SOME p-SCHATTEN NORM INEQUALITIES OF TRAPEZOID TYPE

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Abstract. An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

Assume that $A : [a, b] \to \mathcal{B}_p(H)$, $B : [a, b] \to \mathcal{B}_p(H)$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are continuous and B is strongly differentiable on (a, b), then

$$\left\| \left(\int_a^b A(s)\, ds \right) \frac{B(a) + B(b)}{2} - \int_a^b A(t) B(t)\, dt \right\|_1 \leq \frac{1}{2} \int_a^b \|A(s)\|_p\, ds \int_a^b \|B'(t)\|_q\, dt.$$

Some examples of interest for the inverse and operator exponential are also given.

1. Introduction

In 1999, Cerone and Dragomir proved the following *generalized trapezoid* type inequality for p-norm [6].

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. If $f' \in L_p[a, b]$, then we have the inequality*

$$\left| (b - x) f(b) + (x - a) f(a) - \int_a^b f(t)\, dt \right| \leq \frac{1}{(q + 1)^{1/q}} \left[(x - a)^{q+1} + (b - x)^{q+1} \right]^{1/q} \|f'\|_{[a,b],p}, \tag{1.1}$$

for all $x \in [a, b]$, where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a, b]$, i.e., we recall it

$$\|g\|_{[a,b],p} := \left(\int_a^b |g(t)|^p\, dt \right)^{1/p}.$$

From (1.1) we get the following *trapezoid inequality*

$$\left| (b - a) \frac{f(a) + f(b)}{2} - \int_a^b f(t)\, dt \right| \leq \frac{1}{2(q + 1)^{1/q}} (b - a)^{1+1/q} \|f'\|_{[a,b],p}, \tag{1.2}$$

and $\frac{1}{2}$ is a best possible constant.

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For a survey on scalar trapezoid inequality, see [6]. For recent papers on this inequality see also [1]-[5] and [7].

In order to extend these results for p-Schatten norms, we need the following preparations.

Let $(H; \langle .,. \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.3}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i \rangle, \tag{1.4}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.4) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.5}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.6}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on $\mathcal{B}_1(H)$ with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ then AB, $BA \in \mathcal{B}_1(H)$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the *p-Schatten norm* is finite [14, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{1.7}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{1.8}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [14, p. 60-64],

$$\|A\|_p = \|A^*\|_p, \ A \in \mathcal{B}_p(H) \tag{1.9}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \ A, B \in \mathcal{B}_p(H) \tag{1.10}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \ \|BA\|_p \leq \|B\| \|A\|_p, \ A \in \mathcal{B}_p(H), \ B \in \mathcal{B}(H). \tag{1.11}$$

This implies that

$$\|CAB\|_p \le \|C\| \, \|A\|_p \, \|B\| \, , \; A \in \mathcal{B}_p(H) \, , \; B, \; C \in \mathcal{B}(H) \, . \tag{1.12}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\mathrm{tr}(AB)| \le) \, \|AB\|_1 \le \|A\|_p \, \|B\|_q \, , \; A \in \mathcal{B}_p(H) \, , \; B \in \mathcal{B}_q(H) \, . \tag{1.13}$$

For the theory of trace functionals and their applications the reader is referred to [13] and [14].

For some classical trace inequalities see [8], [9], and [11], which are continuations of the work of Bellman [2].

2. Main Results

We have the following weighted version of generalized trapezoid inequality for two functions with values in Banach algebras:

Theorem 3. *For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ be continuous and B is strongly differentiable on (a,b), then*

$$\begin{aligned} &\left\| \left(\int_a^b A(s)\, ds \right) \frac{B(a) + B(b)}{2} - \int_a^b A(t) B(t)\, dt \right\|_1 \\ &\le \frac{1}{2} \int_a^b \left\| \int_a^t A(s)\, ds - \int_t^b A(s)\, ds \right\|_p \|B'(t)\|_q \, dt =: K_{p,q}(A,B) \\ &\le \frac{1}{2} \int_a^b \|A(s)\|_p \, ds \int_a^b \|B'(t)\|_q \, dt. \end{aligned} \tag{2.1}$$

We also have the bounds,

$$\begin{aligned} &K_{p,q}(A,B) \\ &\le \frac{1}{2} \times \begin{cases} \sup_{t\in[a,b]} \left\| \int_a^t A(s)\, ds - \int_t^b A(s)\, ds \right\|_p \int_a^b \|B'(t)\|_q \, dt, \\ \\ \left(\int_a^b \left\| \int_a^t A(s)\, ds - \int_t^b A(s)\, ds \right\|_p^\alpha dt \right)^{1/\alpha} \left(\int_a^b \|B'(t)\|_q^\beta \, dt \right)^{1/\beta}, \\ \text{for } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \int_a^b \left\| \int_a^t A(s)\, ds - \int_t^b A(s)\, ds \right\|_p dt \sup_{t\in[a,b]} \|B'(t)\|_q \, , \end{cases} \\ &\le \frac{1}{2} \times \begin{cases} \int_a^b \|A(s)\|_p \, dt \int_a^b \|B'(t)\|_q \, dt, \\ \\ (b-a)^{1/\alpha} \int_a^b \|A(s)\|_p \, dt \left(\int_a^b \|B'(t)\|_q^\beta \, dt \right)^{1/\beta}, \\ \text{for } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ (b-a) \int_a^b \|A(s)\|_p \, dt \sup_{t\in[a,b]} \|B'(t)\|_q \, . \end{cases} \end{aligned} \tag{2.2}$$

Proof. Using the integration by parts formula for Bochner integral, we have

$$\begin{aligned}
&\int_a^b \left[\int_a^t A(s)\,ds - \frac{1}{2}\int_a^b A(s)\,ds\right] B'(t)\,dt \\
&= \left[\int_a^t A(s)\,ds - \frac{1}{2}\int_a^b A(s)\,ds\right] B(t)\Big|_a^b \\
&- \int_a^b \left[\int_a^t A(s)\,ds - \frac{1}{2}\int_a^b A(s)\,ds\right]' B(t)\,dt \\
&= \left[\int_a^b A(s)\,ds - \frac{1}{2}\int_a^b A(s)\,ds\right] B(b) \\
&- \left[\int_a^a A(s)\,ds - \frac{1}{2}\int_a^b A(s)\,ds\right] B(a) - \int_a^b A(t)\,B(t)\,dt \\
&= \left(\int_a^b A(s)\,ds\right) \frac{B(a)+B(b)}{2} - \int_a^b A(t)\,B(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left[\int_a^t A(s)\,ds - \frac{1}{2}\int_a^b A(s)\,ds\right] B'(t)\,dt \\
&= \frac{1}{2}\int_a^b \left(\int_a^t A(s)\,ds - \int_t^b A(s)\,ds\right) B'(t)\,dt.
\end{aligned}$$

Therefore we have the following identity of interest

$$\begin{aligned}
&\left(\int_a^b A(s)\,ds\right) \frac{B(a)+B(b)}{2} - \int_a^b A(t)\,B(t)\,dt \\
&= \frac{1}{2}\int_a^b \left(\int_a^t A(s)\,ds - \int_t^b A(s)\,ds\right) B'(t)\,dt.
\end{aligned} \tag{2.3}$$

Taking the 1-Schatten norm in (2.3), using the integral's properties and the Hölder's inequality (1.13), we get

$$\begin{aligned}
&\left\|\left(\int_a^b A(s)\,ds\right) \frac{B(a)+B(b)}{2} - \int_a^b A(t)\,B(t)\,dt\right\|_1 \\
&\le \frac{1}{2}\int_a^b \left\|\left(\int_a^t A(s)\,ds - \int_t^b A(s)\,ds\right) B'(t)\right\|_1 dt \\
&\le \frac{1}{2}\int_a^b \left\|\int_a^t A(s)\,ds - \int_t^b A(s)\,ds\right\|_p \|B'(t)\|_q\,dt
\end{aligned}$$

$$\le \frac{1}{2}\int_a^b \left(\left\| \int_a^t A(s)\,ds \right\|_p + \left\| \int_t^b A(s)\,ds \right\|_p \right) \|B'(t)\|_q \,dt$$
$$\le \frac{1}{2}\int_a^b \left(\int_a^t \|A(s)\|_p \,ds + \int_t^b \|A(s)\|_p \,ds \right) \|B'(t)\|_q \,dt$$
$$= \frac{1}{2}\int_a^b \|A(s)\|_p \,ds \int_a^b \|B'(t)\|_q \,dt.$$

The first inequality in (2.2) follows by Hölder's inequality applied to the integral of the product

$$\int_a^b \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p \|B'(t)\|_q \,dt.$$

The last part follows by the fact that

$$\left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p \le \int_a^b \|A(s)\|_p \,ds$$

for all $t \in [a,b]$.

This implies that

$$\sup_{t\in[a,b]} \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p \le \int_a^b \|A(s)\|_p \,ds,$$

$$\int_a^b \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p^\alpha dt \le \int_a^b \left(\int_a^b \|A(s)\|_p \,ds \right)^\alpha dt$$
$$= (b-a)\left(\int_a^b \|A(s)\|_p \,ds \right)^\alpha$$

and

$$\int_a^b \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p dt \le (b-a)\int_a^b \|A(s)\|_p \,ds,$$

and the second part of (2.2) is thus proved. □

Corollary 1. *Assume that $A:[a,b]\to\mathcal{B}_p(H)$, $C:[a,b]\to\mathcal{B}_q(H)$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ are continuous and C is strongly differentiable on (a,b) and such that*

$$\|C'(t)-V\|_q \le M \text{ for all } t\in(a,b)$$

for some element $V\in\mathcal{B}_q(H)$ and $M>0$, then

$$\left\| \left(\int_a^b A(s)\,ds \right) \frac{C(a)+C(b)}{2} + \int_a^b \left(t-\frac{a+b}{2} \right) A(t)\,dtV - \int_a^b A(t)\,C(t)\,dt \right\|_1$$
$$\le \frac{1}{2}M\int_a^b \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p dt \le \frac{1}{2}M(b-a)\int_a^b \|A(s)\|_p \,ds. \tag{2.4}$$

Proof. Put $B(t) = C(t) - tV$, $t \in [0,1]$, then

$$\begin{aligned}
&\left(\int_a^b A(s)\,ds\right)\frac{B(a)+B(b)}{2} - \int_a^b A(t)\,B(t)\,dt \\
&= \left(\int_a^b A(s)\,ds\right)\frac{C(a)-aV+C(b)-bV}{2} - \int_a^b A(t)\,[C(t)-tV]\,dt \\
&= \left(\int_a^b A(s)\,ds\right)\frac{C(a)+C(b)}{2} - \frac{a+b}{2}\left(\int_a^b A(s)\,ds\right)V \\
&- \int_a^b A(t)\,C(t)\,dt + \int_a^b tA(t)\,V\,dt \\
&= \left(\int_a^b A(s)\,ds\right)\frac{C(a)+C(b)}{2} + \int_a^b tA(t)\,V\,dt \\
&- \frac{a+b}{2}\left(\int_a^b A(s)\,ds\right)V - \int_a^b A(t)\,C(t)\,dt \\
&= \left(\int_a^b A(s)\,ds\right)\frac{C(a)+C(b)}{2} + \int_a^b \left(t - \frac{a+b}{2}\right)A(t)\,V\,dt - \int_a^b A(t)\,C(t)\,dt.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_a^b \left\|\int_a^t A(s)\,ds - \int_t^b A(s)\,ds\right\|_p \|B'(t)\|_q\,dt \\
&= \int_a^b \left\|\int_a^t A(s)\,ds - \int_t^b A(s)\,ds\right\|_p \|C'(t)-V\|_q\,dt \\
&\le M\int_a^b \left\|\int_a^t A(s)\,ds - \int_t^b A(s)\,ds\right\|_p dt \le M(b-a)\int_a^b \|A(s)\|_p\,ds.
\end{aligned}$$

By utilising (2.1) we derive the desired result (2.4). □

Remark 1. *If we take $A(s) = A \in \mathcal{B}_p(H)$, $s \in [a,b]$, then by (2.1) we get*

$$\begin{aligned}
(2.5)\qquad &\left\|A\left[(b-a)\frac{B(a)+B(b)}{2} - \int_a^b B(t)\,dt\right]\right\|_1 \\
&\le \|A\|_p \int_a^b \left|t-\frac{a+b}{2}\right| \|B'(t)\|_q\,dt \le \frac{1}{4}(b-a)^2\|A\|_p \sup_{t\in[a,b]}\|B'(t)\|_q,
\end{aligned}$$

for $B : [a,b] \to \mathcal{B}_q(H)$ strongly differentiable on (a,b).

If $C : [a,b] \to \mathcal{B}_q(H)$ is strongly differentiable on (a,b) and

$$\|C'(t) - V\|_q \le M \text{ for all } t \in (a,b)$$

for some element $V \in \mathcal{B}_q(H)$ and $M > 0$, then

$$(2.6)\qquad \left\|A\left[(b-a)\frac{C(a)+C(b)}{2} - \int_a^b C(t)\,dt\right]\right\|_1 \le \frac{1}{4}(b-a)^2 M.$$

The case of p-Schatten norm is as follows:

Theorem 4. *Let A, $B : [a,b] \to \mathcal{B}_p(H)$ be continuous and B is strongly differentiable on (a,b), then*

$$\begin{aligned}(2.7)\quad &\left\| \left(\int_a^b A(s)\,ds \right) \frac{B(a)+B(b)}{2} - \int_a^b A(t)\,B(t)\,dt \right\|_p \\ &\le \frac{1}{2} \int_a^b \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p \|B'(t)\|_p \,dt =: K_p(A,B) \\ &\le \frac{1}{2} \int_a^b \|A(s)\|_p \,ds \int_a^b \|B'(t)\|_p \,dt.\end{aligned}$$

We also have the bounds,

$$(2.8)\quad K_p(A,B)$$

$$\le \frac{1}{2} \times \begin{cases} \sup_{t\in[a,b]} \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p \int_a^b \|B'(t)\|_p \,dt, \\ \\ \left(\int_a^b \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p^\alpha dt \right)^{1/\alpha} \left(\int_a^b \|B'(t)\|_p^\beta \,dt \right)^{1/\beta}, \\ \text{for } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \int_a^b \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p dt \sup_{t\in[a,b]} \|B'(t)\|_p, \end{cases}$$

$$\le \frac{1}{2} \times \begin{cases} \int_a^b \|A(s)\|_p \,dt \int_a^b \|B'(t)\|_p \,dt, \\ \\ (b-a)^{1/\alpha} \int_a^b \|A(s)\|_p \,dt \left(\int_a^b \|B'(t)\|_p^\beta \,dt \right)^{1/\beta}, \\ \text{for } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ (b-a) \int_a^b \|A(s)\|_p \,dt \sup_{t\in[a,b]} \|B'(t)\|_p. \end{cases}$$

Proof. Taking the p-Schatten norm in (2.3), using the integral's properties and the inequality (1.10), we get

$$\begin{aligned}&\left\| \left(\int_a^b A(s)\,ds \right) \frac{B(a)+B(b)}{2} - \int_a^b A(t)\,B(t)\,dt \right\|_p \\ &\le \frac{1}{2} \int_a^b \left\| \left(\int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right) B'(t) \right\|_p dt \\ &\le \frac{1}{2} \int_a^b \left\| \int_a^t A(s)\,ds - \int_t^b A(s)\,ds \right\|_p \|B'(t)\|_p \,dt\end{aligned}$$

$$\leq \frac{1}{2}\int_a^b \left(\left\|\int_a^t A(s)\,ds\right\|_p + \left\|\int_t^b A(s)\,ds\right\|_p\right)\|B'(t)\|_p\,dt$$
$$\leq \frac{1}{2}\int_a^b \left(\int_a^t \|A(s)\|_p\,ds + \int_t^b \|A(s)\|_p\,ds\right)\|B'(t)\|_p\,dt$$
$$= \frac{1}{2}\int_a^b \|A(s)\|_p\,ds \int_a^b \|B'(t)\|_p\,dt.$$

The rest follows as in the proof of Theorem 3 and we omit the details. □

We also have:

Corollary 2. *Assume that* $A,\ C : [a,b] \to \mathcal{B}_p(H)$, $p > 1$ *are continuous and* C *is strongly differentiable on* (a,b) *and such that*

$$\|C'(t) - V\|_p \leq M \text{ for all } t \in (a,b)$$

for some element $V \in \mathcal{B}_p(H)$ *and* $M > 0$, *then*

$$\left\|\left(\int_a^b A(s)\,ds\right)\frac{C(a)+C(b)}{2} + \int_a^b \left(t - \frac{a+b}{2}\right)A(t)\,dtV - \int_a^b A(t)\,C(t)\,dt\right\|_p \leq \frac{1}{2}M\int_a^b \left\|\int_a^t A(s)\,ds - \int_t^b A(s)\,ds\right\|_p dt \leq \frac{1}{2}M(b-a)\int_a^b \|A(s)\|_p\,ds. \tag{2.9}$$

3. Some Examples

Consider the function $B(t) = \exp(tT)$, where $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. Then $B'(t) = T\exp(tT)$, for $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. By utilising (2.1) we get for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\left\|\left(\int_a^b A(s)\,ds\right)\frac{\exp(aT)+\exp(bT)}{2} - \int_a^b A(t)\exp(tT)\,dt\right\|_1 \leq \frac{1}{2}\int_a^b \|A(s)\|_p\,ds \int_a^b \|T\exp(tT)\|_q\,dt \leq \frac{1}{2}\|T\|_q \int_a^b \|A(s)\|_p\,ds \int_a^b \|\exp(tT)\|_q\,dt \tag{3.1}$$

provided that $A : [a,b] \to \mathcal{B}_p(H)$ and $T \in \mathcal{B}_q(H)$.

Assume that $0 \leq a < b$, then $\|\exp(tT)\|_q \leq \exp\left(t\|T\|_q\right)$ and since

$$\int_a^b \exp\left(t\|T\|_q\right)dt = \frac{\exp\left(b\|T\|_q\right) - \exp\left(a\|T\|_q\right)}{\|T\|_q},$$

then by (3.1) we obtain

$$\left\|\left(\int_a^b A(s)\,ds\right)\frac{\exp(aT)+\exp(bT)}{2}-\int_a^b A(t)\exp(tT)\,dt\right\|_1 \tag{3.2}$$
$$\le \frac{1}{2}\int_a^b \|A(s)\|_p\,ds\int_a^b \|T\exp(tT)\|_q\,dt$$
$$\le \frac{1}{2}\left[\exp\left(b\|T\|_q\right)-\exp\left(a\|T\|_q\right)\right]\int_a^b \|A(s)\|_p\,ds$$

provided that $A:[a,b]\to\mathcal{B}_p(H)$, $T\in\mathcal{B}_q(H)$ and $0\le a<b$.

We use the follow inequality that follows by

$$\left\|\left(\int_0^1 A(s)\,ds\right)\frac{B(0)+B(1)}{2}-\int_0^1 A(t)B(t)\,dt\right\|_1 \tag{3.3}$$
$$\le \frac{1}{2}\int_0^1 \|A(s)\|_p\,ds\int_0^1 \|B'(t)\|_q\,dt.$$

provided that $A:[0,1]\to\mathcal{B}_p(H)$, $B:[0,1]\to\mathcal{B}_q(H)$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, are continuous and B is strongly differentiable on $(0,1)$.

Further, let $C,\ D\in\mathcal{B}(H)$ such that $(1-t)C+tD$ is invertible for all $t\in[0,1]$. For this to happen, it is enough to assume that $C,\ D>0$ in the operator order of $\mathcal{B}(H)$. Consider the function $B(t):=((1-t)C+tD)^{-1}$, $t\in[0,1]$ and observe that

$$B'(t)=-((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1},\ t\in[0,1].$$

If $D,\ C\in\mathcal{B}_p(H)$ then by utilising property (1.12) we get

$$\|B'(t)\|_q\le\left\|((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\right\|_q$$
$$\le\|D-C\|_q\left\|((1-t)C+tD)^{-1}\right\|^2$$

for $t\in(0,1)$.

By (3.3) we derive

$$\left\|\left(\int_0^1 A(s)\,ds\right)\frac{D^{-1}+C^{-1}}{2}-\int_0^1 A(t)((1-t)C+tD)^{-1}\,dt\right\|_1 \tag{3.4}$$
$$\le\frac{1}{2}\|D-C\|_q\sup_{t\in[0,1]}\left\|((1-t)C+tD)^{-1}\right\|^2\int_0^1\|A(s)\|_p\,ds,$$

provided that $A:[0,1]\to\mathcal{B}_p(H)$ and $D,\ C\in\mathcal{B}_p(H)$.

Now, if $C,\ D\ge m>0$ with $D,\ C\in\mathcal{B}_p(H)$, then by (3.4) we derive

$$\left\|\left(\int_0^1 A(s)\,ds\right)\frac{D^{-1}+C^{-1}}{2}-\int_0^1 A(t)((1-t)C+tD)^{-1}\,dt\right\|_1 \tag{3.5}$$
$$\le\frac{1}{2m^2}\|D-C\|_q\int_0^1\|A(s)\|_p\,ds,$$

provided that $A:[0,1]\to\mathcal{B}_p(H)$ and $D,\ C\in\mathcal{B}_p(H)$.

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