SOME LIPSCHITZ TYPE p-SCHATTEN NORM INEQUALITIES FOR THE LERCH TRANSFORM OF SELFADJOINT OPERATORS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For a selfadjoint operator T on a complex Hilbert space H with the spectrum $\operatorname{Sp}(T) \subset (-1,1)$, we define the *Lerch transform*

$$\Phi(T,s,v) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left(e^t - T\right)^{-1} dt$$

for $\operatorname{Re} v > 0$ and $\operatorname{Re} s > 0$. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := \left[\operatorname{tr}\left(|A|^p\right)\right]^{1/p} < \infty.$$

Assume that A, $B \in \mathcal{B}_p(H)$ with $-1 < A \le m_1 < 1$ and $-1 < B \le m_2 < 1$, then for $v > 0$ and $s > 0$,

$$\begin{aligned} &\|\Phi_\mu(B,s,v) - \Phi_\mu(A,s,v)\|_p \\ &\le \|B-A\|_p \times \begin{cases} \frac{\Phi_\mu(m_2,s,v)-\Phi_\mu(m_1,s,v)}{m_2-m_1}, & \text{if } m_1 \ne m_2 \\ \\ \frac{\partial \Phi_\mu(m,s,v)}{\partial z}, & \text{if } m_1 = m_2 = m, \end{cases} \end{aligned}$$

where $\frac{\partial \Phi_\mu}{\partial z}$ is the partial derivative of Φ_μ in the first variable.

Applications for Jensen's gap inequalities and some particular examples of interest are also given.

1. Introduction

The *Lerch transcendent* function is given by the series

$$\Phi(z,s,\alpha) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s}, \ \ |z| < 1, \ \alpha \ne 0, -1, -2, ... \tag{1.1}$$

see for instance [31, Section 1.11, p. 27] or [2, Section 25.14]. This function, defined by Mathias Lerch in 1887 in his paper [42], includes as special cases of the parameters; the Hurwitz, Riemann zeta functions and the polylogarithms, among others. Therefore the transcendent has applications ranging from number theory to physics.

The *Hurwitz zeta* function, formally defined for complex arguments s with $\operatorname{Re}(s) > 1$ and α with $\operatorname{Re}(\alpha) > 0$ by

$$\zeta(s,\alpha) := \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}$$

1991 *Mathematics Subject Classification.* 47A63, 47A60.

Key words and phrases. Operator inequalities, Lipschitz type inequalities, Jensen gap inequalities, p-Schatten norm. Lerch function.

RGMIA Res. Rep. Coll. 24 (2021), Art. 107, 12 pp. Received 24/08/21

is a special case, given by

$$\zeta(s,\alpha)=\Phi(1,s,\alpha).$$

For $\alpha=1$ we have the *Riemann zeta* function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

The *polylogarithm* function $\operatorname{Li}(s,z)$ is defined by a power series in z, which is also a *Dirichlet series* in s:

$$\operatorname{Li}(z,s):=\sum_{n=1}^{\infty}\frac{z^n}{n^s}=z\Phi(z,s,1). \tag{1.2}$$

This definition is valid for arbitrary complex order s and for all complex arguments z with $|z|<1$; it can be extended to $|z|\geq 1$ by the process of analytic continuation. The special case $s=1$ involves the ordinary natural logarithm, $\operatorname{Li}(z,1)=-\ln(1-z)$, while the special cases $s=2$ and $s=3$ are called the *dilogarithm* (also referred to as *Spence's function*) and *trilogarithm* respectively.

The *Legendre chi* function is a special case, given by

$$\chi_s(z)=2^{-s}z\Phi\left(z^2,s,1/2\right). \tag{1.3}$$

The *Legendre chi* function is a special function whose Taylor series is also a Dirichlet series, given by

$$\chi_s(z):=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)^s}. \tag{1.4}$$

The following integral representations are valid [31, p. 27]

$$\Phi(z,s,v)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-vt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^t-z}dt \tag{1.5}$$

for $\operatorname{Re}v>0$ and either $|z|\leq 1$, $z\neq 1$, $\operatorname{Re}s>0$ or $z=1$, $\operatorname{Re}s>1$. Here $\Gamma(\cdot)$ is Euler's Gama function

$$\Gamma(s):=\int_0^{\infty}e^{-t}t^{z-1}dt,\ \ \operatorname{Re}s>0.$$

For a selfadjoint operator T with $\operatorname{Sp}(T)\subset(-1,1)$ and by utilising the continuous functional calculus for selfadjoint operators, we define the transform

$$\Phi(T,s,v):=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-(v-1)t}\left(e^t-T\right)^{-1}dt \tag{1.6}$$

for $\operatorname{Re}v>0$ and $\operatorname{Re}s>0$.

We can also define the related transforms

$$\operatorname{Li}(T,s):=T\Phi(T,s,1)\ \text{ and }\ \chi_s(T):=2^{-s}T\Phi\left(T^2,s,1/2\right) \tag{1.7}$$

for a selfadjoint operator T with $\operatorname{Sp}(T)\subset(-1,1)$ and $\operatorname{Re}s>0$.

Now, for a positive measure μ on $[0,\infty)$, we define the *Lerch operator transform* of the selfadjoint operator T with $\operatorname{Sp}(T)\subset(-1,1)$, by

$$\Phi_{\mu}(T,s,v):=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}e^{-(v-1)t}\left(e^t-T\right)^{-1}d\mu(t), \tag{1.8}$$

for $\operatorname{Re}v>0$ and $\operatorname{Re}s>0$, provided that the integral exists. When μ is Lebesgue measure, then we recapture (1.5).

Let $\mathcal{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

It is known that [7] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathcal{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A| - |B|\| \leq L \|A - B\|$$

for any $A,\ B \in \mathcal{B}(H)$.

However, as shown by Farforovskaya in [32], [33] and Kato in [40], the following inequality holds

$$\||A| - |B|\| \leq \frac{2}{\pi} \|A - B\| \left(2 + \log\left(\frac{\|A\| + \|B\|}{\|A - B\|}\right)\right) \tag{1.9}$$

for any $A,\ B \in \mathcal{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^*C)^{1/2}$ of an operator C, then the following inequality is true [3]

$$\||A| - |B|\|_{HS} \leq \sqrt{2} \|A - B\|_{HS} \tag{1.10}$$

for any $A, B \in \mathcal{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [7] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A| - |B|\| \leq a_1 \|A - B\| + a_2 \|A - B\|^2 + O\left(\|A - B\|^3\right), \tag{1.11}$$

where

$$a_1 = \left\|A^{-1}\right\| \|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3 \|A\|^2.$$

In [6] the author also obtained the following *Lipschitz type inequality*

$$\|f(A) - f(B)\| \leq f'(a) \|A - B\| \tag{1.12}$$

where f is an *operator monotone function* on $[0, \infty)$ and $A,\ B \geq aI_H > 0$. Recall that $A \geq B$ means that $\langle (A - B)x, x\rangle \geq 0$ for all $x \in H$ and f is an *operator monotone function* on $[0, \infty)$ if $f(A) \geq f(B)$ for $A \geq B \geq 0$.

One of the problems in perturbation theory is to find bounds for $\|f(A) - f(B)\|$ in terms of $\|A - B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [8], [34] and the references therein.

In order to obtain similar results for the *p-Schatten norm* and *Lerch operator transform* we need the following preparations.

2. Some Preliminary Facts on p-Schatten Norm

Let $(H; \langle .,. \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{2.1}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{2.2}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{2.3}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{2.4}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [51, p. 60-64]

$$\|A\|_p:=\left[\operatorname{tr}(|A|^p)\right]^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{2.5}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{2.6}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [51, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{2.7}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{2.8}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{2.9}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{2.10}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\,\|AB\|_1\le\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{2.11}$$

For the theory of trace functionals and their applications the reader is referred to [49] and [51].

For some classical trace inequalities see [10], [12], and [46], which are continuations of the work of Bellman [4]. For related works the reader can refer to [1], [5], [10], [36], [43], [44], [45], [48], [50] and the recent papers [13]-[29].

3. Main Results

We have the following representation result:

Lemma 1. *For* A, B *with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$, $\operatorname{Re} v > 0$ *and* $\operatorname{Re} s > 0$ *we have*

$$\begin{aligned} &\Phi_\mu(B,s,v) - \Phi_\mu(A,s,v) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left[\int_0^1 \left(e^t - [(1-u)B + uA]\right)^{-1} (B-A) \right. \\ &\quad \left. \times \left(e^t - [(1-u)B + uA]\right)^{-1} du \right] d\mu(t). \end{aligned} \tag{3.1}$$

Proof. For A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1,1)$, we have by (1.6) that

$$\begin{aligned} &\Phi_\mu(B,s,v) - \Phi_\mu(A,s,v) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(v-1)t} \left[\left(e^t - B\right)^{-1} - \left(e^t - A\right)^{-1} \right] d\mu(t) \end{aligned} \tag{3.2}$$

for $\operatorname{Re} v > 0$ and $\operatorname{Re} s > 0$.

The function $g(u) = -u^{-1}$ is operator monotonic on $(0,\infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla g_T(S) := \lim_{u \to 0} \left[\frac{g(T+uS) - g(T)}{u} \right] = T^{-1} S T^{-1} \tag{3.3}$$

for T, $S > 0$.

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0,1]$ by

$$g_{C,D}(u) = g((1-u)C + uD), \ u \in [0,1].$$

If $g_{C,D}$ is Gâteaux differentiable on the segment $[C,D] := \{(1-u)C + uD, \ u \in [0,1]\}$, then we have, by the properties of the Bochner integral, that

$$g(D) - g(C) = \int_0^1 \frac{d}{du}(g_{C,D}(u))\, du = \int_0^1 \nabla g_{(1-u)C+uD}(D-C)\, du. \tag{3.4}$$

If we write this equality for the function $g(u) = -u^{-1}$ and C, $D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 ((1-u)C + uD)^{-1} (D-C) ((1-u)C + uD)^{-1}\, du. \tag{3.5}$$

By (3.5) for $C = e^t - B$, $D = e^t - A$, $t \in [0, \infty)$ we have

$$\begin{aligned}(3.6)\quad & \left(e^t - B\right)^{-1} - \left(e^t - A\right)^{-1} \\ &= \int_0^1 \left((1-u)\left(e^t - B\right) + u\left(e^t - A\right)\right)^{-1}\left(e^t - A - e^t + B\right) \\ &\times \left((1-u)\left(e^t - B\right) + u\left(e^t - A\right)\right)^{-1} du \\ &= \int_0^1 \left(e^t - [(1-u)B + uA]\right)^{-1}(B - A)\left(e^t - [(1-u)B + uA]\right)^{-1} du.\end{aligned}$$

If we multiply this equality by $t^{s-1}e^{-(v-1)t}$ and integrate, then we obtain (3.1). □

Observe that

$$\begin{aligned}(3.7)\quad \operatorname{Li}(T, s) = T\Phi(T, s, 1) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} T\left(e^t - T\right)^{-1} dt \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(T - e^t + e^t\right)\left(e^t - T\right)^{-1} dt \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left[e^t\left(e^t - T\right)^{-1} - 1\right] dt\end{aligned}$$

for $\operatorname{Re} s > 0$ and T with $\operatorname{Sp}(T) \subset (-1, 1)$.

Corollary 1. *For A, B with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset (-1, 1)$ and $\operatorname{Re} s > 0$, we have*

$$\begin{aligned}(3.8)\quad & \operatorname{Li}(B, s) - \operatorname{Li}(A, s) \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^t \\ &\times \left[\int_0^1 \left(e^t - [(1-u)B + uA]\right)^{-1}(B - A)\left(e^t - [(1-u)B + uA]\right)^{-1} du\right] dt.\end{aligned}$$

Proof. By (3.7) we have

$$\operatorname{Li}(B, s) - \operatorname{Li}(A, s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^t\left[\left(e^t - B\right)^{-1} - \left(e^t - A\right)^{-1}\right] dt$$

for $\operatorname{Re} s > 0$ and by employing the identity (3.6) we derive (3.8). □

We have the following Lipschitz type inequalities for the p-Schatten norm:

Theorem 2. *Assume that $A, B \in \mathcal{B}_p(H)$ with $-1 < A \leq m_1 < 1$ and $-1 < B \leq m_2 < 1$, then for $v > 0$ and $s > 0$,*

$$\begin{aligned}(3.9)\quad & \left\|\Phi_\mu(B, s, v) - \Phi_\mu(A, s, v)\right\|_p \\ &\leq \|B - A\|_p \times \begin{cases} \frac{\Phi_\mu(m_2, s, v) - \Phi_\mu(m_1, s, v)}{m_2 - m_1}, & \text{if } m_1 \neq m_2 \\ \\ \frac{\partial \Phi_\mu(m, s, v)}{\partial z}, & \text{if } m_1 = m_2 = m,\end{cases}\end{aligned}$$

where $\frac{\partial \Phi_\mu}{\partial z}$ is the partial derivative of Φ_μ in the first variable.

Proof. By taking the p-Schatten norm in (3.1), using the properties of the integral and the property (2.10)

$$\begin{aligned}
&\left\|\Phi_{\mu}(B,s,v)-\Phi_{\mu}(A,s,v)\right\|_{p} \\
&\leq \|B-A\|_{p} \\
&\times \frac{1}{\Gamma(s)}\int_{0}^{\infty} t^{s-1}e^{-(v-1)t}\left(\int_{0}^{1}\left\|\left(e^{t}-[(1-u)B+uA]\right)^{-1}\right\|^{2}du\right)d\mu(t) \\
&= \|B-A\|_{p} \\
&\times \frac{1}{\Gamma(s)}\int_{0}^{\infty} t^{s-1}e^{-(v-1)t}\left(\int_{0}^{1}\left\|\left(e^{t}-[uB+(1-u)A]\right)^{-1}\right\|^{2}du\right)d\mu(t).
\end{aligned} \tag{3.10}$$

Since $-1<A\leq m_1<1$ and $-1<B\leq m_2<1$, hence

$$(1-u)m_1+um_2\geq uB+(1-u)A$$

for $u\in[0,1]$, namely

$$0<e^{t}-[(1-u)m_1+um_2]\leq e^{t}-[uB+(1-u)A]$$

for $u\in[0,1]$, $t\geq 0$.

This implies that

$$\left(e^{t}-[uB+(1-u)A]\right)^{-1}\leq\left(e^{t}-[(1-u)m_1+um_2]\right)^{-1}$$

for $u\in[0,1]$, $t\geq 0$.

By taking the norm, we get

$$\left\|\left(e^{t}-[uB+(1-u)A]\right)^{-1}\right\|\leq\left(e^{t}-[(1-u)m_1+um_2]\right)^{-1},$$

namely

$$\left\|\left(e^{t}-[uB+(1-u)A]\right)^{-1}\right\|^{2}\leq\left(e^{t}-[(1-u)m_1+um_2]\right)^{-2},$$

for $u\in[0,1]$, $t\geq 0$.

By taking the integral over u in $[0,1]$, multiplying by $t^{s-1}e^{-(v-1)t}$ and taking the integral on $[0,\infty)$ over the positive measure $d\mu(t)$, we get

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_{0}^{\infty} t^{s-1}e^{-(v-1)t}\left(\int_{0}^{1}\left\|\left(e^{t}-[uB+(1-u)A]\right)^{-1}\right\|^{2}du\right)d\mu(t) \\
&\leq \frac{1}{\Gamma(s)}\int_{0}^{\infty} t^{s-1}e^{-(v-1)t}\left(\int_{0}^{1}\left(e^{t}-[(1-u)m_1+um_2]\right)^{-2}du\right)d\mu(t).
\end{aligned} \tag{3.11}$$

Assume that $m_1<m_2$. If we use the identity (3.1), change the variable, by replacing u with $1-u$ and choose $A=m_1$, $B=m_2$, then we get

$$\begin{aligned}
&\Phi_{\mu}(m_2,s,v)-\Phi_{\mu}(m_1,s,v) \\
&= \frac{m_2-m_1}{\Gamma(s)} \\
&\times \int_{0}^{\infty} t^{s-1}e^{-(v-1)t}\left[\int_{0}^{1}\left(e^{t}-[um_2+(1-u)m_1]\right)^{-2}du\right]d\mu(t),
\end{aligned}$$

which gives that

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(v-1)t}\left(\int_0^1\left(e^t-[(1-u)m_1+um_2]\right)^{-2}du\right)d\mu(t) = \frac{\Phi_\mu(m_2,s,v)-\Phi_\mu(m_1,s,v)}{m_2-m_1}. \tag{3.12}$$

By making use of (3.10), (3.11) and (3.12) we get the first inequality in (3.9).

The case $m_1 > m_2$ goes in a similar way and we omit the details.

Let $m_1 = m_2 = m$ and $\varepsilon > 0$. Then $B \leq m+\varepsilon = m_2$ and $A \leq m = m_1$. By using the first inequality in (3.9) for $m_2 > m_1$ we get

$$\|\Phi_\mu(B,s,v)-\Phi_\mu(A,s,v)\|_p \leq \|B-A\|_p \frac{\Phi_\mu(m+\varepsilon,s,v)-\Phi_\mu(m,s,v)}{\varepsilon}. \tag{3.13}$$

By taking the limit over $\varepsilon \to 0+$ in (3.13), we derive the second branch of the inequality (3.9). □

Corollary 2. *Assume that* A, $B \in \mathcal{B}_p(H)$ *with* $-1 < A \leq m_1 < 1$, $-1 < B \leq m_2 < 1$ *and* $\operatorname{Re} s > 0$, *we have*

$$\begin{aligned} &\|\operatorname{Li}(B,s)-\operatorname{Li}(A,s)\|_p \\ &\leq \|B-A\|_p \times \begin{cases} \frac{\operatorname{Li}(m_2,s)-\operatorname{Li}(m_1,s)}{m_2-m_1}, & \text{if } m_1 \neq m_2 \\ \\ \frac{\partial \operatorname{Li}(m,s)}{\partial z}, & \text{if } m_1 = m_2 = m. \end{cases}\end{aligned} \tag{3.14}$$

4. Some Discrete Inequalities

Assume A, $B \in \mathcal{B}_p(H)$ with $-1 < A_k \leq m < 1$ and $w_k \geq 0$, for $k \in \{1,...,n\}$ with $\sum_{k=1}^n w_k = 1$. Then $-1 < \sum_{j=1}^n w_j A_j \leq m < 1$ and by the second inequality in (3.9), we get

$$\left\|\Phi_\mu(A_k,s,v)-\Phi_\mu\left(\sum_{j=1}^n w_jA_j,s,v\right)\right\|_p \leq \frac{\partial\Phi_\mu(m,s,v)}{\partial z}\left\|A_k-\sum_{j=1}^n w_jA_j\right\|_p \tag{4.1}$$

for $k \in \{1,...,n\}$.

If we multiply (4.1) by $w_k \geq 0$ and sum, then we get

$$\begin{aligned} &\sum_{k=1}^n w_k\left\|\Phi_\mu(A_k,s,v)-\Phi_\mu\left(\sum_{j=1}^n w_jA_j,s,v\right)\right\|_p \\ &\leq \frac{\partial\Phi_\mu(m,s,v)}{\partial z}\sum_{k=1}^n w_k\left\|A_k-\sum_{j=1}^n w_jA_j\right\|_p. \end{aligned} \tag{4.2}$$

By using the triangle inequality, we have

$$\begin{aligned} &\left\|\sum_{k=1}^n w_k\Phi_\mu(A_k,s,v)-\Phi_\mu\left(\sum_{j=1}^n w_jA_j,s,v\right)\right\|_p \\ &\leq \sum_{k=1}^n w_k\left\|\Phi_\mu(A_k,s,v)-\Phi_\mu\left(\sum_{j=1}^n w_jA_j,s,v\right)\right\|_p. \end{aligned} \tag{4.3}$$

By making use of (4.2) and (4.3), we then derive the following upper bound for the *Jensen's gap*

$$\left\| \sum_{k=1}^{n} w_k \Phi_\mu \left(A_k, s, v\right) - \Phi_\mu \left(\sum_{j=1}^{n} w_j A_j, s, v \right) \right\|_p \leq \frac{\partial \Phi_\mu \left(m, s, v\right)}{\partial z} \sum_{k=1}^{n} w_k \left\| A_k - \sum_{j=1}^{n} w_j A_j \right\|_p . \tag{4.4}$$

Denote

$$K(p, A) := \sum_{k=1}^{n} w_k \left\| A_k - \sum_{j=1}^{n} w_j A_j \right\|_p .$$

By making use of Hölder's inequality, we have the bounds

$$K(w, A) \leq \begin{cases} \max_{k \in \{1,\ldots,n\}} \left\| A_k - \sum_{j=1}^{n} w_j A_j \right\|_p \\ \\ \left(\sum_{k=1}^{n} w_k \left\| A_k - \sum_{j=1}^{n} w_j A_j \right\|_p^r \right)^{1/r}, \ r > 1 \\ \\ \max_{k \in \{1,\ldots,n\}} \{w_k\} \sum_{k=1}^{n} \left\| A_k - \sum_{j=1}^{n} w_j A_j \right\|_p . \end{cases} \tag{4.5}$$

By the triangle inequality we also have

$$\begin{aligned} \sum_{k=1}^{n} w_k \left\| A_k - \sum_{j=1}^{n} w_j A_j \right\|_p &= \sum_{k=1}^{n} w_k \left\| \sum_{j=1}^{n} w_j \left(A_k - A_j\right) \right\|_p \\ &\leq \sum_{k=1}^{n} w_k \sum_{j=1}^{n} w_j \left\| A_k - A_j \right\|_p \\ &= 2 \sum_{1 \leq j < k \leq n}^{n} w_k w_j \left\| A_k - A_j \right\|_p =: 2G\left(w, A\right). \end{aligned}$$

Observe that

$$\begin{aligned} G\left(w, A\right) &\leq \max_{1 \leq j < k \leq n} \left\| A_k - A_j \right\|_p \sum_{1 \leq j < k \leq n}^{n} w_k w_j \\ &= \max_{1 \leq j < k \leq n} \left\| A_k - A_j \right\|_p \frac{1}{2} \left(\sum_{k,j=1}^{n} w_k w_j - \sum_{k=1}^{n} w_k^2 \right) \\ &= \max_{1 \leq j < k \leq n} \left\| A_k - A_j \right\|_p \frac{1}{2} \left(1 - \sum_{k=1}^{n} w_k^2 \right), \end{aligned}$$

which implies that

$$K(w, A) \leq \max_{1 \leq j < k \leq n} \left\| A_k - A_j \right\|_p \left(1 - \sum_{k=1}^{n} w_k^2 \right). \tag{4.6}$$

Utilising Hölder's inequality, we also have for r, $q>1$ with $\frac{1}{r}+\frac{1}{q}=1$, that

$$\begin{aligned} G(w,A) &\le \left(\sum_{1\le j<k\le n}^{n}(w_k w_j)^q\right)^{1/q}\left(\sum_{1\le j<k\le n}^{n}\|A_k-A_j\|_p^r\right)^{1/r} \\ &= \frac{1}{2^{1/q}}\left(\sum_{k,j=1}^{n}w_k^q w_j^q-\sum_{k=1}^{n}w_k^{2q}\right)^{1/q}\left(\sum_{1\le j<k\le n}^{n}\|A_k-A_j\|_p^r\right)^{1/r} \\ &= \frac{1}{2^{1/q}}\left(\left(\sum_{k=1}^{n}w_k^q\right)^2-\sum_{k=1}^{n}w_k^{2q}\right)^{1/q}\left(\sum_{1\le j<k\le n}^{n}\|A_k-A_j\|_p^r\right)^{1/r}, \end{aligned}$$

which implies that

$$K(w,A)\le 2^{1/r}\left[\left(\sum_{k=1}^{n}w_k^q\right)^2-\sum_{k=1}^{n}w_k^{2q}\right]^{1/q}\left(\sum_{1\le j<k\le n}^{n}\|A_k-A_j\|_p^r\right)^{1/r}. \tag{4.7}$$

References

[1] T. Ando, Matrix Young inequalities, *Oper. Theory Adv. Appl.* **75** (1995), 33–38.

[2] T. M. Apostol, Zeta and related functions, in: F. W. F. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (Eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, National Institute of Standards and Technology, Cambridge, Washington, DC, 2010, pp. 601–616. Available online in http://dlmf.nist.gov/25.

[3] H. Araki and S. Yamagami, An inequality for Hilbert-Schmidt norm, *Commun. Math. Phys.* **81** (1981), 89-96.N: 0-387-94846-5.

[4] R. Bellman, Some inequalities for positive definite matrices, in: E.F. Beckenbach (Ed.), *General Inequalities 2*, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[5] E. V. Belmega, M. Jungers and S. Lasaulce, A generalization of a trace inequality for positive definite matrices. *Aust. J. Math. Anal. Appl.* **7** (2010), no. 2, Art. 26, 5 pp.

[6] R. Bhatia, First and second order perturbation bounds for the operator absolute value, *Linear Algebra Appl.* **208/209** (1994), 367-376.

[7] R. Bhatia, Perturbation bounds for the operator absolute value. *Linear Algebra Appl.* **226/228** (1995), 639–645.

[8] R. Bhatia, D. Singh and K. B. Sinha, Differentiation of operator functions and perturbation bounds. *Comm. Math. Phys.* **191** (1998), no. 3, 603–611.

[9] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISB

[10] D. Chang, A matrix trace inequality for products of Hermitian matrices, *J. Math. Anal. Appl.* **237** (1999) 721–725.

[11] L. Chen and C. Wong, Inequalities for singular values and traces, *Linear Algebra Appl.* **171** (1992), 109–120.

[12] I. D. Coop, On matrix trace inequalities and related topics for products of Hermitian matrix, *J. Math. Anal. Appl.* **188** (1994) 999–1001.

[13] S. S. Dragomir, Some trace inequalities of Čebyšev type for functions of operators in Hilbert spaces. *Linear Multilinear Algebra* **64** (2016), no. 9, 1800–1813.

[14] S. S. Dragomir, Some trace inequalities for convex functions of selfadjoint operators in Hilbert spaces. *Korean J. Math.* **24** (2016), no. 2, 273–296.

[15] S. S. Dragomir, Some Grüss' type inequalities for trace of operators in Hilbert spaces. *Oper. Matrices* **10** (2016), no. 4, 923–943.

[16] S. S. Dragomir, Jensen's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces. *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 5, 981–998.

[17] S. S. Dragomir, Additive reverses of Schwarz and Grüss type trace inequalities for operators in Hilbert spaces. *J. Math. Tokushima Univ.* **50** (2016), 15–42.
[18] S. S. Dragomir, Some Slater's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces. *Toyama Math. J.* **38** (2016), 75–99
[19] S. S. Dragomir, Reverse Jensen's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces. *Ann. Math. Sil.* **30** (2016), no. 1, 39–62.
[20] S. S. Dragomir, Integral inequalities for Lipschitzian mappings between two Banach spaces and applications. *Kodai Math. J.* **39** (2016), no. 1, 227–251.
[21] S. S. Dragomir, Sever Recent developments of Schwarz's type trace inequalities for operators in Hilbert spaces. *Adv. Oper. Theory* **1** (2016), no. 1, 15–91.
[22] S. S. Dragomir, Trace inequalities of Shisha-Mond type for operators in Hilbert spaces. *Concr. Oper.* **4** (2017), no. 1, 32–47.
[23] S. S. Dragomir, Some trace inequalities for operators in Hilbert spaces. *Kragujevac J. Math.* **41** (2017), no. 1, 33–55.
[24] S. S. Dragomir,. Trace inequalities of Cassels and Grüss type for operators in Hilbert spaces. *Acta Univ. Sapientiae Math.* 9 (2017), no. 1, 74–93.
[25] S. S. Dragomir, Some inequalities for trace class operators via a Kato's result. *Asian-Eur. J. Math.* **11** (2018), no. 1, 1850004, 24 pp.
[26] S. S. Dragomir, Trace inequalities of Lipschitz type for power series of operators on Hilbert spaces. *Extracta Math.* **32** (2017), no. 1, 25–54.
[27] S. S. Dragomir, Trace inequalities for positive operators via recent refinements and reverses of Young's inequality. *Spec. Matrices* **6** (2018), 180–192.
[28] S. S. Dragomir, Power and Hölder type trace inequalities for positive operators in Hilbert spaces. *Studia Sci. Math. Hungar.* **55** (2018), no. 3, 383–406.
[29] S. S. Dragomir, On some Hölder type trace inequalities for operator weighted geometric mean. *Acta Comment. Univ. Tartu. Math.* **24** (2020), no. 2, 271–280
[30] S. S. Dragomir, Operator monotonicity of an integral transform of positive operators in Hilbert spaces with applications, Preprint *RGMIA Res. Rep. Coll.* **23** (2020), Art. 65, 15 pp. [Online https://rgmia.org/papers/v23/v23a65.pdf].
[31] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.
[32] Yu. B. Farforovskaya, Estimates of the closeness of spectral decompositions of self-adjoint operators in the Kantorovich-Rubinshtein metric (in Russian), *Vesln. Leningrad. Gos. Univ. Ser. Mat. Mekh. Astronom.* **4** (1967), 155-156.
[33] Yu. B. Farforovskaya, An estimate of the norm $\|f(B) - f(A)\|$ for self-adjoint operators A and B (in Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst.* **56** (1976), 143-162 .
[34] Yu. B. Farforovskaya and L. Nikolskaya, Modulus of continuity of operator functions. *Algebra i Analiz* **20** (2008), no. 3, 224–242; translation in St. Petersburg Math. J. **20** (2009), no. 3, 493–506.
[35] J. I. Fujii, Y. Seo, On parametrized operator means dominated by power ones, *Sci. Math.* **1** (1998) 301–306.
[36] S. Furuichi and M. Lin, Refinements of the trace inequality of Belmega, Lasaulce and Debbah. *Aust. J. Math. Anal. Appl.* **7** (2010), no. 2, Art. 23, 4 pp.
[37] T. Furuta, Concrete examples of operator monotone functions obtained by an elementary method without appealing to Löwner integral representation, *Linear Algebra Appl.* **429** (2008) 972–980.
[38] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.
[39] E. Heinz, Beiträge zur Störungsteorie der Spektralzerlegung, *Math. Ann.* **123** (1951) 415–438.
[40] T. Kato, Continuity of the map $S \to |S|$ for linear operators, *Proc. Japan Acad.* **49** (1973), 143-162.
[41] A. Laforgia and P. Natalini, Turán-type inequalities for some special functions, *J. Inequal. Pure & Appl. Math.*, **7** (2006),Issue 1, Article 22.
[42] M. Lerch, Note sur la fonction $\mathfrak{k}(w, x, s) = \sum_{k=0}^{\infty} e^{2\pi ikx} (w + k)^{-s}$, *Acta Math.* **11** (1887) 19–24.
[43] H. D. Lee, On some matrix inequalities, *Korean J. Math.* **16** (2008), No. 4, pp. 565-571.
[44] L. Liu, A trace class operator inequality, *J. Math. Anal. Appl.* 328 (2007) 1484–1486.
[45] S. Manjegani, Hölder and Young inequalities for the trace of operators, *Positivity* **11** (20

[46] H. Neudecker, A matrix trace inequality, *J. Math. Anal. Appl.* **166** (1992) 302–303.
[47] M. B. Ruskai, Inequalities for traces on von Neumann algebras, *Commun. Math. Phys.* **26**(1972), 280—289.
[48] K. Shebrawi and H. Albadawi, Operator norm inequalities of Minkowski type, *J. Inequal. Pure Appl. Math.* **9**(1) (2008), 1–10, article 26.
[49] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.
[50] Z. Ulukök and R. Türkmen, On some matrix trace inequalities. *J. Inequal. Appl.* **2010**, Art. ID 201486, 8 pp.
[51] V. A. Zagrebvov, *Gibbs Semigroups,* Operator Theory: Advances and Applications, Volume 273, Birkhäuser, 2019

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Johannesburg, South Africa.