p-SCHATTEN NORM INEQUALITIES OF OSTROWSKI'S TYPE

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ABSTRACT. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

Assume that $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_p(H)$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are continuous and B is strongly differentiable on (a,b), then

$$\left\| \int_a^b A(t) B(t) \, dt - \left(\int_a^b A(s) \, ds \right) B(u) \right\|_1 \le \sup_{t \in [a,b]} \|B'(t)\|_q \times \begin{cases} \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right] \int_a^b \|A(t)\|_p \, dt, \\ \\ \left[\frac{(u-a)^{\beta+1} + (b-u)^{\beta+1}}{\beta+1} \right]^{1/\beta} \left(\int_a^b \|A(t)\|_p^\alpha \, dt \right)^{1/\alpha}, \\ \text{for } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \left[\frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2} \right)^2 \right] \sup_{t \in [a,b]} \|A(t)\|_p, \end{cases}$$

for all $u \in [a,b]$.

Some examples of interest for the inverse and operator exponential are also given.

1. Introduction

In 1938, A. Ostrowski [7], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a} \int_a^b f(t) \, dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1 (Ostrowski, 1938 [7])**.** *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

Let X be a Banach space and $-\infty < a < b < \infty$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms of vectors or operators acting on X will be denoted by $\|\cdot\|$.

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A function $f:[a,b]\to X$ is called *measurable* if there exists a sequence of simple functions $f_n:[a,b]\to X$ which converges punctually almost everywhere on $[a,b]$ at f. We recall also that a measurable function $f:[a,b]\to X$ is *Bochner integrable* if and only if its norm function (i.e. the function $t\mapsto \|f(t)\|:[a,b]\to\mathbb{R}_+$) is Lebesgue integrable on $[a,b]$.

The following generalization of Ostrowski scalar inequality holds:

Theorem 2 ([2])**.** *Assume that $B:[a,b]\to\mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e.,*

$$\|B(t)-B(s)\|\le H|t-s|^{\alpha}\quad \text{for all}\quad t,\ s\in[a,b], \tag{1.2}$$

where $H>0$ and $\alpha\in(0,1]$.

If $f:[a,b]\to X$ is Bochner integrable on $[a,b]$, then we have the inequality:

$$\left\|\int_a^b B(s)f(s)\,ds - B(t)\int_a^b f(s)\,ds\right\| \le H\int_a^b |t-s|^{\alpha}\|f(s)\|\,ds \tag{1.3}$$

$$\le H\times\begin{cases} \dfrac{(b-t)^{\alpha+1}+(t-a)^{\alpha+1}}{\alpha+1}\operatorname{esssup}_{t\in[a,b]}\|f(t)\|, \\[2ex] \left[\dfrac{(b-t)^{q\alpha+1}+(t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}}\left(\int_a^b\|f(t)\|^p\,dt\right)^{1/p}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[2ex] \left[\dfrac{1}{2}(b-a)+\left|t-\dfrac{a+b}{2}\right|\right]^{\alpha}\int_a^b\|f(t)\|\,dt \end{cases}$$

for any $t\in[a,b]$, provided the integrals and $\operatorname{esssup}_{t\in[a,b]}$ *from the right hand side are finite.*

For a recent survey on Ostrowski type scalar inequalities, see [5].

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A|e_i,e_i\rangle<\infty. \tag{1.4}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.5}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.5) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.6}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \text{ and }\ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.7}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [9, p. 60-64]

$$\|A\|_p:=\left[\operatorname{tr}(|A|^p)\right]^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.8}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{1.9}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [9, p. 60-64], for $p\ge1$,

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{1.10}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{1.11}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{1.12}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{1.13}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\ \|AB\|_1\le\|A\|_p\|B\|_q,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}_q(H). \tag{1.14}$$

For the theory of trace functionals and their applications the reader is referred to [8] and [9].

For some classical trace inequalities see [3], [4], and [6], which are continuations of the work of Bellman [1].

2. 1-Schatten Norm Ostrowski Inequalities

We have the following weighted version of Ostrowski's inequality for two functions with values in Banach algebra $\mathcal{B}(H)$:

Theorem 4. *Assume that* $A:[a,b]\to\mathcal{B}_p(H)$, $B:[a,b]\to\mathcal{B}_p(H)$, p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *are continuous and* B *is strongly differentiable on* (a,b), *then for all* $u\in[a,b]$

$$\left\|\int_a^b A(t)\,B(t)\,dt-\left(\int_a^b A(s)\,ds\right)B(u)\right\|_1\le B_{p,q}(A,B,u)\,, \tag{2.1}$$

where

$$B_{p,q}(A,B,u):=\int_u^b\left(\int_t^b\|A(s)\|_p\,ds\right)\|B'(t)\|_q\,dt+\int_a^u\left(\int_a^t\|A(s)\|_p\,ds\right)\|B'(t)\|_q\,dt.$$

We also have the bounds

$$\begin{aligned}B_{p,q}(A,B,u)&\\ \le&\begin{cases}\int_u^b\|A(s)\|_p\,ds\int_u^b\|B'(t)\|_q\,dt,\\[2mm] \left[\int_u^b\left(\int_t^b\|A(s)\|_p\,ds\right)^\alpha dt\right]^{1/\alpha}\left(\int_u^b\|B'(t)\|_q^\beta\,dt\right)^{1/\beta},\\[2mm] \int_u^b\left(\int_t^b\|A(s)\|_p\,ds\right)dt\,\sup_{t\in[u,b]}\|B'(t)\|_q\,,\end{cases}\\ +&\begin{cases}\int_a^u\|A(s)\|_p\,ds\int_a^u\|B'(t)\|_q\,dt,\\[2mm] \left[\int_a^u\left(\int_a^t\|A(s)\|_p\,ds\right)^\alpha dt\right]^{1/\alpha}\left(\int_a^u\|B'(t)\|_q^\beta\,dt\right)^{1/\beta},\\[2mm] \int_a^u\left(\int_a^t\|A(s)\|_p\,ds\right)dt\,\sup_{t\in[a,u]}\|B'(t)\|_q\,,\end{cases}\end{aligned} \tag{2.2}$$

where α, $\beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Proof. Let $u\in[a,b]$. Using the integration by parts formula for Bochner integral, we have

$$\begin{aligned}&\int_u^b\left(\int_t^b A(s)\,ds\right)B'(t)\,dt\\ &=\left(\int_t^b A(s)\,ds\right)B(t)\Big]_u^b+\int_u^b A(t)\,B(t)\,dt\\ &=-\left(\int_u^b A(s)\,ds\right)B(u)+\int_u^b A(t)\,B(t)\,dt\end{aligned}$$

and

$$\begin{aligned}&\int_a^u\left(\int_a^t A(s)\,ds\right)B'(t)\,dt\\&=\left(\int_a^t A(s)\,ds\right)B(t)\Big]_a^u-\int_a^u A(t)\,B(t)\,dt\\&=\left(\int_a^u A(s)\,ds\right)B(u)-\int_a^u A(t)\,B(t)\,dt.\end{aligned}$$

By subtracting the second identity from the first, we get

$$\begin{aligned}&\int_u^b\left(\int_t^b A(s)\,ds\right)B'(t)\,dt-\int_a^u\left(\int_a^t A(s)\,ds\right)B'(t)\,dt\\&=\int_u^b A(t)\,B(t)\,dt+\int_a^u A(t)\,B(t)\,dt\\&-\left(\int_u^b A(s)\,ds\right)B(u)-\left(\int_a^u A(s)\,ds\right)B(u)\\&=\int_a^b A(t)\,B(t)\,dt-\left(\int_a^b A(s)\,ds\right)B(u).\end{aligned}$$

Therefore, we get the following identity of interest

$$\begin{aligned}&\int_a^b A(t)\,B(t)\,dt-\left(\int_a^b A(s)\,ds\right)B(u)\\&=\int_u^b\left(\int_t^b A(s)\,ds\right)B'(t)\,dt-\int_a^u\left(\int_a^t A(s)\,ds\right)B'(t)\,dt\end{aligned}\tag{2.3}$$

for all $u\in[a,b]$.

If we take the 1-Schatten norm in (2.3), then we get

$$\begin{aligned}&\left\|\int_a^b A(t)\,B(t)\,dt-\left(\int_a^b A(s)\,ds\right)B(u)\right\|_1\\&\le\left\|\int_u^b\left(\int_t^b A(s)\,ds\right)B'(t)\,dt\right\|_1+\left\|\int_a^u\left(\int_a^t A(s)\,ds\right)B'(t)\,dt\right\|_1\\&\le\int_u^b\left\|\left(\int_t^b A(s)\,ds\right)B'(t)\right\|_1 dt+\int_a^u\left\|\left(\int_a^t A(s)\,ds\right)B'(t)\right\|_1 dt\\&=:C(A,B,u).\end{aligned}\tag{2.4}$$

Using the Hölder's type inequality (1.14) for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we get

$$
\begin{aligned}
& C(A,B,u). \\
& \leq \int_u^b \left\Vert \int_t^b A(s)\,ds \right\Vert_p \left\Vert B'(t)\right\Vert_q dt + \int_a^u \left\Vert \int_a^t A(s)\,ds \right\Vert_p \left\Vert B'(t)\right\Vert_q dt \\
& \leq \int_u^b \left(\int_t^b \left\Vert A(s)\right\Vert_p ds \right) \left\Vert B'(t)\right\Vert_q dt + \int_a^u \left(\int_a^t \left\Vert A(s)\right\Vert_p ds \right) \left\Vert B'(t)\right\Vert_q dt \\
& = B_{p,q}(A,B,u),
\end{aligned}
$$

which proves (2.1).

Using Hölder's inequality, we get for α, $\beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, that

$$
\begin{aligned}
& \int_u^b \left(\int_t^b \left\Vert A(s)\right\Vert_p ds \right) \left\Vert B'(t)\right\Vert_q dt \\
& \leq \left\{ \begin{array}{l}
\sup_{t\in[u,b]} \left(\int_t^b \left\Vert A(s)\right\Vert_p ds \right) \int_u^b \left\Vert B'(t)\right\Vert_q dt, \\ \\
\left[\int_u^b \left(\int_t^b \left\Vert A(s)\right\Vert_p ds \right)^\alpha dt \right]^{1/\alpha} \left(\int_u^b \left\Vert B'(t)\right\Vert_q^\beta dt \right)^{1/\beta}, \\ \\
\int_u^b \left(\int_t^b \left\Vert A(s)\right\Vert_p ds \right) dt \sup_{t\in[u,b]} \left\Vert B'(t)\right\Vert_q,
\end{array} \right. \\
& = \left\{ \begin{array}{l}
\left(\int_u^b \left\Vert A(s)\right\Vert_p ds \right) \int_u^b \left\Vert B'(t)\right\Vert_q dt, \\ \\
\left[\int_u^b \left(\int_t^b \left\Vert A(s)\right\Vert_p ds \right)^\alpha dt \right]^{1/\alpha} \left(\int_u^b \left\Vert B'(t)\right\Vert_q^\beta dt \right)^{1/\beta}, \\ \\
\int_u^b \left(\int_t^b \left\Vert A(s)\right\Vert_p ds \right) dt \sup_{t\in[u,b]} \left\Vert B'(t)\right\Vert_q
\end{array} \right.
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_a^u \left(\int_a^t \left\Vert A(s)\right\Vert_p ds \right) \left\Vert B'(t)\right\Vert_q dt \\
& \leq \left\{ \begin{array}{l}
\sup_{t\in[a,u]} \left(\int_a^t \left\Vert A(s)\right\Vert_p ds \right) \int_a^u \left\Vert B'(t)\right\Vert_q dt, \\ \\
\left[\int_a^u \left(\int_a^t \left\Vert A(s)\right\Vert_p ds \right)^\alpha dt \right]^{1/\alpha} \left(\int_a^u \left\Vert B'(t)\right\Vert_q^\beta dt \right)^{1/\beta}, \\ \\
\int_a^u \left(\int_a^t \left\Vert A(s)\right\Vert_p ds \right) dt \sup_{t\in[a,u]} \left\Vert B'(t)\right\Vert_q,
\end{array} \right. \\
& = \left\{ \begin{array}{l}
\left(\int_a^u \left\Vert A(s)\right\Vert_p ds \right) \int_a^u \left\Vert B'(t)\right\Vert_q dt, \\ \\
\left[\int_a^u \left(\int_a^t \left\Vert A(s)\right\Vert_p ds \right)^\alpha dt \right]^{1/\alpha} \left(\int_a^u \left\Vert B'(t)\right\Vert_q^\beta dt \right)^{1/\beta}, \\ \\
\int_a^u \left(\int_a^t \left\Vert A(s)\right\Vert_p ds \right) dt \sup_{t\in[a,u]} \left\Vert B'(t)\right\Vert_q .
\end{array} \right.
\end{aligned}
$$

By making use of (2.4), we derive (2.2). □

Corollary 1. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}(2.5)\qquad & \left\| \int_a^b A(t)\,B(t)\,dt - \left(\int_a^b A(s)\,ds\right) B(u)\right\|_1 \\ & \leq \int_u^b \|A(s)\|_p\,ds \int_u^b \|B'(t)\|_q\,dt + \int_a^u \|A(s)\|_p\,ds \int_a^u \|B'(t)\|_q\,dt \\ & \leq \begin{cases} \max\left\{\int_u^b \|A(s)\|_p\,ds, \int_a^u \|A(s)\|_p\,ds\right\} \int_a^b \|B'(t)\|_q\,dt \\ \\ \int_a^b \|A(s)\|_p\,ds \max\left\{\int_u^b \|B'(t)\|_q\,dt, \int_a^u \|B'(t)\|_q\,dt\right\} \end{cases} \\ & \leq \int_a^b \|A(s)\|_p\,ds \int_a^b \|B'(t)\|_q\,dt,\end{aligned}$$

for all $u \in [a,b]$.

The proof follows by the first branches in the bounds (2.2).

Remark 1. *If* $m \in (a,b)$ *is such that*

$$(2.6)\qquad \int_a^u \|A(s)\|_p\,ds = \int_u^b \|A(s)\|_p\,ds = \frac{1}{2}\int_a^b \|A(s)\|_p\,ds,$$

then by (2.5) we get

$$\begin{aligned}(2.7)\qquad & \left\| \int_a^b A(t)\,B(t)\,dt - \left(\int_a^b A(s)\,ds\right) B(m)\right\|_1 \\ & \leq \frac{1}{2}\int_a^b \|A(s)\|_p\,ds \int_a^b \|B'(t)\|_q\,dt.\end{aligned}$$

Corollary 2. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}(2.8)\qquad & \left\| \int_a^b A(t)\,B(t)\,dt - \left(\int_a^b A(s)\,ds\right) B(u)\right\|_1 \\ & \leq \int_u^b \left(\int_t^b \|A(s)\|_p\,ds\right) dt \sup_{t\in[u,b]} \|B'(t)\|_q \\ & + \int_a^u \left(\int_a^t \|A(s)\|_p\,ds\right) dt \sup_{t\in[a,u]} \|B'(t)\|_q \\ & \leq \sup_{t\in[a,b]} \|B'(t)\|_q \int_a^b |t-u|\,\|A(t)\|_p\,dt,\end{aligned}$$

for all $u \in [a,b]$.

Proof. From the third branches in the bounds (2.2) we derive

$$\begin{aligned}
&\left\| \int_a^b A(t) B(t)\,dt - \left(\int_a^b A(s)\,ds\right) B(u) \right\|_1 \qquad (2.9)\\
&\le \int_u^b \left(\int_t^b \|A(s)\|_p\,ds\right) dt \sup_{t\in[u,b]} \|B'(t)\|_q \\
&+ \int_a^u \left(\int_a^t \|A(s)\|_p\,ds\right) dt \sup_{t\in[a,u]} \|B'(t)\|_q \\
&\le \sup_{t\in[a,b]} \|B'(t)\|_q \left[\int_u^b \left(\int_t^b \|A(s)\|_p\,ds\right) dt + \int_a^u \left(\int_a^t \|A(s)\|_p\,ds\right) dt\right].
\end{aligned}$$

Using integration by parts, we have for $u \in [a,b]$ that

$$\begin{aligned}
\int_u^b \left(\int_t^b \|A(s)\|_p\,ds\right) dt &= \left(\int_t^b \|A(s)\|_p\,ds\right) t\Big|_u^b + \int_u^b t\,\|A(t)\|_p\,dt \\
&= \int_u^b t\,\|A(t)\|_p\,dt - \left(\int_u^b \|A(s)\|_p\,ds\right) u \\
&= \int_u^b (t-u)\,\|A(t)\|_p\,dt = \int_u^b |t-u|\,\|A(t)\|_p\,dt
\end{aligned}$$

and

$$\begin{aligned}
\int_a^u \left(\int_a^t \|A(s)\|_p\,ds\right) dt &= \left(\int_a^t \|A(s)\|_p\,ds\right) t\Big|_a^u - \int_a^u t\,\|A(t)\|_p\,dt \\
&= \left(\int_a^u \|A(s)\|_p\,ds\right) u - \int_a^u t\,\|A(t)\|_p\,dt \\
&= \int_a^u (u-t)\,\|A(t)\|_p\,dt = \int_a^u |t-u|\,\|A(t)\|_p\,dt,
\end{aligned}$$

which gives that

$$\begin{aligned}
&\int_u^b \left(\int_t^b \|A(s)\|\,ds\right) dt + \int_a^u \left(\int_a^t \|A(s)\|\,ds\right) dt \\
&= \int_u^b |t-u|\,\|A(t)\|\,dt + \int_a^u |t-u|\,\|A(t)\|\,dt = \int_a^b |t-u|\,\|A(t)\|\,dt.
\end{aligned}$$

By making use of (2.9) we derive (2.8). □

Remark 2. *By making use of Hölder's integral inequality, we have for α, $\beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ that*

$$\int_a^b |t-u|\,\|A(t)\|_p\,dt \le \begin{cases} \sup_{t\in[a,b]} |t-u| \int_a^b \|A(t)\|_p\,dt, \\[2ex] \left(\int_a^b |t-u|^\beta\,dt\right)^{1/\beta} \left(\int_a^b \|A(t)\|_p^\alpha\,dt\right)^{1/\alpha}, \\[2ex] \int_a^b |t-u|\,dt \sup_{t\in[a,b]} \|A(t)\|_p. \end{cases}$$

Since

$$\sup_{t\in[a,b]}|t-u|=\max\{u-a,b-u\}=\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|,$$

$$\left(\int_a^b|t-u|^\beta dt\right)^{1/\beta}=\left[\frac{(u-a)^{\beta+1}+(b-u)^{\beta+1}}{\beta+1}\right]^{1/\beta}$$

and

$$\int_a^b|t-u|\,dt=\frac{(u-a)^2+(b-u)^2}{2}=\frac{1}{4}(b-a)^2+\left(u-\frac{a+b}{2}\right)^2.$$

Then by (2.8) we derive the non-commutative Ostrowski type inequalities

$$\left\|\int_a^b A(t)\,B(t)\,dt-\left(\int_a^b A(s)\,ds\right)B(u)\right\|_1 \tag{2.10}$$
$$\leq\sup_{t\in[a,b]}\|B'(t)\|_q\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_a^b\|A(t)\|_p\,dt,\\[2mm] \left[\frac{(u-a)^{\beta+1}+(b-u)^{\beta+1}}{\beta+1}\right]^{1/\beta}\left(\int_a^b\|A(t)\|_p^\alpha dt\right)^{1/\alpha},\\[2mm] \left[\frac{1}{4}(b-a)^2+\left(u-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\|A(t)\|_p,\end{cases}$$

for all $u\in[a,b]$.

We also have:

Corollary 3. *With the assumptions of Theorem 4, we have for all* $u\in[a,b]$,

$$\left\|\int_a^b A(t)\,B(t)\,dt-\left(\int_a^b A(s)\,ds\right)B(u)\right\|_1 \tag{2.11}$$
$$\leq\left[\left(\int_u^b\|A(s)\|_p\,ds\right)^\alpha(b-u)+\left(\int_a^u\|A(s)\|_p\,ds\right)^\alpha(u-a)\right]^{1/\alpha}$$
$$\times\left(\int_a^b\|B'(t)\|_q^\beta dt\right)^{1/\beta}$$
$$\leq(b-a)^{1/\alpha}\left[\left(\int_u^b\|A(s)\|_p\,ds\right)^\alpha+\left(\int_a^u\|A(s)\|_p\,ds\right)^\alpha\right]^{1/\alpha}$$
$$\times\left(\int_a^b\|B'(t)\|_q^\beta dt\right)^{1/\beta}$$

for $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Proof. Observe that, by the elementary inequality for $a,\ b,\ c,\ d\geq 0$ and $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$,

$$(ab+cd)\leq(a^\alpha+c^\alpha)^{1/\alpha}\left(b^\beta+d^\beta\right)^{1/\beta}$$

we have

$$\begin{aligned}
(2.12)\qquad & \left[\int_u^b \left(\int_t^b \|A(s)\|_p\,ds\right)^{\alpha} dt\right]^{1/\alpha} \left(\int_u^b \|B'(t)\|_q^{\beta}\,dt\right)^{1/\beta} \\
&+ \left[\int_a^u \left(\int_a^t \|A(s)\|_p\,ds\right)^{\alpha} dt\right]^{1/\alpha} \left(\int_a^u \|B'(t)\|_q^{\beta}\,dt\right)^{1/\beta} \\
&\le \left[\int_u^b \left(\int_t^b \|A(s)\|_p\,ds\right)^{\alpha} dt + \int_a^u \left(\int_a^t \|A(s)\|_p\,ds\right)^{\alpha} dt\right]^{1/\alpha} \\
&\times \left[\int_u^b \|B'(t)\|_q^{\beta}\,dt + \int_a^u \|B'(t)\|_q^{\beta}\,dt\right]^{1/\beta}
\end{aligned}$$

$$\begin{aligned}
&= \left[\int_u^b \left(\int_t^b \|A(s)\|_p\,ds\right)^{\alpha} dt + \int_a^u \left(\int_a^t \|A(s)\|_p\,ds\right)^{\alpha} dt\right]^{1/\alpha} \\
&\times \left(\int_a^b \|B'(t)\|_q^{\beta}\,dt\right)^{1/\beta} \\
&\le \left[\left(\int_u^b \|A(s)\|_p\,ds\right)^{\alpha} \int_u^b dt + \left(\int_a^u \|A(s)\|_p\,ds\right)^{\alpha} \int_a^u dt\right]^{1/\alpha} \\
&\times \left(\int_a^b \|B'(t)\|_q^{\beta}\,dt\right)^{1/\beta}
\end{aligned}$$

$$\begin{aligned}
&= \left[\left(\int_u^b \|A(s)\|_p\,ds\right)^{\alpha} (b-u) + \left(\int_a^u \|A(s)\|_p\,ds\right)^{\alpha} (u-a)\right]^{1/\alpha} \\
&\times \left(\int_a^b \|B'(t)\|_q^{\beta}\,dt\right)^{1/\beta} \\
&\le (b-a)^{1/\alpha} \left[\left(\int_u^b \|A(s)\|_p\,ds\right)^{\alpha} + \left(\int_a^u \|A(s)\|_p\,ds\right)^{\alpha}\right]^{1/\alpha} \\
&\times \left(\int_a^b \|B'(t)\|_q^{\beta}\,dt\right)^{1/\beta},
\end{aligned}$$

which proves (2.11). □

Remark 3. *If $m \in (a,b)$ is such that (2.6) is valid, then by (2.11) we get*

$$\begin{aligned}
(2.13)\qquad & \left\|\int_a^b A(t)\,B(t)\,dt - \left(\int_a^b A(s)\,ds\right) B(m)\right\|_1 \\
&\le \frac{1}{2}(b-a)^{1/\alpha} \int_a^b \|A(s)\|_p\,ds \left(\int_a^b \|B'(t)\|_q^{\beta}\,dt\right)^{1/\beta}.
\end{aligned}$$

Remark 4. *With the assumptions of Theorem 4 we have the mid-point inequality*

$$\left\| \int_a^b A(t) B(t)\, dt - \left(\int_a^b A(s)\, ds \right) B\left(\frac{a+b}{2} \right) \right\|_1 \le M_{p,q}(A,B), \tag{2.14}$$

where

$$M_{p,q}(A,B) := \int_{\frac{a+b}{2}}^b \left(\int_t^b \|A(s)\|_p\, ds \right) \|B'(t)\|_q\, dt + \int_a^{\frac{a+b}{2}} \left(\int_a^t \|A(s)\|_p\, ds \right) \|B'(t)\|_q\, dt.$$

We also have the bounds

$$M_p(A,B) \le \begin{cases} \left(\int_{\frac{a+b}{2}}^b \|A(s)\|_p\, ds \right) \int_{\frac{a+b}{2}}^b \|B'(t)\|_q\, dt, \\ \left[\int_{\frac{a+b}{2}}^b \left(\int_t^b \|A(s)\|_p\, ds \right)^\alpha dt \right]^{1/\alpha} \left(\int_{\frac{a+b}{2}}^b \|B'(t)\|_q^\beta\, dt \right)^{1/\beta}, \\ \int_{\frac{a+b}{2}}^b \left(\int_t^b \|A(s)\|_p\, ds \right) dt \sup_{t\in\left[\frac{a+b}{2},b\right]} \|B'(t)\|_q, \end{cases} + \begin{cases} \left(\int_a^{\frac{a+b}{2}} \|A(s)\|_p\, ds \right) \int_a^{\frac{a+b}{2}} \|B'(t)\|_q\, dt, \\ \left[\int_a^{\frac{a+b}{2}} \left(\int_a^t \|A(s)\|_p\, ds \right)^\alpha dt \right]^{1/\alpha} \left(\int_a^{\frac{a+b}{2}} \|B'(t)\|_q^\beta\, dt \right)^{1/\beta}, \\ \int_a^{\frac{a+b}{2}} \left(\int_a^t \|A(s)\|_p\, ds \right) dt \sup_{t\in\left[a,\frac{a+b}{2}\right]} \|B'(t)\|_q. \end{cases} \tag{2.15}$$

Making use of (2.5), we get

$$\begin{aligned} &\left\| \int_a^b A(t) B(t)\, dt - \left(\int_a^b A(s)\, ds \right) B\left(\frac{a+b}{2} \right) \right\|_1 \\ &\le \left(\int_{\frac{a+b}{2}}^b \|A(s)\|_p\, ds \right) \int_{\frac{a+b}{2}}^b \|B'(t)\|_q\, dt \\ &+ \left(\int_a^{\frac{a+b}{2}} \|A(s)\|_p\, ds \right) \int_a^{\frac{a+b}{2}} \|B'(t)\|_q\, dt \\ &\le \begin{cases} \max\left\{ \int_{\frac{a+b}{2}}^b \|A(s)\|_p\, ds, \int_a^{\frac{a+b}{2}} \|A(s)\|_p\, ds \right\} \int_a^b \|B'(t)\|_q\, dt \\ \int_a^b \|A(s)\|_p\, ds \max\left\{ \int_{\frac{a+b}{2}}^b \|B'(t)\|_q\, dt, \int_a^{\frac{a+b}{2}} \|B'(t)\|_q\, dt \right\} \end{cases} \\ &\le \int_a^b \|A(s)\|_p\, ds \int_a^b \|B'(t)\|_q\, dt \end{aligned} \tag{2.16}$$

and by (2.8),

$$\left\| \int_a^b A(t) B(t)\, dt - \left(\int_a^b A(s)\, ds \right) B\left(\frac{a+b}{2} \right) \right\|_1$$

$$
\begin{aligned}
&\leq \int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}\|A(s)\|_{p}\,ds\right)dt \sup_{t\in\left[\frac{a+b}{2},b\right]}\|B'(t)\|_{q}\\
&+\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}\|A(s)\|_{p}\,ds\right)dt \sup_{t\in\left[a,\frac{a+b}{2}\right]}\|B'(t)\|_{q}\\
&\leq \sup_{t\in[a,b]}\|B'(t)\|_{q}\int_{a}^{b}\left|t-\frac{a+b}{2}\right|\|A(t)\|_{p}\,dt.
\end{aligned}
$$

From (2.10) we derive the non-commutative mid-point type inequalities

$$
\begin{aligned}
&\left\|\int_{a}^{b}A(t)B(t)\,dt-\left(\int_{a}^{b}A(s)\,ds\right)B\left(\frac{a+b}{2}\right)\right\|_{1} \quad (2.17)\\
&\leq \sup_{t\in[a,b]}\|B'(t)\|_{q}\times\begin{cases}\frac{1}{2}(b-a)\int_{a}^{b}\|A(t)\|_{p}\,dt,\\[2mm]
\frac{(b-a)^{1+1/\beta}}{2(\beta+1)^{1/\beta}}\left(\int_{a}^{b}\|A(t)\|_{p}^{\alpha}\,dt\right)^{1/\alpha},\\[2mm]
\frac{1}{4}(b-a)^{2}\sup_{t\in[a,b]}\|A(t)\|_{p}.\end{cases}
\end{aligned}
$$

By (2.11) we obtain that

$$
\begin{aligned}
&\left\|\int_{a}^{b}A(t)B(t)\,dt-\left(\int_{a}^{b}A(s)\,ds\right)B\left(\frac{a+b}{2}\right)\right\|_{1} \quad (2.18)\\
&\leq\frac{(b-a)^{1/\alpha}}{2^{1/\alpha}}\left[\left(\int_{\frac{a+b}{2}}^{b}\|A(s)\|_{p}\,ds\right)^{\alpha}+\left(\int_{a}^{\frac{a+b}{2}}\|A(s)\|_{p}\,ds\right)^{\alpha}\right]^{1/\alpha}\\
&\times\left(\int_{a}^{b}\|B'(t)\|_{q}^{\beta}\,dt\right)^{1/\beta}.
\end{aligned}
$$

If we consider the case when $A(t)=A\in\mathcal{B}_{p}(H)$ $t\in[a,b]$, then by (2.1) we get

$$
\left\|A\left(\int_{a}^{b}B(t)\,dt-(b-a)B(u)\right)\right\|\leq\|A\|_{p}B_{q}(B,u), \quad (2.19)
$$

where

$$
B_{q}(B,u):=\int_{u}^{b}(b-t)\|B'(t)\|_{q}\,dt+\int_{a}^{u}(t-a)\|B'(t)\|_{q}\,dt.
$$

Subsequently, by (2.2) we also have the bounds

$$
\begin{aligned}
&B_q(B,u) \\
&\leq \begin{cases} (b-u)\int_u^b \|B'(t)\|_q\,dt, \\ \\ \frac{1}{(\alpha+1)^{1/\alpha}}(b-u)^{1+1/\alpha}\left(\int_u^b \|B'(t)\|_q^\beta\,dt\right)^{1/\beta}, \\ \\ \frac{1}{2}(b-u)^2\sup_{t\in[u,b]}\|B'(t)\|_q, \end{cases} \\
&+ \begin{cases} (u-a)\int_a^u \|B'(t)\|_q\,dt, \\ \\ \frac{1}{(\alpha+1)^{1/\alpha}}(u-a)^{1+1/\alpha}\left(\int_a^u \|B'(t)\|_q^\beta\,dt\right)^{1/\beta}, \\ \\ \frac{1}{2}(u-a)^2\sup_{t\in[a,u]}\|B'(t)\|_q. \end{cases}
\end{aligned} \tag{2.20}
$$

From (2.5) we get

$$
\begin{aligned}
&\left\|A\left(\int_a^b B(t)\,dt-(b-a)B(u)\right)\right\|_1 \\
&\leq \|A\|_p\left[(b-u)\int_u^b \|B'(t)\|_q\,dt+(u-a)\int_a^u \|B'(t)\|_q\,dt\right] \\
&\leq \begin{cases} \left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\|A\|_p\int_a^b \|B'(t)\|_q\,dt, \\ \\ (b-a)\|A\|_p\max\left\{\int_u^b \|B'(t)\|_q\,dt,\int_a^u \|B'(t)\|_q\,dt\right\}, \end{cases}
\end{aligned} \tag{2.21}
$$

where $A\in\mathcal{B}_p(H)$ and $B:[a,b]\to\mathcal{B}_q(H)$ is strongly differentiable and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

By (2.9) we also have the Ostrowski's inequality

$$
\begin{aligned}
&\left\|A\left(\int_a^b B(t)\,dt-(b-a)B(u)\right)\right\|_1 \\
&\leq \left[\frac{1}{4}(b-a)^2+\left(u-\frac{a+b}{2}\right)^2\right]\|A\|_p\sup_{t\in[a,b]}\|B'(t)\|_q
\end{aligned} \tag{2.22}
$$

for $u\in[a,b]$, where $A\in\mathcal{B}_p(H)$ and $B:[a,b]\to\mathcal{B}_q(H)$ is strongly differentiable and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

3. p-Schatten Norm Ostrowski Inequalities

We also have the p-Schatten norm version of Ostrowski's inequality:

Theorem 5. *Assume that $A,\ B:[a,b]\to\mathcal{B}_p(H)$ are continuous and B is strongly differentiable on (a,b), then for all $u\in[a,b]$*

$$
\left\|\int_a^b A(t)B(t)\,dt-\left(\int_a^b A(s)\,ds\right)B(u)\right\|_p\leq B_p(A,B,u), \tag{3.1}
$$

where

$$B_p(A,B,u) := \int_u^b \left(\int_t^b \|A(s)\|_p \, ds \right) \|B'(t)\|_p \, dt + \int_a^u \left(\int_a^t \|A(s)\|_p \, ds \right) \|B'(t)\|_p \, dt.$$

We also have the bounds

$$B_p(A,B,u) \leq \begin{cases} \int_u^b \|A(s)\|_p \, ds \int_u^b \|B'(t)\|_p \, dt, \\ \left[\int_u^b \left(\int_t^b \|A(s)\|_p \, ds \right)^\alpha dt \right]^{1/\alpha} \left(\int_u^b \|B'(t)\|_p^\beta \, dt \right)^{1/\beta}, \\ \int_u^b \left(\int_t^b \|A(s)\|_p \, ds \right) dt \sup_{t\in[u,b]} \|B'(t)\|_p, \end{cases} + \begin{cases} \int_a^u \|A(s)\|_p \, ds \int_a^u \|B'(t)\|_p \, dt, \\ \left[\int_a^u \left(\int_a^t \|A(s)\|_p \, ds \right)^\alpha dt \right]^{1/\alpha} \left(\int_a^u \|B'(t)\|_p^\beta \, dt \right)^{1/\beta}, \\ \int_a^u \left(\int_a^t \|A(s)\|_p \, ds \right) dt \sup_{t\in[a,u]} \|B'(t)\|_p, \end{cases} \tag{3.2}$$

where α, $\beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. Let $u \in [a,b]$. If we take the p-Schatten norm in (2.3), then we get

$$\begin{aligned} &\left\| \int_a^b A(t) B(t) \, dt - \left(\int_a^b A(s) \, ds \right) B(u) \right\|_p \\ &\leq \left\| \int_u^b \left(\int_t^b A(s) \, ds \right) B'(t) \, dt \right\|_p + \left\| \int_a^u \left(\int_a^t A(s) \, ds \right) B'(t) \, dt \right\|_p \\ &\leq \int_u^b \left\| \left(\int_t^b A(s) \, ds \right) B'(t) \right\|_p dt + \int_a^u \left\| \left(\int_a^t A(s) \, ds \right) B'(t) \right\|_p dt \\ &=: D(A,B,u). \end{aligned} \tag{3.3}$$

Using the inequality (1.11) for $p > 1$ we get

$$\begin{aligned} &D(A,B,u). \\ &\leq \int_u^b \left\| \int_t^b A(s) \, ds \right\|_p \|B'(t)\|_p \, dt + \int_a^u \left\| \int_a^t A(s) \, ds \right\|_p \|B'(t)\|_p \, dt \\ &\leq \int_u^b \left(\int_t^b \|A(s)\|_p \, ds \right) \|B'(t)\|_p \, dt + \int_a^u \left(\int_a^t \|A(s)\|_p \, ds \right) \|B'(t)\|_p \, dt \\ &= B_p(A,B,u), \end{aligned}$$

which proves (3.1).

Using Hölder's inequality, we get for α, $\beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, that

$$\begin{aligned}&\int_u^b\left(\int_t^b\|A(s)\|_p\,ds\right)\|B'(t)\|_p\,dt\\&\leq\begin{cases}\sup_{t\in[u,b]}\left(\int_t^b\|A(s)\|_p\,ds\right)\int_u^b\|B'(t)\|_p\,dt,\\ \\ \left[\int_u^b\left(\int_t^b\|A(s)\|_p\,ds\right)^\alpha dt\right]^{1/\alpha}\left(\int_u^b\|B'(t)\|_p^\beta\,dt\right)^{1/\beta},\\ \\ \int_u^b\left(\int_t^b\|A(s)\|_p\,ds\right)dt\sup_{t\in[u,b]}\|B'(t)\|_p\,,\end{cases}\\&=\begin{cases}\left(\int_u^b\|A(s)\|_p\,ds\right)\int_u^b\|B'(t)\|_p\,dt,\\ \\ \left[\int_u^b\left(\int_t^b\|A(s)\|_p\,ds\right)^\alpha dt\right]^{1/\alpha}\left(\int_u^b\|B'(t)\|_p^\beta\,dt\right)^{1/\beta},\\ \\ \int_u^b\left(\int_t^b\|A(s)\|_p\,ds\right)dt\sup_{t\in[u,b]}\|B'(t)\|_p\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\int_a^u\left(\int_a^t\|A(s)\|_p\,ds\right)\|B'(t)\|_p\,dt\\&\leq\begin{cases}\sup_{t\in[a,u]}\left(\int_a^t\|A(s)\|_p\,ds\right)\int_a^u\|B'(t)\|_p\,dt,\\ \\ \left[\int_a^u\left(\int_a^t\|A(s)\|_p\,ds\right)^\alpha dt\right]^{1/\alpha}\left(\int_a^u\|B'(t)\|_p^\beta\,dt\right)^{1/\beta},\\ \\ \int_a^u\left(\int_a^t\|A(s)\|_p\,ds\right)dt\sup_{t\in[a,u]}\|B'(t)\|_p\,,\end{cases}\end{aligned}$$

$$=\begin{cases}\left(\int_a^u\|A(s)\|_p\,ds\right)\int_a^u\|B'(t)\|_p\,dt,\\ \\ \left[\int_a^u\left(\int_a^t\|A(s)\|_p\,ds\right)^\alpha dt\right]^{1/\alpha}\left(\int_a^u\|B'(t)\|_p^\beta\,dt\right)^{1/\beta},\\ \\ \int_a^u\left(\int_a^t\|A(s)\|_p\,ds\right)dt\sup_{t\in[a,u]}\|B'(t)\|_p\,.\end{cases}$$

By making use of (2.4), we derive (2.2). □

Corollary 4. *With the assumptions of Theorem 5, we have*

$$\begin{aligned}
(3.4)\qquad & \left\| \int_a^b A(t)\, B(t)\, dt - \left(\int_a^b A(s)\, ds \right) B(u) \right\|_p \\
& \leq \int_u^b \|A(s)\|_p\, ds \int_u^b \|B'(t)\|_p\, dt + \int_a^u \|A(s)\|_p\, ds \int_a^u \|B'(t)\|_p\, dt \\
& \leq \begin{cases} \max\left\{ \int_u^b \|A(s)\|_p\, ds, \int_a^u \|A(s)\|_p\, ds \right\} \int_a^b \|B'(t)\|_p\, dt \\ \\ \int_a^b \|A(s)\|_p\, ds \max\left\{ \int_u^b \|B'(t)\|_p\, dt, \int_a^u \|B'(t)\|_p\, dt \right\} \end{cases} \\
& \leq \int_a^b \|A(s)\|_p\, ds \int_a^b \|B'(t)\|_p\, dt,
\end{aligned}$$

for all $u \in [a, b]$.

The proof follows by the first branches in the bounds (3.1).

Remark 5. *If* $m \in (a, b)$ *is such that (2.6) is valid, then by (3.4) we get*

$$\begin{aligned}
(3.5)\qquad & \left\| \int_a^b A(t)\, B(t)\, dt - \left(\int_a^b A(s)\, ds \right) B(m) \right\|_p \\
& \leq \frac{1}{2} \int_a^b \|A(s)\|_p\, ds \int_a^b \|B'(t)\|_p\, dt.
\end{aligned}$$

Corollary 5. *With the assumptions of Theorem 5, we have*

$$\begin{aligned}
(3.6)\qquad & \left\| \int_a^b A(t)\, B(t)\, dt - \left(\int_a^b A(s)\, ds \right) B(u) \right\|_p \\
& \leq \int_u^b \left(\int_t^b \|A(s)\|_p\, ds \right) dt \sup_{t \in [u,b]} \|B'(t)\|_p \\
& + \int_a^u \left(\int_a^t \|A(s)\|_p\, ds \right) dt \sup_{t \in [a,u]} \|B'(t)\|_p \\
& \leq \sup_{t \in [a,b]} \|B'(t)\|_p \int_a^b |t - u|\, \|A(t)\|_p\, dt,
\end{aligned}$$

for all $u \in [a, b]$.

Other similar inequalities may be also stated, however we omit the details.

4. Some Examples

We use the follow inequality that follows by (2.17),

$$\left\|\int_0^1 A(t)\,B(t)\,dt-\left(\int_0^1 A(s)\,ds\right)B\left(\frac{1}{2}\right)\right\|_1 \tag{4.1}$$
$$\leq \sup_{t\in[0,1]}\left\|B'(t)\right\|_q \times \begin{cases} \frac{1}{2}\int_0^1 \left\|A(t)\right\|_p dt, \\ \\ \frac{1}{2(\beta+1)^{1/\beta}}\left(\int_0^1 \left\|A(t)\right\|_p^{\alpha} dt\right)^{1/\alpha}, \\ \\ \frac{1}{4}\sup_{t\in[0,1]}\left\|A(t)\right\|_p, \end{cases}$$

provided that $A:[0,1]\to\mathcal{B}_p(H)$, $B:[0,1]\to\mathcal{B}_q(H)$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, are continuous and B is strongly differentiable on $(0,1)$.

Further, let $C,\ D\in\mathcal{B}(H)$ such that $(1-t)\,C+tD$ is invertible for all $t\in[0,1]$. For this to happen, it is enough to assume that $C,\ D>0$ in the operator order of $\mathcal{B}(H)$. Consider the function $B(t):=((1-t)\,C+tD)^{-1}$, $t\in[0,1]$ and observe that

$$B'(t)=-((1-t)\,C+tD)^{-1}(D-C)((1-t)\,C+tD)^{-1},\ t\in[0,1].$$

If $D,\ C\in\mathcal{B}_p(H)$ then by utilising property (1.13) we get

$$\begin{aligned}\left\|B'(t)\right\|_q &\leq \left\|((1-t)\,C+tD)^{-1}(D-C)((1-t)\,C+tD)^{-1}\right\|_q \\ &\leq \left\|D-C\right\|_q \left\|((1-t)\,C+tD)^{-1}\right\|^2\end{aligned}$$

for $t\in(0,1)$.

By (4.1) we derive

$$\left\|\int_0^1 A(t)\,((1-t)\,C+tD)^{-1}dt-\left(\int_0^1 A(s)\,ds\right)\left(\frac{C+D}{2}\right)^{-1}\right\|_1 \tag{4.2}$$
$$\leq \left\|D-C\right\|_q \sup_{t\in[0,1]}\left\|((1-t)\,C+tD)^{-1}\right\|^2$$
$$\times\begin{cases} \frac{1}{2}\int_0^1 \left\|A(t)\right\|_p dt, \\ \\ \frac{1}{2(\beta+1)^{1/\beta}}\left(\int_0^1 \left\|A(t)\right\|_p^{\alpha} dt\right)^{1/\alpha}, \\ \\ \frac{1}{4}\sup_{t\in[0,1]}\left\|A(t)\right\|_p, \end{cases}$$

provided that $A:[0,1]\to\mathcal{B}_p(H)$ and $D,\ C\in\mathcal{B}_p(H)$.

Now, if $C,\ D\geq m>0$ with $D,\ C\in\mathcal{B}_p(H)$, then by (4.2) we derive

$$\left\|\int_0^1 A(t)\left((1-t)C+tD\right)^{-1}dt-\left(\int_0^1 A(s)\,ds\right)\left(\frac{C+D}{2}\right)^{-1}\right\|_1 \tag{4.3}$$
$$\leq\frac{1}{m^2}\left\|D-C\right\|_q\times\begin{cases}\frac{1}{2}\int_0^1\left\|A(t)\right\|_p dt,\\[2ex] \frac{1}{2(\beta+1)^{1/\beta}}\left(\int_0^1\left\|A(t)\right\|_p^\alpha dt\right)^{1/\alpha},\\[2ex] \frac{1}{4}\sup_{t\in[0,1]}\left\|A(t)\right\|_p,\end{cases}$$

provided that $A:[0,1]\to\mathcal{B}_p(H)$ and $D,\ C\in\mathcal{B}_p(H)$.

Consider the function $B(t)=\exp(tT)$, where $t\in\mathbb{R}$ and $T\in\mathcal{B}(H)$. Then $B'(t)=T\exp(tT)$, for $t\in\mathbb{R}$ and $T\in\mathcal{B}(H)$. By utilising (2.17) we get for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\left\|\int_a^b A(t)\exp(tT)\,dt-\left(\int_a^b A(s)\,ds\right)\exp\left(\frac{a+b}{2}T\right)\right\|_1 \tag{4.4}$$
$$\leq\sup_{t\in[a,b]}\left\|T\exp(tT)\right\|_q\times\begin{cases}\frac{1}{2}(b-a)\int_a^b\left\|A(t)\right\|_p dt,\\[2ex] \frac{(b-a)^{1+1/\beta}}{2(\beta+1)^{1/\beta}}\left(\int_a^b\left\|A(t)\right\|_p^\alpha dt\right)^{1/\alpha},\\[2ex] \frac{1}{4}(b-a)^2\sup_{t\in[a,b]}\left\|A(t)\right\|_p\end{cases}$$
$$\leq\left\|T\right\|_q\sup_{t\in[a,b]}\left\|\exp(tT)\right\|_q\times\begin{cases}\frac{1}{2}(b-a)\int_a^b\left\|A(t)\right\|_p dt,\\[2ex] \frac{(b-a)^{1+1/\beta}}{2(\beta+1)^{1/\beta}}\left(\int_a^b\left\|A(t)\right\|_p^\alpha dt\right)^{1/\alpha},\\[2ex] \frac{1}{4}(b-a)^2\sup_{t\in[a,b]}\left\|A(t)\right\|_p,\end{cases}$$

provided that $A:[a,b]\to\mathcal{B}_p(H)$ and $T\in\mathcal{B}_q(H)$.

Since

$$\left\|\exp(tT)\right\|_q\leq\exp\left[|t|\left\|T\right\|_q\right]\leq\max\left\{\exp\left[|a|\left\|T\right\|_q\right],\exp\left[|b|\left\|T\right\|_q\right]\right\}$$
$$=\exp\left[\max\{|a|,|b|\}\left\|T\right\|_q\right],$$

then by (4.4) we derive

$$\left\|\int_a^b A(t)\exp(tT)\,dt-\left(\int_a^b A(s)\,ds\right)\exp\left(\frac{a+b}{2}T\right)\right\|_1 \tag{4.5}$$
$$\leq\exp\left[\max\{|a|,|b|\}\left\|T\right\|_q\right]\times\begin{cases}\frac{1}{2}(b-a)\int_a^b\left\|A(t)\right\|_p dt,\\[2ex] \frac{(b-a)^{1+1/\beta}}{2(\beta+1)^{1/\beta}}\left(\int_a^b\left\|A(t)\right\|_p^\alpha dt\right)^{1/\alpha},\\[2ex] \frac{1}{4}(b-a)^2\sup_{t\in[a,b]}\left\|A(t)\right\|_p\end{cases}$$

provided that $A:[a,b]\to\mathcal{B}_p(H)$ and $T\in\mathcal{B}_q(H)$.

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