p-SCHATTEN NORM QUADRATIC LIPSCHITZIAN INEQUALITIES FOR THE $\mathcal{AT}$-INTEGRAL TRANSFORM OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. For a continuous and positive function $p(\lambda)$, $\lambda \in (0,1)$ and μ a positive measure on $(0,1)$ define the $\mathcal{AT}$*-operator transform* by

$$\mathcal{AT}(p,\mu)(T) := \int_0^1 \frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2 T^2+1\right)^{-1}\right] d\mu(\lambda),$$

where the integral is assumed to exist for T a selfadjoint operator on a complex Hilbert space H. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := \left[\operatorname{tr}\left(|A|^p\right)\right]^{1/p} < \infty.$$

For A, B selfadjoint operators with $A^2 \ge m_1^2$, $B^2 \ge m_2^2$ where $m_1, m_2 > 0$ and $A^2, B^2 \in \mathcal{B}_p(H)$, we have the p-Schatten norm inequality

$$\|\mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A)\|_p$$
$$\le \left\|B^2 - A^2\right\|_p \times \begin{cases} \frac{\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1)}{m_2^2-m_1^2} & \text{if } m_2 \ne m_1, \\ \\ \frac{1}{2m}\mathcal{AT}'(p,\mu)(m) & \text{if } m_2 = m_1 = m, \end{cases}$$

where $\mathcal{AT}'(p,\mu)(t)$ is the derivative of $\mathcal{AT}(p,\mu)$ as a function of t.

Applications for Hermite-Hadamard type inequalities and some particular examples of interest are also given.

1. Introduction

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [6] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \le L\|A - B\|$$

for any $A, B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [31], [32] and Kato in [39], the following inequality holds

$$\||A| - |B|\| \le \frac{2}{\pi}\|A - B\|\left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.1}$$

for any $A, B \in \mathcal{B}(H)$ with $A \ne B$.

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If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\||A| - |B|\|_{HS} \le \sqrt{2}\,\|A - B\|_{HS}$$

for any A, $B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [6] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \le a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right)$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2 .$$

Consider a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. An operator T is said to be positive (denoted by $T \ge 0$) if $\langle Tx, x\rangle \ge 0$ for all $x \in H$ and also an operator T is said to be *strictly positive* (denoted by $T > 0$) if T is positive and invertible.

In [5] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \le f'(a)\|A - B\|$$

where f is an *operator monotone function* on $(0, \infty)$ and A, $B \ge a > 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [7], [33] and the references therein.

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [8, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \int_0^\infty \frac{\lambda^{r-1}}{\lambda + t} d\lambda. \tag{1.2}$$

Observe that for $t > 0$, $t \ne 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.3}$$

for all $t > 0$.

Motivated by these representations, we introduced in [29], for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w, \mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\mu(\lambda), \ t > 0, \tag{1.4}$$

where μ is a positive measure on $(0, \infty)$ and the integral (1.4) exists for all $t > 0$.

For μ the Lebesgue measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda,\ t > 0. \tag{1.5}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0,1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t),\ t > 0. \tag{1.6}$$

For the same measure, if we take the kernel $w_{\ln}(\lambda) = (\lambda + 1)^{-1}$, $t > 0$, then we have the representation

$$\ln t = (t-1)\mathcal{D}(w_{\ln})(t),\ t > 0. \tag{1.7}$$

Assume that $T > 0$, then by the continuous functional calculus for selfadjoint operators, we can define the positive operator [29]

$$\mathcal{D}(w,\mu)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\mu(\lambda), \tag{1.8}$$

where w and μ are as above. Also, when μ is the usual Lebesgue measure, then

$$\mathcal{D}(w)(T) := \int_0^\infty w(\lambda)(\lambda + T)^{-1} d\lambda, \tag{1.9}$$

for $T > 0$.

We also defined the *logarithmic transform* for a continuous and positive function $w(\lambda)$, $\lambda > 0$ by [30]

$$\mathcal{L}og(w,\mu)(t) := \int_0^\infty w(\lambda)\ln(\lambda + t)\, d\mu(\lambda), \tag{1.10}$$

where μ is a positive measure on $(0,\infty)$ and the integral (1.4) exists for all $t > 0$.Also, when μ is the usual Lebesgue measure, then

$$\mathcal{L}og(w)(t) := \int_0^\infty w(\lambda)\ln(\lambda + t)\, d\lambda. \tag{1.11}$$

Now, for a continuous and positive function $p(\lambda)$, $\lambda \in [0,1]$ and a positive measure μ on $[0,1]$, we define the integral transform

$$\mathcal{AT}(p,\mu)(t) := \begin{cases} \int_0^1 \frac{p(\lambda)}{\lambda^2 + t^{-2}} d\mu(\lambda),\ t \neq 0, \\ 0,\ t = 0. \end{cases} \tag{1.12}$$

If μ is the usual Lebesgue measure, then we put

$$\mathcal{AT}(p)(t) := \begin{cases} \int_0^1 \frac{p(\lambda)}{\lambda^2 + t^{-2}} d\lambda,\ t \neq 0, \\ 0,\ t = 0. \end{cases} \tag{1.13}$$

For $p(\lambda) = 1$, $\lambda \in [0,1]$ and μ is the usual Lebesgue measure, we have

$$\mathcal{AT}(t) := \begin{cases} \int_0^1 \frac{1}{\lambda^2 + t^{-2}} d\lambda,\ t \neq 0, \\ 0,\ t = 0. \end{cases} = t\arctan t,\ t \in \mathbb{R}. \tag{1.14}$$

In the case when $p(\lambda)=\ell^2(\lambda)=\lambda^2$, we derive

$$\mathcal{AT}\left(\ell^2\right)(t)=\begin{cases}\int_0^1\frac{\lambda^2}{\lambda^2+t^{-2}}d\lambda,\ t\neq 0,\\ \\ 0,\ t=0.\end{cases}=\begin{cases}1-\frac{\arctan t}{t},\ t\neq 0,\\ \\ 0,\ t=0.\end{cases}$$

We observe that (1.12) can be written in equivalent form as

$$\begin{aligned}\mathcal{AT}(p,\mu)(t)&:=\int_0^1\frac{t^2p(\lambda)}{t^2\lambda^2+1}d\mu(\lambda)\\&=\int_0^1\left(1-\frac{1}{t^2\lambda^2+1}\right)\frac{p(\lambda)}{\lambda^2}d\mu(\lambda),\ t\in\mathbb{R}.\end{aligned}\tag{1.15}$$

For a selfadjoint operator T we have that λ^2T^2+1 is invertible for all $\lambda\in[0,1]$ and by the continuous functional calculus for selfadjoint operators we can define the $\mathcal{AT}$-*operator transform* by

$$\mathcal{AT}(p,\mu)(T):=\int_0^1\frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2T^2+1\right)^{-1}\right]d\mu(\lambda),\tag{1.16}$$

and, in particular, for the Lebesgue measure

$$\mathcal{AT}(p)(T):=\int_0^1\frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2T^2+1\right)^{-1}\right]d\lambda.\tag{1.17}$$

We observe that

$$\mathcal{AT}(T)=\int_0^1\frac{1}{\lambda^2}\left[1-\left(\lambda^2T^2+1\right)^{-1}\right]d\lambda=T\arctan T,$$

for any selfadjoint operator T.

If T is invertible, then

$$\mathcal{AT}\left(\ell^2\right)(T)=\int_0^1\left[1-\left(\lambda^2T^2+1\right)^{-1}\right]d\lambda=1-T^{-1}\arctan T.$$

2. Some Preliminary on Trace

Let $(H;\langle.,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle|A|e_i,e_i\rangle<\infty.\tag{2.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle,\tag{2.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{2.3}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{2.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [49, p. 60-64]

$$\|A\|_p:=[\operatorname{tr}(|A|^p)]^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{2.5}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{2.6}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [49, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{2.7}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{2.8}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{2.9}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{2.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\,\|AB\|_1\le\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{2.11}$$

For the theory of trace functionals and their applications the reader is referred to [47] and [49].

For some classical trace inequalities see [9], [11], and [44], which are continuations of the work of Bellman [3]. For related works the reader can refer to [1], [4], [9], [35], [40], [41], [42], [46], [48] and the recent papers [12]-[28].

3. Main Results

We have the following identity of interest for the difference $\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)$.

Lemma 1. *Let A, B be selfadjoint operators, then*

$$\begin{aligned}\mathcal{AT}(p,\mu)(B)&-\mathcal{AT}(p,\mu)(A)\\&=\int_0^1 p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\\&\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda).\end{aligned}\tag{3.1}$$

Proof. Let A, B selfadjoint operators. By employing (1.16) we have

$$\begin{aligned}\mathcal{AT}(p,\mu)(B)&-\mathcal{AT}(p,\mu)(A)\\&=\int_0^1\frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda)\\&-\int_0^1\frac{p(\lambda)}{\lambda^2}\left[1-\left(\lambda^2A^2+1\right)^{-1}\right]d\mu(\lambda)\\&=\int_0^1\frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda).\end{aligned}\tag{3.2}$$

The function $g(t)=-t^{-1}$ is operator monotone on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla g_T(S):=\lim_{t\to 0}\left[\frac{g(T+tS)-g(T)}{t}\right]=T^{-1}ST^{-1}\tag{3.3}$$

for T, $S>0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for C, D selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$g_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the Bochner integral, that

$$g(D)-g(C)=\int_0^1\frac{d}{dt}\left(g_{C,D}(t)\right)dt=\int_0^1\nabla g_{(1-t)C+tD}(D-C)\,dt.\tag{3.4}$$

If we write this equality for the function $g(t)=-t^{-1}$ and C, $D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt.\tag{3.5}$$

If in (3.5) we take $C=\lambda^2A^2+1$ and $D=\lambda^2B^2+1$, then we get

$$
\begin{aligned}
&\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1} \\
&=\int_0^1\left((1-t)\left(\lambda^2A^2+1\right)+t\left(\lambda^2B^2+1\right)\right)^{-1}\left(\lambda^2B^2-\lambda^2A^2\right) \\
&\times\left((1-t)\left(\lambda^2A^2+1\right)+t\left(\lambda^2B^2+1\right)\right)^{-1}dt \\
&=\lambda^2\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right) \\
&\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt
\end{aligned}
\tag{3.6}
$$

for all $t,\ \lambda\in[0,1]$.

Therefore

$$
\begin{aligned}
&\int_0^1\frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda) \\
&=\int_0^1\frac{p(\lambda)}{\lambda^2}\left[\left(\lambda^2A^2+1\right)^{-1}-\left(\lambda^2B^2+1\right)^{-1}\right]d\mu(\lambda) \\
&=\int_0^1p(\lambda)\left[\int_0^1\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right. \\
&\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}dt\right]d\mu(\lambda)
\end{aligned}
$$

and by (3.2) we get the representation (3.1). □

We have the following main result:

Theorem 2. *For A, B selfadjoint operators with $A^2\geq m_1^2$, $B^2\geq m_2^2$ where m_1, $m_2>0$ and A^2, $B^2\in\mathcal{B}_p(H)$, we have the p-Schatten norm inequality*

$$
\begin{aligned}
&\left\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\right\|_p \\
&\leq\left\|B^2-A^2\right\|_p\times\begin{cases}\frac{\mathcal{AT}(p,\mu)(m_2)-\mathcal{AT}(p,\mu)(m_1)}{m_2^2-m_1^2} & \text{if } m_2\neq m_1,\\ \\ \frac{1}{2m}\mathcal{AT}'(p,\mu)(m) & \text{if } m_2=m_1=m,\end{cases}
\end{aligned}
\tag{3.7}
$$

where $\mathcal{AT}'(p,\mu)(t)$ is the derivative of $\mathcal{AT}(p,\mu)$ as a function of t.

Proof. From the identity (3.1), we get, by taking the norm, that

$$
\begin{aligned}
&\left\|\mathcal{AT}(p,\mu)(B)-\mathcal{AT}(p,\mu)(A)\right\|_p \\
&\leq\int_0^1p(\lambda)\left[\int_0^1\left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\left(B^2-A^2\right)\right.\right. \\
&\left.\left.\times\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\|_pdt\right]d\mu(\lambda) \\
&\leq\left\|B^2-A^2\right\|_p \\
&\times\int_0^1p(\lambda)\left(\int_0^1\left\|\left(1+\lambda^2\left[(1-t)A^2+tB^2\right]\right)^{-1}\right\|^2dt\right)d\mu(\lambda)
\end{aligned}
\tag{3.8}
$$

for all A, B with A^2, $B^2\in\mathcal{B}_p(H)$, where for the last inequality we used the property (2.10).

Assume that $m_1^2 < m_2^2$, then

$$(1-t)\, m_1^2 + t m_2^2 \le (1-t)\, A^2 + tB^2$$

for $t \in [0,1]$, which implies that

$$1 + \lambda^2 \left[(1-t)\, m_1^2 + t m_2^2 \right] \le 1 + \lambda^2 \left[(1-t)\, A^2 + tB^2 \right],$$

namely

$$\left(1 + \lambda^2 \left[(1-t)\, A^2 + tB^2 \right]\right)^{-1} \le \left(1 + \lambda^2 \left[(1-t)\, m_1^2 + t m_2^2 \right]\right)^{-1}$$

for $t,\ \lambda \in [0,1]$.

This implies that

$$\left\| \left(1 + \lambda^2 \left[(1-t)\, A^2 + tB^2 \right]\right)^{-1} \right\| \le \left(1 + \lambda^2 \left[(1-t)\, m_1^2 + t m_2^2 \right]\right)^{-1},$$

which implies that

$$\left\| \left(1 + \lambda^2 \left[(1-t)\, A^2 + tB^2 \right]\right)^{-1} \right\|^2 \le \left(1 + \lambda^2 \left[(1-t)\, m_1^2 + t m_2^2 \right]\right)^{-2}, \tag{3.9}$$

for $t,\ \lambda \in [0,1]$.

If we integrate (3.9) over $t \in [0,1]$, then multiply by $p(\lambda) \ge 0$ and integrate over the positive measure $d\mu(\lambda)$ on $[0,1]$, we have by (3.1) that,

$$\begin{aligned} &\int_0^1 p(\lambda) \left(\int_0^1 \left\| \left(1 + \lambda^2 \left[(1-t)\, A^2 + tB^2 \right]\right)^{-1} \right\|^2 dt \right) d\mu(\lambda) \\ &\le \int_0^1 p(\lambda) \left(\int_0^1 \left(1 + \lambda^2 \left[(1-t)\, m_1^2 + t m_2^2 \right]\right)^{-2} dt \right) d\mu(\lambda). \end{aligned} \tag{3.10}$$

If we use the identity (3.1) for $A = m_1$ and $B = m_2$, then we get

$$\begin{aligned} &\mathcal{AT}(p,\mu)(m_2) - \mathcal{AT}(p,\mu)(m_1) \\ &= \int_0^1 p(\lambda) \left[\int_0^1 \left(1 + \lambda^2 \left[(1-t)\, m_1^2 + t m_2^2 \right]\right)^{-1} \left(m_2^2 - m_1^2\right) \right. \\ &\left. \times \left(1 + \lambda^2 \left[(1-t)\, m_1^2 + t m_2^2 \right]\right)^{-1} dt \right] d\mu(\lambda) \\ &= \left(m_2^2 - m_1^2\right) \int_0^1 p(\lambda) \left(\int_0^1 \left(1 + \lambda^2 \left[(1-t)\, m_1^2 + t m_2^2 \right]\right)^{-2} dt \right) d\mu(\lambda), \end{aligned}$$

which gives that

$$\begin{aligned} &\int_0^1 p(\lambda) \left(\int_0^1 \left(1 + \lambda^2 \left[(1-t)\, m_1^2 + t m_2^2 \right]\right)^{-2} dt \right) d\mu(\lambda) \\ &= \frac{\mathcal{AT}(p,\mu)(m_2) - \mathcal{AT}(p,\mu)(m_1)}{m_2^2 - m_1^2} \end{aligned}$$

and by (3.8) and (3.10), we get the first inequality in (3.7).

If $m_1^2 > m_2^2$, then we can prove (3.7) in a similar way.

Assume that $m_1^2 = m_2^2 = m^2$ with $m > 0$. Let $\varepsilon > 0$ such that $m^2 - \varepsilon > 0$. Then $B^2 \ge m^2$ and $A^2 \ge m^2 > m^2 - \varepsilon > 0$. If we take $m_2^2 := m^2$, $m_1^2 = m^2 - \varepsilon > 0$, then $m_2^2 > m_1^2$ and by the first inequality in (3.7,)

$$\begin{aligned} &\left\| \mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A) \right\|_p \\ &\le \left\| B^2 - A^2 \right\|_p \frac{\mathcal{AT}(p,\mu)(m) - \mathcal{AT}(p,\mu)\left(\sqrt{m^2 - \varepsilon}\right)}{\varepsilon} \end{aligned}$$

and by taking the limit over $\varepsilon \to 0+$, then we get

$$\begin{aligned}
(3.11)\qquad & \left\| \mathcal{AT}(p,\mu)(B) - \mathcal{AT}(p,\mu)(A) \right\|_p \\
& \leq \left\| B^2 - A^2 \right\|_p \lim_{\varepsilon \to 0+} \frac{\mathcal{AT}(p,\mu)(m) - \mathcal{AT}(p,\mu)\left(\sqrt{m^2-\varepsilon}\right)}{\varepsilon} \\
& = \left\| B^2 - A^2 \right\|_p \lim_{\varepsilon \to 0+} \frac{\mathcal{AT}(p,\mu)\,\mathcal{AT}(p,\mu)(m) - \mathcal{AT}(p,\mu)\left(\sqrt{m^2-\varepsilon}\right)}{m - \sqrt{m^2-\varepsilon}} \\
& \times \lim_{\varepsilon \to 0+} \frac{m - \sqrt{m^2-\varepsilon}}{\varepsilon}.
\end{aligned}$$

Since

$$\begin{aligned}
& \lim_{\varepsilon \to 0+} \frac{\mathcal{AT}(p,\mu)(m) - \mathcal{AT}(p,\mu)\left(\sqrt{m^2-\varepsilon} - m + m\right)}{m - \sqrt{m^2-\varepsilon}} \\
& = \lim_{h \to 0+} \frac{\mathcal{AT}(p,\mu)(m) - \mathcal{AT}(p,\mu)(m-h)}{h} = \mathcal{AT}'(p,\mu)(m)
\end{aligned}$$

and

$$\lim_{\varepsilon \to 0+} \frac{m - \sqrt{m^2-\varepsilon}}{\varepsilon} = \lim_{\varepsilon \to 0+} \frac{1}{m + \sqrt{m^2-\varepsilon}} = \frac{1}{2m},$$

hence by (3.11) we obtain the second part of (3.7). □

Remark 1. *For A, B selfadjoint operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ where m_1, $m_2 > 0$ and A^2, $B^2 \in \mathcal{B}_p(H)$, we have the p-Schatten norm inequality*

$$\begin{aligned}
(3.12)\qquad & \left\| B \arctan(B) - A \arctan(A) \right\|_p \\
& \leq \left\| B^2 - A^2 \right\|_p \times \begin{cases} \frac{m_2 \arctan(m_2) - m_1 \arctan(m_1)}{m_2^2 - m_1^2} & \text{if } m_2 \neq m_1, \\[2ex] \frac{1}{2m}\left(\arctan m + \frac{m}{m^2+1}\right) & \text{if } m_2 = m_1 = m. \end{cases}
\end{aligned}$$

For any A, B selfadjoint and invertible operators with $A^2 \geq m_1^2$, $B^2 \geq m_2^2$ where m_1, $m_2 > 0$ and A^2, $B^2 \in \mathcal{B}_p(H)$,

$$\begin{aligned}
(3.13)\qquad & \left\| A^{-1} \arctan A - B^{-1} \arctan B \right\|_p \\
& \leq \left\| B^2 - A^2 \right\|_p \times \begin{cases} \frac{m_1^{-1} \arctan(m_1) - m_2^{-1} \arctan(m_2)}{m_2^2 - m_1^2} & \text{if } m_2 \neq m_1, \\[2ex] \frac{1}{2m^3}\left(\arctan m - \frac{m}{m^2+1}\right) & \text{if } m_2 = m_1 = m. \end{cases}
\end{aligned}$$

4. A Hermite-Hadamard Type Inequality

We have:

Proposition 1. *For A, B selfadjoint operators with A, $B \in \mathcal{B}_p(H)$, assume that $((1-s)A+sB)^2 \geq m^2$ for $s \in [0,1]$ with $m>0$, then*

$$\left\| \int_0^1 \mathcal{AT}(p,\mu)((1-s)A+sB)\,dt - \mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\|_p \leq \frac{1}{2m}\mathcal{AT}'(p,\mu)(m)\int_0^1 \left\|((1-s)A+sB)^2 - \left(\frac{A+B}{2}\right)^2\right\|_p ds \leq \frac{1}{8m}\mathcal{AT}'(p,\mu)(m)\left[\|B-A\|_p\left(\|A\|_p+\|B\|_p\right)+\|BA-AB\|_p\right]. \tag{4.1}$$

Proof. Since $\left(\frac{A+B}{2}\right)^2 \geq m^2$, then by (3.7) we have

$$\left\|\mathcal{AT}(p,\mu)((1-s)A+sB) - \mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\|_p \leq \frac{1}{2m}\left\|((1-s)A+sB)^2 - \left(\frac{A+B}{2}\right)^2\right\|_p \mathcal{AT}'(p,\mu)(m),$$

for $s \in [0,1]$.

If we take the integral and use the norm properties, then we get

$$\left\| \int_0^1 \mathcal{AT}(p,\mu)((1-s)A+sB)\,ds - \mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\|_p \leq \int_0^1 \left\|\mathcal{AT}(p,\mu)((1-s)A+sB) - \mathcal{AT}(p,\mu)\left(\frac{A+B}{2}\right)\right\|_p ds \leq \frac{1}{2m}\mathcal{AT}'(p,\mu)(m)\int_0^1 \left\|((1-s)A+sB)^2 - \left(\frac{A+B}{2}\right)^2\right\|_p ds. \tag{4.2}$$

Now, observe that

$$X^2 - Y^2 = (X-Y)(X+Y) + YX - XY,$$

which gives that

$$\left\|X^2 - Y^2\right\|_p = \|(X-Y)(X+Y)+YX-XY\|_p \leq \|X-Y\|_p\|X+Y\|_p + \|YX-XY\|_p$$

if X, $Y \in \mathcal{B}_p(H)$.

Then

$$
\begin{aligned}
&\left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\|_p \\
&\le \left\|(1-s)A+sB-\frac{A+B}{2}\right\|_p\left\|(1-s)A+sB+\frac{A+B}{2}\right\|_p \\
&+\left\|((1-s)A+sB)\frac{A+B}{2}-\frac{A+B}{2}((1-s)A+sB)\right\|_p \\
&=\left|s-\frac{1}{2}\right|\|B-A\|_p\left\|\left(\frac{3}{2}-s\right)A+\left(s+\frac{1}{2}\right)B\right\|_p \\
&+\frac{1}{2}\|((1-s)A+sB)(A+B)-(A+B)((1-s)A+sB)\|_p \\
&=\left|s-\frac{1}{2}\right|\|B-A\|_p\left\|\left(\frac{3}{2}-s\right)A+\left(s+\frac{1}{2}\right)B\right\|_p \\
&+\left|s-\frac{1}{2}\right|\|BA-AB\|_p \\
&\le\left(\left|s-\frac{1}{2}\right|\left(\frac{3}{2}-s\right)\|A\|_p+\left|s-\frac{1}{2}\right|\left(s+\frac{1}{2}\right)\|B\|_p\right)\|B-A\|_p \\
&+\left|s-\frac{1}{2}\right|\|BA-AB\|_p
\end{aligned}
$$

Since

$$
\int_0^1\left|s-\frac{1}{2}\right|\left(\frac{3}{2}-s\right)ds=\frac{1}{4},\quad \int_0^1\left|s-\frac{1}{2}\right|\left(s+\frac{1}{2}\right)ds=\frac{1}{4},\quad \int_0^1\left|s-\frac{1}{2}\right|ds=\frac{1}{4},
$$

hence

$$
\begin{aligned}
&\int_0^1\left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\|_p \\
&\le\frac{1}{4}\left[\|B-A\|_p\left(\|A\|_p+\|B\|_p\right)+\|BA-AB\|_p\right],
\end{aligned}
$$

which proves the last part of (4.1). □

Remark 2. *We observe that, if* $A,\ B\ge m>0$ *with* $A,\ B\in\mathcal{B}_p(H)$, *then* $(1-s)A+sB\ge m>0$, *which implies that* $((1-s)A+sB)^2\ge m^2$ *for* $s\in[0,1]$.

With the assumptions in Proposition 1, we then obtain

$$
\begin{aligned}
&\left\|\int_0^1((1-s)A+sB)\arctan((1-s)A+sB)\,ds\right. \\
&\left.-\left(\frac{A+B}{2}\right)\arctan\left(\frac{A+B}{2}\right)\right\|_p \\
&\le\frac{1}{2m}\left(\arctan m+\frac{m}{m^2+1}\right)\int_0^1\left\|((1-s)A+sB)^2-\left(\frac{A+B}{2}\right)^2\right\|_p ds \\
&\le\frac{1}{8m}\left(\arctan m+\frac{m}{m^2+1}\right)\left[\|B-A\|_p\left(\|A\|_p+\|B\|_p\right)+\|BA-AB\|_p\right].
\end{aligned}\tag{4.3}
$$

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