambda)\mathcal{DL}og(w,\mu)[(1-t)((1-\lambda)A + \lambda B) + tB] \\ &+ \lambda \mathcal{DL}og(w,\mu)[(1-t)A + t((1-\lambda)A + \lambda B)], \end{aligned} \tag{4.3}$$

where $t \in [0,1]$ and $\lambda \in [0,1]$.

By utilising Lemma 4 and Theorem 3 we can state:

Proposition 1. *Assume that $A,\ B \in \mathcal{B}_p(H)$, $p \ge 1$ and $A,\ B \ge m > 0$, then*

$$\begin{aligned} &\left\| \Phi_{\mathcal{DL}og(w,\mu),A,B,\lambda}(t) - \int_0^1 \mathcal{DL}og(w,\mu)[sB + (1-s)A]\,ds \right\|_p \\ &\le 2\mathcal{D}(w,\mu)(m)\|A - B\|_p \\ &\times \left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right], \end{aligned} \tag{4.4}$$

for any $t \in [0,1]$ and $\lambda \in [0,1]$.

In particular, we have

$$\begin{aligned} &\left\| \frac{1}{2}\left[\mathcal{DL}og(w,\mu)\left(\frac{3A+B}{4}\right) + \mathcal{DL}og(w,\mu)\left(\frac{A+3B}{4}\right)\right] \right. \\ &\left. - \int_0^1 \mathcal{DL}og(w,\mu)[sB + (1-s)A]\,ds \right\|_p \\ &\le \frac{1}{8}\mathcal{D}(w,\mu)(m)\|A - B\|_p, \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\left\| \mathcal{DL}og(w,\mu)[(1-t)A + tB] - \int_0^1 \mathcal{DL}og(w,\mu)[sB + (1-s)A]\,ds \right\|_p \\ &\le \mathcal{D}(w,\mu)(m)\left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\|A - B\|_p, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\left\| \mathcal{DL}og(w,\mu)\left(\frac{A+B}{2}\right) - \int_0^1 \mathcal{DL}og(w,\mu)[sB + (1-s)A]\,ds \right\|_p \\ &\le \frac{1}{4}\mathcal{D}(w,\mu)(m)\|A - B\|_p \end{aligned} \tag{4.7}$$

and

$$\left\| \frac{1}{2}\left[\mathcal{DL}og\left(w,\mu\right)\left[\left(1-t\right)\frac{A+B}{2}+tB\right] \right.\right. \\ \left.+\mathcal{DL}og\left(w,\mu\right)\left[\left(1-t\right)A+t\frac{A+B}{2}\right]\right] \\ \left.-\int_0^1 \mathcal{DL}og\left(w,\mu\right)\left[sB+\left(1-s\right)A\right]ds\right\|_p \\ \leq \frac{1}{2}\mathcal{D}\left(w,\mu\right)\left(m\right)\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\left\|A-B\right\|_p \tag{4.8}$$

for any $t\in[0,1]$.

We can also consider another associated function $\Psi_{F,x,y,\lambda}:[0,1]\to Y$, where x, $y\in C$, $\lambda\in[0,1]$, defined by [19]

$$\Psi_{F,x,y,\lambda}\left(t\right):=\left(1-\lambda\right)F\left[\left(1-t\right)\left(\left(1-\lambda\right)x+\lambda y\right)+ty\right] \\ +\lambda F\left[tx+\left(1-t\right)\left(\left(1-\lambda\right)x+\lambda y\right)\right]. \tag{4.9}$$

We observe that for $\lambda=0$ and $\lambda=1$ we have

$$\Psi_{F,x,y,0}\left(t\right)=F\left[\left(1-t\right)x+ty\right],\ \Psi_{F,x,y,1}\left(t\right)=F\left[tx+\left(1-t\right)y\right]$$

and

$$\Psi_{F,x,y,\frac{1}{2}}\left(t\right)=\frac{1}{2}\left(F\left[\left(1-t\right)\frac{x+y}{2}+ty\right]+F\left[tx+\left(1-t\right)\frac{x+y}{2}\right]\right),$$

where x, $y\in B$.

In [19] we also obtained the following result:

Lemma 5. *Let* $F:C\subset X\to Y$ *be a Lipschitzian mapping with the constant* $L>0$ *on the convex subset* C *of* X. *If* x, $y\in C$, *then we have*

$$\left\|\Psi_{F,x,y,\lambda}\left(t\right)-\int_0^1 F\left[sy+\left(1-s\right)x\right]ds\right\|_Y \\ \leq 2L\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\left\|x-y\right\|_X \tag{4.10}$$

for any $t\in[0,1]$ *and* $\lambda\in[0,1]$.

We define, for the positive operators A, $B>0$ and the transform $\mathcal{DL}og\left(w,\mu\right)$,

$$\Psi_{\mathcal{DL}og(w,\mu),A,B,\lambda}\left(t\right) \\ :=\left(1-\lambda\right)\mathcal{DL}og\left(w,\mu\right)\left[\left(1-t\right)\left(\left(1-\lambda\right)A+\lambda B\right)+tB\right] \\ +\lambda\mathcal{DL}og\left(w,\mu\right)\left[tA+\left(1-t\right)\left(\left(1-\lambda\right)A+\lambda B\right)\right]. \tag{4.11}$$

By utilising Lemma 5 and Theorem 3 we can also state:

Proposition 2. *Assume that* $A, B \in \mathcal{B}_p(H)$, $p \geq 1$ *and* $A, B \geq m > 0$, *then*

$$\left\| \Psi_{\mathcal{DL}og(w,\mu),A,B,\lambda}(t) - \int_0^1 \mathcal{DL}og(w,\mu)[sB + (1-s)A]\,ds \right\|_p \leq 2\mathcal{D}(w,\mu)(m)\|A-B\|_p \times \left[\frac{1}{4} + \left(t - \frac{1}{2}\right)^2\right]\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \tag{4.12}$$

for any $t \in [0,1]$ *and* $\lambda \in [0,1]$.

In particular,

$$\left\| \frac{1}{2}[\mathcal{DL}og(w,\mu)(A) + \mathcal{DL}og(w,\mu)(B)] - \int_0^1 \mathcal{DL}og(w,\mu)[sB + (1-s)A]\,ds \right\|_p \leq \frac{1}{4}\mathcal{D}(w,\mu)(m)\|A-B\|_p. \tag{4.13}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.