

# $p$ -SCHATTEN NORM INEQUALITIES OF GRÜSS' TYPE

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ABSTRACT. An operator  $A \in \mathcal{B}(H)$  is said to belong to the *von Neumann-Schatten class*  $\mathcal{B}_p(H)$ ,  $1 \leq p < \infty$  if the *p-Schatten norm* is finite

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For two continuous functions  $A, B : [a, b] \rightarrow \mathcal{B}(H)$  we define the *noncommutative Čebyšev functional*

$$D(A, B) := (b-a) \int_a^b A(t) B(t) dt - \int_a^b A(t) dt \int_a^b B(t) dt.$$

If  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , let  $A : [a, b] \rightarrow \mathcal{B}_p(H)$ ,  $B : [a, b] \rightarrow \mathcal{B}_q(H)$  be strongly differentiable functions on the interval  $(a, b)$ , then

$$\begin{aligned} \|D(A, B)\|_1 &\leq D\left(\int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_q du\right) \\ &\leq \frac{1}{4} (b-a)^2 \int_a^b \|A'(u)\|_p du \int_a^b \|B'(u)\|_q du. \end{aligned}$$

Some examples of interest for the operator exponential are also given.

## 1. INTRODUCTION

For two Lebesgue integrable functions  $f, g : [a, b] \rightarrow \mathbb{C}$ , in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f, g) := (b-a) \int_a^b f(t) g(t) dt - \int_a^b f(t) dt \int_a^b g(t) dt.$$

In 1934, G. Grüss [10] showed that

$$(1.1) \quad |D(f, g)| \leq \frac{1}{4} (b-a)^2 (M-m)(N-n),$$

provided  $m, M, n, N$  are real numbers with the property that

$$(1.2) \quad -\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a, b].$$

The constant  $\frac{1}{4}$  is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for  $D(f, g)$  was derived in 1882 by Čebyšev [7] under the assumption that  $f', g'$  exist and are continuous on  $[a, b]$ , and is given by

$$(1.3) \quad |D(f, g)| \leq \frac{1}{12} \|f'\|_{\infty} \|g'\|_{\infty} (b-a)^4,$$

where  $\|f'\|_{\infty} := \sup_{t \in [a, b]} |f'(t)| < \infty$ .

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The constant  $\frac{1}{12}$  cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if  $f, g : [a, b] \rightarrow \mathbb{R}$  are assumed to be absolutely continuous and  $f', g' \in L_\infty[a, b]$ .

In 1970, A. M. Ostrowski [14] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$(1.4) \quad |D(f, g)| \leq \frac{1}{8} (b-a)^3 (M-m) \|g'\|_\infty,$$

provided  $f$  is Lebesgue integrable on  $[a, b]$  and satisfying (1.2) while  $g : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous and  $g' \in L_\infty[a, b]$ . Here the constant  $\frac{1}{8}$  is also sharp.

In 1973, A. Lupaş [11] (see also [12, p. 210]) obtained the following result as well:

$$(1.5) \quad |D(f, g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a)^3,$$

provided  $f, g$  are absolutely continuous and  $f', g' \in L_2[a, b]$ .

Here the constant  $\frac{1}{\pi^2}$  is the best possible as well.

In [3], P. Cerone and S. S. Dragomir proved the following inequalities:

$$(1.6) \quad |D(f, g)| \leq (b-a) \times \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt, \\ \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \left( \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|^p dt \right)^{\frac{1}{p}} \\ \text{where } p > 1, 1/p + 1/q = 1. \end{cases}$$

For  $\gamma = 0$ , we get from the first inequality in (1.6)

$$(1.7) \quad |D(f, g)| \leq (b-a) \|g\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt$$

for which the constant 1 cannot be replaced by a smaller constant.

If  $m \leq g \leq M$  for a.e.  $x \in [a, b]$ , then  $\|g - \frac{m+M}{2}\|_\infty \leq \frac{1}{2}(M-m)$  and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [8]

$$(1.8) \quad |D(f, g)| \leq \frac{1}{2} (b-a) (M-m) \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt.$$

The constant  $\frac{1}{2}$  is best in (1.8) as shown by Cerone and Dragomir in [4].

The following result holds [9].

**Theorem 1.** *Let  $f : [a, b] \rightarrow \mathbb{C}$  be of bounded variation on  $[a, b]$  and  $g : [a, b] \rightarrow \mathbb{C}$  a Lebesgue integrable function on  $[a, b]$ . Then*

$$(1.9) \quad |D(f, g)| \leq \frac{1}{2} (b-a) \bigvee_a^b(f) \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt$$

where  $\bigvee_a^b(f)$  denotes the total variation of  $f$  on the interval  $[a, b]$ . The constant  $\frac{1}{2}$  is best possible in (1.9).

In order to extend some of the above results for the  $p$ -Schatten norm of bounded linear operators in complex Hilbert spaces we need the following preparations.

## 2. SOME PRELIMINARY FACTS ON $p$ -SCHATTEN NORM

Let  $(H; \langle \cdot, \cdot \rangle)$  be a complex Hilbert space and  $\mathcal{B}(H)$  the Banach algebra of all bounded linear operators on  $H$ . If  $\{e_i\}_{i \in I}$  an orthonormal basis of  $H$ , we say that  $A \in \mathcal{B}(H)$  is of *trace class* if

$$(2.1) \quad \|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty.$$

The definition of  $\|A\|_1$  does not depend on the choice of the orthonormal basis  $\{e_i\}_{i \in I}$ . We denote by  $\mathcal{B}_1(H)$  the set of trace class operators in  $\mathcal{B}(H)$ .

We define the *trace* of a trace class operator  $A \in \mathcal{B}_1(H)$  to be

$$(2.2) \quad \text{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle,$$

where  $\{e_i\}_{i \in I}$  an orthonormal basis of  $H$ . Note that this coincides with the usual definition of the trace if  $H$  is finite-dimensional. We observe that the series (2.2) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

**Theorem 2.** *We have:*

(i) *If  $A \in \mathcal{B}_1(H)$  then  $A^* \in \mathcal{B}_1(H)$  and*

$$(2.3) \quad \text{tr}(A^*) = \overline{\text{tr}(A)};$$

(ii) *If  $A \in \mathcal{B}_1(H)$  and  $T \in \mathcal{B}(H)$ , then  $AT, TA \in \mathcal{B}_1(H)$  and*

$$(2.4) \quad \text{tr}(AT) = \text{tr}(TA) \text{ and } |\text{tr}(AT)| \leq \|A\|_1 \|T\|;$$

(iii)  *$\text{tr}(\cdot)$  is a bounded linear functional on  $\mathcal{B}_1(H)$  with  $\|\text{tr}\| = 1$ ;*

(iv) *If  $A, B \in \mathcal{B}_2(H)$  then  $AB, BA \in \mathcal{B}_1(H)$  and  $\text{tr}(AB) = \text{tr}(BA)$ ;*

(v)  *$\mathcal{B}_{fin}(H)$  is a dense subspace of  $\mathcal{B}_1(H)$ .*

An operator  $A \in \mathcal{B}(H)$  is said to belong to the *von Neumann-Schatten class*  $\mathcal{B}_p(H)$ ,  $1 \leq p < \infty$  if the  $p$ -Schatten norm is finite [16, p. 60-64]

$$\|A\|_p := [\text{tr}(|A|^p)]^{1/p} < \infty.$$

For  $1 < p < q < \infty$  we have that

$$(2.5) \quad \mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H)$$

and

$$(2.6) \quad \|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|.$$

For  $p \geq 1$  the functional  $\|\cdot\|_p$  is a *norm* on the  $*$ -ideal  $\mathcal{B}_p(H)$  and  $(\mathcal{B}_p(H), \|\cdot\|_p)$  is a Banach space.

Also, see for instance [16, p. 60-64],

$$(2.7) \quad \|A\|_p = \|A^*\|_p, \quad A \in \mathcal{B}_p(H)$$

$$(2.8) \quad \|AB\|_p \leq \|A\|_p \|B\|_p, \quad A, B \in \mathcal{B}_p(H)$$

and

$$(2.9) \quad \|AB\|_p \leq \|A\|_p \|B\|, \quad \|BA\|_p \leq \|B\| \|A\|_p, \quad A \in \mathcal{B}_p(H), \quad B \in \mathcal{B}(H).$$

This implies that

$$(2.10) \quad \|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \quad A \in \mathcal{B}_p(H), \quad B, C \in \mathcal{B}(H).$$

In terms of  $p$ -Schatten norm we have the Hölder inequality for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$

$$(2.11) \quad (|\operatorname{tr}(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \quad A \in \mathcal{B}_p(H), \quad B \in \mathcal{B}_q(H).$$

For the theory of trace functionals and their applications the reader is referred to [15] and [16].

For some classical trace inequalities see [5], [6], and [13], which are continuations of the work of Bellman [1].

### 3. 1-SCHATTEN NORM INEQUALITIES

For two continuous functions  $A, B : [a, b] \rightarrow \mathcal{B}(H)$  we define the *noncommutative Čebyšev functional*

$$D(A, B) := (b-a) \int_a^b A(t) B(t) dt - \int_a^b A(t) dt \int_a^b B(t) dt.$$

We have the following result of interest:

**Theorem 3.** For  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , let  $A : [a, b] \rightarrow \mathcal{B}_p(H)$ ,  $B : [a, b] \rightarrow \mathcal{B}_q(H)$  be strongly differentiable functions on the interval  $(a, b)$ . Then

$$(3.1) \quad \|D(A, B)\|_1 \leq D \left( \int_a^b \|A'(u)\|_p du, \int_a^b \|B'(u)\|_q du \right) \\ \leq \frac{1}{4} (b-a)^2 \int_a^b \|A'(u)\|_p du \int_a^b \|B'(u)\|_q du.$$

*Proof.* Observe that

$$\begin{aligned} & \int_a^b \int_a^b [A(t) - A(s)] [B(t) - B(s)] dt ds \\ &= \int_a^b \int_a^b (A(t) B(t) - A(s) B(t) - A(t) B(s) + A(s) B(s)) dt ds \\ &= (b-a) \int_a^b A(t) B(t) dt - \int_a^b A(s) ds \int_a^b B(t) dt \\ & \quad - \int_a^b A(t) dt \int_a^b B(s) ds + (b-a) \int_a^b A(s) B(s) ds \\ &= 2(b-a) \int_a^b A(t) B(t) dt - 2 \int_a^b A(t) dt \int_a^b B(t) dt = 2D(A, B), \end{aligned}$$

which give the Korkine's noncommutative identity for functions with values in Banach algebra  $\mathcal{B}(H)$

$$(3.2) \quad D(A, B) = \frac{1}{2} \int_a^b \int_a^b [A(t) - A(s)] [B(t) - B(s)] dt ds.$$

For Korkine's classical identity for real-valued functions, see [12, p. 242].

If we take the 1-Schatten norm, use the integral's properties and employ Hölder's inequality (2.11) for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then we get

$$(3.3) \quad \begin{aligned} \|D(A, B)\|_1 &\leq \frac{1}{2} \int_a^b \int_a^b \|[A(t) - A(s)][B(t) - B(s)]\|_1 dt ds \\ &\leq \frac{1}{2} \int_a^b \int_a^b \|A(t) - A(s)\|_p \|B(t) - B(s)\|_q dt ds. \end{aligned}$$

Observe that for  $s, t \in [a, b]$

$$A(t) - A(s) = \int_s^t A'(u) du, \quad B(t) - B(s) = \int_s^t B'(u) du,$$

which implies that

$$\begin{aligned} \|A(t) - A(s)\|_p \|B(t) - B(s)\|_q &= \left\| \int_s^t A'(u) du \right\|_p \left\| \int_s^t B'(u) du \right\|_q \\ &\leq \left| \int_s^t \|A'(u)\|_p du \right| \left| \int_s^t \|B'(u)\|_q du \right| \\ &= \left| \int_s^t \|A'(u)\|_p du \int_s^t \|B'(u)\|_q du \right| \\ &= \int_s^t \|A'(u)\|_p du \int_s^t \|B'(u)\|_q du, \end{aligned}$$

for all  $s, t \in [a, b]$ .

By (3.3) we get

$$(3.4) \quad \|D(A, B)\|_1 \leq \frac{1}{2} \int_a^b \int_a^b \left( \int_s^t \|A'(u)\|_p du \right) \left( \int_s^t \|B'(u)\|_q du \right) dt ds$$

for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .

Since

$$\begin{aligned} &\int_s^t \|A'(u)\|_p du \int_s^t \|B'(u)\|_q du \\ &= \left( \int_a^t \|A'(u)\|_p du - \int_a^s \|A'(u)\|_p du \right) \left( \int_a^t \|B'(u)\|_q du - \int_a^s \|B'(u)\|_q du \right), \end{aligned}$$

hence by Korkine's identity for real valued functions  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_q du$ , we have

$$\begin{aligned}
(3.5) \quad & \frac{1}{2} \int_a^b \int_a^b \left( \int_a^t \|A'(u)\|_p du - \int_a^s \|A'(u)\|_p du \right) \\
& \times \left( \int_a^t \|B'(u)\|_q du - \int_a^s \|B'(u)\|_q du \right) dt ds \\
& = (b-a) \int_a^b \left( \int_a^t \|A'(u)\|_p du \right) \left( \int_a^t \|B'(u)\|_q du \right) dt \\
& - \int_a^b \left( \int_a^t \|A'(u)\|_p du \right) dt \int_a^b \left( \int_a^t \|B'(u)\|_q du \right) dt \\
& = D \left( \int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_q du \right).
\end{aligned}$$

By utilising (3.4) and (3.5), we deduce the first inequality in (3.1).

Observe that

$$0 \leq \int_a^t \|A'(u)\|_p du \leq \int_a^b \|A'(u)\|_p du$$

and

$$0 \leq \int_a^t \|B'(u)\|_q du \leq \int_a^b \|B'(u)\|_q du$$

for all  $t \in [a, b]$ , then by Grüss's inequality for the functions  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_q du$ ,  $t \in [a, b]$ , we get the last part of (3.1).  $\square$

We have the noncommutative Čebyšev's inequality:

**Corollary 1.** *Let  $A : [a, b] \rightarrow \mathcal{B}_p(H)$ ,  $B : [a, b] \rightarrow \mathcal{B}_q(H)$  for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , be strongly differentiable functions on the interval  $(a, b)$ . Then*

$$(3.6) \quad \|D(A, B)\|_1 \leq \frac{1}{12} (b-a)^4 \sup_{u \in (a,b)} \|A'(u)\|_p \sup_{u \in (a,b)} \|B'(u)\|_q,$$

provided that the right hand side in (3.6) is finite.

*Proof.* If we use Čebyšev's inequality (1.3) for  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_q du$ ,  $t \in [a, b]$ , then we get

$$\begin{aligned}
0 & \leq D \left( \int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_q du \right) \\
& \leq \frac{1}{12} (b-a)^4 \|f'\|_{\infty} \|g'\|_{\infty} \\
& = \frac{1}{12} (b-a)^4 \sup_{u \in (a,b)} \|A'(u)\|_p \sup_{u \in (a,b)} \|B'(u)\|_q,
\end{aligned}$$

which by the first inequality in (3.1) gives the desired result (3.6).  $\square$

By the use of Ostrowski's inequality (1.4) we derive:

**Corollary 2.** *Let  $A : [a, b] \rightarrow \mathcal{B}_p(H)$ ,  $B : [a, b] \rightarrow \mathcal{B}_q(H)$  for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , be strongly differentiable functions on the interval  $(a, b)$ , then*

$$(3.7) \quad \|D(A, B)\|_1 \leq \frac{1}{8} (b-a)^3 \sup_{u \in (a, b)} \|B'(u)\|_q \int_a^b \|A'(u)\|_p du.$$

By the use of Lupaş inequality for  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_q du$ ,  $t \in [a, b]$ , we get:

**Corollary 3.** *Let  $A : [a, b] \rightarrow \mathcal{B}_p(H)$ ,  $B : [a, b] \rightarrow \mathcal{B}_q(H)$  for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , be strongly differentiable functions on the interval  $(a, b)$ , then*

$$(3.8) \quad \|D(A, B)\|_1 \leq \frac{1}{\pi^2} (b-a)^3 \int_a^b \|A'(u)\|_p^2 du \int_a^b \|B'(u)\|_q^2 du.$$

We also have:

**Corollary 4.** *Let  $A : [a, b] \rightarrow \mathcal{B}_p(H)$ ,  $B : [a, b] \rightarrow \mathcal{B}_q(H)$  for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , be strongly differentiable functions on the interval  $(a, b)$ , then*

$$(3.9) \quad \|D(A, B)\|_1 \leq \frac{1}{2} \int_a^b \|B'(u)\|_q du \\ \times \left| \int_a^b \left( \int_a^t (u-a) \|A'(u)\|_p du - \int_t^b (b-u) \|A'(u)\|_p du \right) dt \right|.$$

*Proof.* Observe that for  $f(t) = \int_a^t \|A'(u)\|_p du$ , we get, integrating by parts, that

$$\begin{aligned} & \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p du - \frac{1}{b-a} \int_a^b \left( \int_a^s \|A'(u)\|_p du \right) ds \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p du - \frac{1}{b-a} \left( \left( \int_a^b \|A'(u)\|_p du \right) b - \int_a^b \|A'(s)\|_p s ds \right) \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p du - \frac{1}{b-a} \left( \int_a^b (b-u) \|A'(u)\|_p du \right) \right| dt \\ &= \frac{1}{b-a} \int_a^b \left| (b-a) \int_a^t \|A'(u)\|_p du - \int_a^b (b-u) \|A'(u)\|_p du \right| dt \\ &= \frac{1}{b-a} \int_a^b \left| \int_a^t (u-a) \|A'(u)\|_p du - \int_t^b (b-u) \|A'(u)\|_p du \right| dt. \end{aligned}$$

By utilising (1.8) for  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_q du$ ,  $t \in [a, b]$ , we then get (3.9).  $\square$

We also have:

**Corollary 5.** *With the assumptions of Corollary 4,*

$$(3.10) \quad \begin{aligned} \|D(A, B)\|_1 &\leq \int_a^b \|B'(u)\|_q du \int_a^b (b-t)(t-a) \|A'(t)\|_p dt \\ &\leq \frac{1}{4} (b-a)^2 \int_a^b \|B'(u)\|_q du \int_a^b \|A'(t)\|_p dt. \end{aligned}$$

*Proof.* We observe that

$$\begin{aligned} &\int_a^b \left| \int_a^t (u-a) \|A'(u)\|_p du - \int_t^b (b-u) \|A'(u)\|_p du \right| dt \\ &\leq \int_a^b \left[ \left| \int_a^t (u-a) \|A'(u)\|_p du \right| + \left| \int_t^b (b-u) \|A'(u)\|_p du \right| \right] dt \\ &\leq \int_a^b \left[ \int_a^t (u-a) \|A'(u)\|_p du + \int_t^b (b-u) \|A'(u)\|_p du \right] dt \\ &= \left[ \int_a^t (u-a) \|A'(u)\|_p du + \int_t^b (b-u) \|A'(u)\|_p du \right] \Big|_a^b \\ &\quad - \int_a^b t \left( (t-a) \|A'(t)\|_p - (b-t) \|A'(t)\|_p \right) dt \\ &= b \int_a^b (u-a) \|A'(u)\|_p du - a \int_a^b (b-u) \|A'(u)\|_p du \\ &\quad - \int_a^b t \left( (t-a) \|A'(t)\|_p - (b-t) \|A'(t)\|_p \right) dt \\ &= 2 \int_a^b (b-t)(t-a) \|A'(t)\|_p dt \end{aligned}$$

and by (3.9) we get (3.10).  $\square$

We also have:

**Theorem 4.** *Let  $A : [a, b] \rightarrow \mathcal{B}_p(H)$ ,  $B : [a, b] \rightarrow \mathcal{B}_q(H)$  for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , be continuous functions on the interval  $[a, b]$ , then*

$$(3.11) \quad \|D(A, B)\|_1 \leq \begin{cases} \sup_{t \in [a, b]} \|A(t) - V\|_p \int_a^b \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q dt, \\ \left( \int_a^b \|A(t) - V\|_p^\alpha dt \right)^{1/\alpha} \left( \int_a^b \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q^\beta dt \right)^{1/\beta}, \\ \text{if } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \sup_{t \in [a, b]} \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q \int_a^b \|A(t) - V\|_p dt \end{cases}$$

for all  $V \in \mathcal{B}_p(H)$ .



*Proof.* For all  $V \in \mathcal{B}(H)$  we have

$$\begin{aligned}
 & \int_a^b [A(t) - V] \left[ (b-a)B(t) - \int_a^b B(s) ds \right] dt \\
 &= \int_a^b A(t) \left[ (b-a)B(t) - \int_a^b B(s) ds \right] dt \\
 & \quad - V \int_a^b \left[ (b-a)B(t) - \int_a^b B(s) ds \right] dt \\
 &= (b-a) \int_a^b A(t)B(t) dt - \int_a^b A(t) dt \int_a^b B(s) ds \\
 & \quad - V \left[ (b-a) \int_a^b B(t) dt - (b-a) \int_a^b B(s) ds \right] = D(A, B).
 \end{aligned}$$

If we take the 1-Schatten norm, use the integral's properties and employ Hölder's inequality (2.11) for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then we get

$$\begin{aligned}
 \|D(A, B)\|_1 &\leq \int_a^b \left\| [A(t) - V] \left[ (b-a)B(t) - \int_a^b B(s) ds \right] \right\|_1 dt \\
 &\leq \int_a^b \|A(t) - V\|_p \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q dt \\
 &\leq \begin{cases} \sup_{t \in [a, b]} \|A(t) - V\|_p \int_a^b \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q dt, \\ \left( \int_a^b \|A(t) - V\|_p^\alpha dt \right)^{1/\alpha} \left( \int_a^b \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q^\beta dt \right)^{1/\beta}, \\ \int_a^b \|A(t) - V\|_p dt \sup_{t \in [a, b]} \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q \end{cases},
 \end{aligned}$$

for all  $V \in \mathcal{B}(H)$ . □

**Corollary 6.** *With the assumptions of Theorem 4 and if there exists  $V \in \mathcal{B}_p(H)$  and  $M > 0$  such that*

$$\|A(t) - V\|_p \leq M \text{ for all } t \in [a, b],$$

then

$$(3.12) \quad \|D(A, B)\|_1 \leq M \int_a^b \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q dt.$$

The proof is obvious from the first branch of (3.11).

**Corollary 7.** *Let  $A : [a, b] \rightarrow \mathcal{B}_p(H)$ ,  $B : [a, b] \rightarrow \mathcal{B}_q(H)$  for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , be continuous functions on the interval  $[a, b]$  and  $A$  strongly differentiable*

on  $(a, b)$ , then for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ ,

$$\begin{aligned}
(3.13) \quad & \|D(A, B)\|_1 \\
& \leq \sup_{t \in [a, b]} \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q \\
& \times \left[ \int_a^{\frac{a+b}{2}} (t-a) \|A'(t)\|_p dt + \int_{\frac{a+b}{2}}^b (b-t) \|A'(t)\|_p dt \right] \\
& \leq \sup_{t \in [a, b]} \left\| (b-a)B(t) - \int_a^b B(s) ds \right\|_q \\
& \times \begin{cases} \frac{1}{8} (b-a)^2 \left[ \sup_{t \in [a, \frac{a+b}{2}]} \|A'(t)\|_p + \sup_{t \in [\frac{a+b}{2}, b]} \|A'(t)\|_p \right], \\ \frac{(b-a)^{1+1/\beta}}{(\beta+1)^{1/\beta} 2^{1+1/\beta}} \\ \times \left[ \left( \int_a^{\frac{a+b}{2}} \|A'(t)\|_p^\alpha dt \right)^{1/\alpha} + \left( \int_{\frac{a+b}{2}}^b \|A'(t)\|_p^\alpha dt \right)^{1/\alpha} \right], \\ \text{if } \alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \frac{1}{2} (b-a) \int_a^b \|A'(t)\|_p dt. \end{cases}
\end{aligned}$$

*Proof.* We have for  $V = A\left(\frac{a+b}{2}\right)$  that

$$\begin{aligned}
& \int_a^b \left\| A(t) - A\left(\frac{a+b}{2}\right) \right\| dt \\
& = \int_a^b \left\| \int_{\frac{a+b}{2}}^t A'(s) ds \right\| dt \leq \int_a^b \left| \int_{\frac{a+b}{2}}^t \|A'(s)\| ds \right| dt \\
& = \int_a^{\frac{a+b}{2}} \left( \int_t^{\frac{a+b}{2}} \|A'(s)\| ds \right) dt + \int_{\frac{a+b}{2}}^b \left( \int_{\frac{a+b}{2}}^t \|A'(s)\| ds \right) dt \\
& = \left( \int_t^{\frac{a+b}{2}} \|A'(s)\| ds \right) t \Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} t \|A'(t)\| dt \\
& + \left( \int_{\frac{a+b}{2}}^t \|A'(s)\| ds \right) t \Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b t \|A'(t)\| dt \\
& = \int_a^{\frac{a+b}{2}} t \|A'(t)\| dt - a \int_a^{\frac{a+b}{2}} \|A'(s)\| ds \\
& + b \int_{\frac{a+b}{2}}^b \|A'(s)\| ds - \int_{\frac{a+b}{2}}^b t \|A'(t)\| dt \\
& = \int_a^{\frac{a+b}{2}} (t-a) \|A'(t)\| dt + \int_{\frac{a+b}{2}}^b (b-t) \|A'(t)\| dt,
\end{aligned}$$

which, by the third branch of (3.11), gives the first part of (3.13).

The last part follows by Hölder's inequality for the integrals

$$\int_a^{\frac{a+b}{2}} (t-a) \|A'(t)\| dt \text{ and } \int_{\frac{a+b}{2}}^b (b-t) \|A'(t)\| dt.$$

□

#### 4. $p$ -SCHATTEN NORM INEQUALITIES

We also have the following inequalities in terms of  $p$ -Schatten norm:

**Theorem 5.** *Let  $A, B : [a, b] \rightarrow \mathcal{B}_p(H)$  for  $p > 1$ , be strongly differentiable functions on the interval  $(a, b)$ . Then*

$$(4.1) \quad \begin{aligned} \|D(A, B)\|_p &\leq D \left( \int_a^b \|A'(u)\|_p du, \int_a^b \|B'(u)\|_p du \right) \\ &\leq \frac{1}{4} \int_a^b \|A'(u)\|_p du \int_a^b \|B'(u)\|_p du. \end{aligned}$$

*Proof.* If we take the  $p$ -Schatten norm in the Korkine's identity (3.2), use the integral's properties and the property (2.8), then we get

$$(4.2) \quad \begin{aligned} \|D(A, B)\|_p &\leq \frac{1}{2} \int_a^b \int_a^b \|[A(t) - A(s)][B(t) - B(s)]\|_p dt ds \\ &\leq \frac{1}{2} \int_a^b \int_a^b \|A(t) - A(s)\|_p \|B(t) - B(s)\|_p dt ds. \end{aligned}$$

Observe that for  $s, t \in [a, b]$

$$A(t) - A(s) = \int_s^t A'(u) du, \quad B(t) - B(s) = \int_s^t B'(u) du,$$

which implies that

$$\begin{aligned} \|A(t) - A(s)\|_p \|B(t) - B(s)\|_p &= \left\| \int_s^t A'(u) du \right\|_p \left\| \int_s^t B'(u) du \right\|_p \\ &\leq \left| \int_s^t \|A'(u)\|_p du \right| \left| \int_s^t \|B'(u)\|_p du \right| \\ &= \left| \int_s^t \|A'(u)\|_p du \int_s^t \|B'(u)\|_p du \right| \\ &= \int_s^t \|A'(u)\|_p du \int_s^t \|B'(u)\|_p du, \end{aligned}$$

for all  $s, t \in [a, b]$ .

By (4.2) we get

$$(4.3) \quad \|D(A, B)\|_p \leq \frac{1}{2} \int_a^b \int_a^b \left( \int_s^t \|A'(u)\|_p du \right) \left( \int_s^t \|B'(u)\|_p du \right) dt ds$$

for  $p > 1$ .

Since

$$\begin{aligned} & \int_s^t \|A'(u)\|_p du \int_s^t \|B'(u)\|_p du \\ &= \left( \int_a^t \|A'(u)\|_p du - \int_a^s \|A'(u)\|_p du \right) \left( \int_a^t \|B'(u)\|_p du - \int_a^s \|B'(u)\|_p du \right), \end{aligned}$$

hence by Korkine's identity for real valued functions  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_p du$ , we have

$$\begin{aligned} (4.4) \quad & \frac{1}{2} \int_a^b \int_a^b \left( \int_a^t \|A'(u)\|_p du - \int_a^s \|A'(u)\|_p du \right) \\ & \times \left( \int_a^t \|B'(u)\|_p du - \int_a^s \|B'(u)\|_p du \right) dt ds \\ &= (b-a) \int_a^b \left( \int_a^t \|A'(u)\|_p du \right) \left( \int_a^t \|B'(u)\|_p du \right) dt \\ & - \int_a^b \left( \int_a^t \|A'(u)\|_p du \right) dt \int_a^b \left( \int_a^t \|B'(u)\|_p du \right) dt \\ &= D \left( \int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_p du \right). \end{aligned}$$

By utilising (4.3) and (4.4), we deduce the first inequality in (4.1).

Observe that

$$0 \leq \int_a^t \|A'(u)\|_p du \leq \int_a^b \|A'(u)\|_p du$$

and

$$0 \leq \int_a^t \|B'(u)\|_p du \leq \int_a^b \|B'(u)\|_p du$$

for all  $t \in [a, b]$ , then by Grüss's inequality for the functions  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_p du$ ,  $t \in [a, b]$ , we get the last part of (4.1).  $\square$

We have the noncommutative Čebyšev's inequality:

**Corollary 8.** *Let  $A, B : [a, b] \rightarrow \mathcal{B}_p(H)$  for  $p > 1$ , be strongly differentiable functions on the interval  $(a, b)$ . Then*

$$(4.5) \quad \|D(A, B)\|_p \leq \frac{1}{12} (b-a)^4 \sup_{u \in (a,b)} \|A'(u)\|_p \sup_{u \in (a,b)} \|B'(u)\|_p,$$

provided that the right hand side in (4.5) is finite.

*Proof.* If we use Čebyšev's inequality (1.3) for  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_p du$ ,  $t \in [a, b]$ , then we get

$$\begin{aligned} 0 & \leq D \left( \int_a^{\cdot} \|A'(u)\|_p du, \int_a^{\cdot} \|B'(u)\|_p du \right) \\ & \leq \frac{1}{12} (b-a)^4 \|f'\|_{\infty} \|g'\|_{\infty} \\ & = \frac{1}{12} (b-a)^4 \sup_{u \in (a,b)} \|A'(u)\|_p \sup_{u \in (a,b)} \|B'(u)\|_p, \end{aligned}$$

which by the first inequality in (4.1) gives the desired result (4.5).  $\square$

By the use of Ostrowski's inequality (1.4) we derive:

**Corollary 9.** *Let  $A, B : [a, b] \rightarrow \mathcal{B}_p(H)$  for  $p > 1$ , be strongly differentiable functions on the interval  $(a, b)$ , then*

$$(4.6) \quad \|D(A, B)\|_p \leq \frac{1}{8} (b-a)^3 \sup_{u \in (a, b)} \|B'(u)\|_p \int_a^b \|A'(u)\|_p du.$$

By the use of Lupaş inequality for  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_p du$ ,  $t \in [a, b]$ , we get:

**Corollary 10.** *Let  $A, B : [a, b] \rightarrow \mathcal{B}_p(H)$  for  $p > 1$ , be strongly differentiable functions on the interval  $(a, b)$ , then*

$$(4.7) \quad \|D(A, B)\|_p \leq \frac{1}{\pi^2} (b-a)^3 \int_a^b \|A'(u)\|_p^2 du \int_a^b \|B'(u)\|_p^2 du.$$

We also have:

**Corollary 11.** *Let  $A, B : [a, b] \rightarrow \mathcal{B}_p(H)$  for  $p > 1$ , be strongly differentiable functions on the interval  $(a, b)$ , then*

$$(4.8) \quad \|D(A, B)\|_p \leq \frac{1}{2} \int_a^b \|B'(u)\|_p du \\ \times \int_a^b \left| \int_a^t (u-a) \|A'(u)\|_p du - \int_t^b (b-u) \|A'(u)\|_p du \right| dt.$$

*Proof.* Observe that for  $f(t) = \int_a^t \|A'(u)\|_p du$ , we get, integrating by parts, that

$$\begin{aligned} & \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p du - \frac{1}{b-a} \int_a^b \left( \int_a^s \|A'(u)\|_p du \right) ds \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p du - \frac{1}{b-a} \left( \left( \int_a^b \|A'(u)\|_p du \right) b - \int_a^b \|A'(s)\|_p s ds \right) \right| dt \\ &= \int_a^b \left| \int_a^t \|A'(u)\|_p du - \frac{1}{b-a} \left( \int_a^b (b-u) \|A'(u)\|_p du \right) \right| dt \\ &= \frac{1}{b-a} \int_a^b \left| (b-a) \int_a^t \|A'(u)\|_p du - \int_a^b (b-u) \|A'(u)\|_p du \right| dt \\ &= \frac{1}{b-a} \int_a^b \left| \int_a^t (u-a) \|A'(u)\|_p du - \int_t^b (b-u) \|A'(u)\|_p du \right| dt. \end{aligned}$$

By utilising (1.8) for  $f(t) = \int_a^t \|A'(u)\|_p du$  and  $g(t) = \int_a^t \|B'(u)\|_p du$ ,  $t \in [a, b]$ , we then get (4.8).  $\square$

We also have:

**Corollary 12.** *With the assumptions of Corollary 4,*

$$(4.9) \quad \begin{aligned} \|D(A, B)\|_p &\leq \int_a^b \|B'(u)\|_p du \int_a^b (b-t)(t-a) \|A'(t)\|_p dt \\ &\leq \frac{1}{4} (b-a)^2 \int_a^b \|B'(u)\|_p du \int_a^b \|A'(u)\|_p du. \end{aligned}$$

*Proof.* We observe that

$$\begin{aligned} &\int_a^b \left| \int_a^t (u-a) \|A'(u)\|_p du - \int_t^b (b-u) \|A'(u)\|_p du \right| dt \\ &\leq \int_a^b \left[ \left| \int_a^t (u-a) \|A'(u)\|_p du \right| + \left| \int_t^b (b-u) \|A'(u)\|_p du \right| \right] dt \\ &\leq \int_a^b \left[ \int_a^t (u-a) \|A'(u)\|_p du + \int_t^b (b-u) \|A'(u)\|_p du \right] dt \\ &= \left[ \int_a^t (u-a) \|A'(u)\|_p du + \int_t^b (b-u) \|A'(u)\|_p du \right] t \Big|_a^b \\ &\quad - \int_a^b t \left( (t-a) \|A'(t)\|_p - (b-t) \|A'(t)\|_p \right) dt \\ &= b \int_a^b (u-a) \|A'(u)\|_p du - a \int_a^b (b-u) \|A'(u)\|_p du \\ &\quad - \int_a^b t \left( (t-a) \|A'(t)\|_p - (b-t) \|A'(t)\|_p \right) dt \\ &= 2 \int_a^b (b-t)(t-a) \|A'(t)\|_p dt \end{aligned}$$

and by (4.8) we get (4.9).  $\square$

## 5. SOME EXAMPLES

Consider the function  $A(t) = \exp(tT)$ , where  $t \in \mathbb{R}$  and  $T \in \mathcal{B}(H)$ . Then  $A'(t) = T \exp(tT)$ , for  $t \in \mathbb{R}$  and  $T \in \mathcal{B}(H)$ . If  $T$  is invertible, then [2]

$$(5.1) \quad \int_a^b \exp(tT) dt = T^{-1} [\exp(bT) - \exp(aT)].$$

For  $T, V$  commutative, i.e.  $TV = VT$  then

$$\exp(T+V) = \exp T \exp V.$$

Consider the functions

$$A_T(t) = \exp(tT), \quad B_V(t) = \exp(tV), \quad t \in [a, b].$$

For  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , let  $T \in \mathcal{B}_p(H)$ ,  $V \in \mathcal{B}_q(H)$ . Then

$$D(A_T, B_V) := (b-a) \int_a^b \exp[t(T+V)] dt - \int_a^b \exp(tT) dt \int_a^b \exp(tV) dt.$$

By (3.1) we derive

$$(5.2) \quad \begin{aligned} & \|D(A_T, B_V)\|_1 \\ & \leq \frac{1}{4} (b-a)^2 \int_a^b \|T \exp(uT)\|_p \, du \int_a^b \|V \exp(uV)\|_q \, du. \end{aligned}$$

By the property (2.8) we derive

$$\|T \exp(uT)\|_p \leq \|T\|_p \|\exp(uT)\|$$

and

$$\|V \exp(uV)\|_q \leq \|V\|_q \|\exp(uV)\|_q$$

for  $u \in [a, b]$ .

Since  $\|\exp(tT)\|_p \leq \exp(|t| \|T\|_p)$ ,  $t \in \mathbb{R}$ ,  $T \in \mathcal{B}_p(H)$  then by (5.2) we derive

$$(5.3) \quad \begin{aligned} & \|D(A_T, B_V)\|_1 \\ & \leq \frac{1}{4} (b-a)^2 \|T\|_p \|V\|_q \int_a^b \exp(|t| \|T\|_p) \, dt \int_a^b \exp(|u| \|V\|_q) \, du. \end{aligned}$$

If  $0 \leq a \leq b$ , then

$$\begin{aligned} \int_a^b \exp(\|T\|_p |t|) \, dt &= \int_a^b \exp(\|T\|_p t) \, dt = \frac{\exp(\|T\|_p b) - \exp(\|T\|_p a)}{\|T\|_p}, \\ \int_a^b \exp(|u| \|V\|_q) \, du &= \int_a^b \exp(\|V\|_q u) \, du = \frac{\exp(\|V\|_q b) - \exp(\|V\|_q a)}{\|V\|_q} \end{aligned}$$

and by (5.3) we get

$$(5.4) \quad \begin{aligned} \|D(A_T, B_V)\|_1 &\leq \frac{1}{4} (b-a)^2 \left[ \exp(\|T\|_p b) - \exp(\|T\|_p a) \right] \\ &\quad \times \left[ \exp(\|V\|_q b) - \exp(\|V\|_q a) \right]. \end{aligned}$$

Moreover, if  $T$ ,  $V$  and  $T + V$  are invertible, then

$$\begin{aligned} D(A_T, B_V) &= (b-a) (T+V)^{-1} [\exp(b(T+V)) - \exp(a(T+V))] \\ &\quad - T^{-1} [\exp(bT) - \exp(aT)] V^{-1} [\exp(bV) - \exp(aV)] \end{aligned}$$

and by (5.4) we derive

$$(5.5) \quad \begin{aligned} & \left\| (b-a) (T+V)^{-1} [\exp(b(T+V)) - \exp(a(T+V))] \right. \\ & \quad \left. - T^{-1} [\exp(bT) - \exp(aT)] V^{-1} [\exp(bV) - \exp(aV)] \right\|_1 \\ & \leq \frac{1}{4} (b-a)^2 \left[ \exp(\|T\|_p b) - \exp(\|T\|_p a) \right] \\ & \quad \times \left[ \exp(\|V\|_q b) - \exp(\|V\|_q a) \right], \end{aligned}$$

provide that  $T \in \mathcal{B}_p(H)$ ,  $V \in \mathcal{B}_q(H)$  for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , are commutative,  $T$ ,  $V$  and  $T + V$  are invertible while  $0 \leq a \leq b$ .

In particular, for  $a = 0$  and  $b = 1$  we derive from (5.5) the simpler inequality

$$(5.6) \quad \left\| (T + V)^{-1} [\exp((T + V)) - 1] - T^{-1} [\exp(T) - 1] V^{-1} [\exp(V) - 1] \right\|_1 \\ \leq \frac{1}{4} \left[ \exp(\|T\|_p) - 1 \right] \left[ \exp(\|V\|_q) - 1 \right],$$

provide that  $T \in \mathcal{B}_p(H)$ ,  $V \in \mathcal{B}_q(H)$  for  $p, q > 1$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , are commutative, and  $T$ ,  $V$  and  $T + V$  are invertible.

Similar exponential inequalities may be derived by employing the inequalities (3.6)-(3.8) and (3.10) as well as, (4.1), (4.5)-(4.7) and (4.9).

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