p-SCHATTEN NORM INEQUALITIES OF GENERALIZED TRAPEZOID TYPE

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Abstract. An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\leq p<\infty$ if the *p-Schatten norm* is finite

$$\|A\|_p:=\left[\operatorname{tr}\left(|A|^p\right)\right]^{1/p}<\infty.$$

Assume that $A:[a,b]\to\mathcal{B}_p(H)$, $B:[a,b]\to\mathcal{B}_p(H)$, $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, are continuous and B is strongly differentiable on (a,b), then

$$\left\|\left(\int_u^b A(s)\,ds\right)B(b)+\left(\int_a^u A(s)\,ds\right)B(a)-\int_a^b A(t)B(t)\,dt\right\|_1$$
$$\leq\sup_{t\in[a,b]}\left\|B'(t)\right\|_q\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_a^b\|A(t)\|_p\,dt,\\[2ex] \frac{1}{(\beta+1)^{1/\beta}}\left[(b-u)^{\beta+1}+(u-a)^{\beta+1}\right]^{1/\beta}\\ \times\left(\int_a^b\|A(t)\|_p^\alpha\,dt\right)^{1/\alpha},\ \alpha,\beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1,\\[2ex] \left[\frac{1}{4}(b-a)+\left(u-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\|A(t)\|_p\end{cases}$$

for all $u\in[a,b]$.

Some examples of interest for the inverse and operator exponential are also given.

1. Introduction

In 2001, Dragomir et al. [10] obtained the following generalized trapezoid inequality:

Theorem 1. *If $g:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)g(t)\,dt-f(b)\int_u^b g(t)\,dt-f(a)\int_a^u g(t)\,dt\right|\leq\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f)\tag{1.1}$$

for all $u\in[a,b]$, where $\bigvee_a^b(f)$ is the total variation of f on $[a,b]$.

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In particular, we have the mid-point trapezoid inequality

$$\left|\int_a^b f(t)\,g(t)\,dt - f(b)\int_{\frac{a+b}{2}}^b g(t)\,dt - f(a)\int_a^{\frac{a+b}{2}} g(t)\,dt\right| \leq \frac{1}{2}(b-a)\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \tag{1.2}$$

The constant $1/2$ *is sharp in the sense that it cannot be replaced by a smaller quantity.*

In order to extend these results for p-Schatten norms we need the following preparations.

Let $(H;\langle .,.\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i\in I}\langle |A|\,e_i, e_i\rangle < \infty. \tag{1.3}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I}\langle Ae_i, e_i\rangle, \tag{1.4}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.4) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.5}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT, TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1\|T\|; \tag{1.6}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B\in\mathcal{B}_2(H)$ *then* $AB, BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\leq p<\infty$ if the *p-Schatten norm* is finite [14, p. 60-64]

$$\|A\|_p := [\operatorname{tr}(|A|^p)]^{1/p} < \infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{1.7}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{1.8}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [14, p. 60-64],

$$\|A\|_p = \|A^*\|_p\,,\ A \in \mathcal{B}_p(H) \tag{1.9}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p\,,\ A, B \in \mathcal{B}_p(H) \tag{1.10}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|\,,\ \|BA\|_p \leq \|B\| \|A\|_p\,,\ A \in \mathcal{B}_p(H)\,,\ B \in \mathcal{B}(H)\,. \tag{1.11}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|\,,\ A \in \mathcal{B}_p(H)\,,\ B,\ C \in \mathcal{B}(H)\,. \tag{1.12}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$(|\mathrm{tr}\,(AB)| \leq)\ \|AB\|_1 \leq \|A\|_p \|B\|_q\,,\ A \in \mathcal{B}_p(H)\,,\ B \in \mathcal{B}_q(H)\,. \tag{1.13}$$

For the theory of trace functionals and their applications the reader is referred to [13] and [14].

For some classical trace inequalities see [8], [9], and [11], which are continuations of the work of Bellman [2].

2. 1-Schatten Norm General Trapezoid Inequalities

We have the following weighted version of generalized trapezoid inequality for two functions with values in Banach algebras:

Theorem 3. *Assume that* $A : [a, b] \to \mathcal{B}_p(H)$, $B : [a, b] \to \mathcal{B}_p(H)$ *for* $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *are continuous and* B *is strongly differentiable on* (a, b), *then for all* $u \in [a, b]$

$$\left\| \left(\int_u^b A(s)\, ds \right) B(b) + \left(\int_a^u A(s)\, ds \right) B(a) - \int_a^b A(t) B(t)\, dt \right\|_1 \leq C_{p,q}(A, B, u)\,, \tag{2.1}$$

where

$$C_{p,q}(A, B, u) := \int_u^b \left(\int_u^t \|A(s)\|_p\, ds \right) \|B'(t)\|_q\, dt + \int_a^u \left(\int_t^u \|A(s)\|_p\, ds \right) \|B'(t)\|_q\, dt.$$

We also have the bounds

$$C_{p,q}(A,B,u) \leq \begin{cases} \int_u^b \|A(s)\|_p \, ds \int_u^b \|B'(t)\|_q \, dt, \\ \left[\int_u^b \left(\int_u^t \|A(s)\|_p \, ds\right)^\alpha dt\right]^{1/\alpha} \left(\int_u^b \|B'(t)\|_q^\beta \, dt\right)^{1/\beta}, \\ \int_u^b \left(\int_u^t \|A(s)\|_p \, ds\right) dt \sup_{t\in[u,b]} \|B'(t)\|_q, \end{cases} + \begin{cases} \left(\int_a^u \|A(s)\|_p \, ds\right) \int_a^u \|B'(t)\|_q \, dt, \\ \left[\int_a^u \left(\int_t^u \|A(s)\|_p \, ds\right)^\alpha dt\right]^{1/\alpha} \left(\int_a^u \|B'(t)\|_q^\beta \, dt\right)^{1/\beta}, \\ \int_a^u \left(\int_t^u \|A(s)\|_p \, ds\right) dt \sup_{t\in[a,u]} \|B'(t)\|_q, \end{cases} \tag{2.2}$$

where $\alpha,\ \beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. Let $u \in [a,b]$. Using the integration by parts formula for Bochner integral, we have

$$\begin{aligned} &\int_a^b \left(\int_a^t A(s)\,ds - \int_a^u A(s)\,ds\right) B'(t)\,dt \\ &= \left(\int_a^t A(s)\,ds - \int_a^u A(s)\,ds\right) B(t)\Big|_a^b \\ &- \int_a^b \left(\int_a^t A(s)\,ds - \int_a^u A(s)\,ds\right)' B(t)\,dt \\ &= \left(\int_a^b A(s)\,ds - \int_a^u A(s)\,ds\right) B(b) \\ &- \left(\int_a^a A(s)\,ds - \int_a^u A(s)\,ds\right) B(a) \\ &- \int_a^b \left(\int_a^t A(s)\,ds - \int_a^u A(s)\,ds\right)' B(t)\,dt \\ &= \left(\int_u^b A(s)\,ds\right) B(b) + \left(\int_a^u A(s)\,ds\right) B(a) - \int_a^b A(t)\,B(t)\,dt. \end{aligned} \tag{2.3}$$

Also,

$$
\begin{aligned}
(2.4)\qquad & \int_a^b \left(\int_a^t A(s)\,ds - \int_a^u A(s)\,ds\right) B'(t)\,dt \\
&= \int_a^u \left(\int_a^t A(s)\,ds - \int_a^u A(s)\,ds\right) B'(t)\,dt \\
&+ \int_u^b \left(\int_a^t A(s)\,ds - \int_a^u A(s)\,ds\right) B'(t)\,dt \\
&= -\int_a^u \left(\int_t^u A(s)\,ds\right) B'(t)\,dt + \int_u^b \left(\int_u^t A(s)\,ds\right) B'(t)\,dt.
\end{aligned}
$$

By utilising (2.3) and (2.4) we derive the following identity of interest

$$
\begin{aligned}
(2.5)\qquad & \left(\int_u^b A(s)\,ds\right) B(b) + \left(\int_a^u A(s)\,ds\right) B(a) - \int_a^b A(t)\,B(t)\,dt \\
&= \int_u^b \left(\int_u^t A(s)\,ds\right) B'(t)\,dt - \int_a^u \left(\int_t^u A(s)\,ds\right) B'(t)\,dt
\end{aligned}
$$

for all $u \in [a,b]$.

Taking the 1-Schatten norm in (2.5), using the Hölder's inequality (1.13) and the properties of the integral, we get

$$
\begin{aligned}
(2.6)\qquad & \left\| \left(\int_u^b A(s)\,ds\right) B(b) + \left(\int_a^u A(s)\,ds\right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_1 \\
&\le \left\| \int_u^b \left(\int_u^t A(s)\,ds\right) B'(t)\,dt \right\|_1 + \left\| \int_a^u \left(\int_t^u A(s)\,ds\right) B'(t)\,dt \right\|_1 \\
&\le \int_u^b \left\| \left(\int_u^t A(s)\,ds\right) B'(t) \right\|_1 dt + \int_a^u \left\| \left(\int_t^u A(s)\,ds\right) B'(t) \right\|_1 dt \\
&\le \int_u^b \left\| \int_u^t A(s)\,ds \right\|_p \|B'(t)\|_q\,dt + \int_a^u \left\| \int_t^u A(s)\,ds \right\|_p \|B'(t)\|_q\,dt \\
&\le \int_u^b \left(\int_u^t \|A(s)\|_p\,ds \right) \|B'(t)\|_q\,dt + \int_a^u \left(\int_t^u \left\|A(s)_p\right\|\,ds \right) \|B'(t)\|_q\,dt \\
&= C_{p,q}(A,B,u),
\end{aligned}
$$

which proves (2.1).

Using Hölder's inequality, we get for α, $\beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, that

$$\begin{aligned}
&\int_u^b \left(\int_u^t \|A(s)\|_p \, ds \right) \|B'(t)\|_q \, dt \\
&\leq \begin{cases} \sup_{t\in[u,b]} \left(\int_u^t \|A(s)\|_p \, ds \right) \int_u^b \|B'(t)\|_q \, dt, \\ \\ \left[\int_u^b \left(\int_u^t \|A(s)\|_p \, ds \right)^{\alpha} dt \right]^{1/\alpha} \left(\int_u^b \|B'(t)\|_q^{\beta} \, dt \right)^{1/\beta}, \\ \\ \int_u^b \left(\int_u^t \|A(s)\|_p \, ds \right) dt \sup_{t\in[u,b]} \|B'(t)\|_q, \end{cases} \\
&= \begin{cases} \left(\int_u^b \|A(s)\|_p \, ds \right) \int_u^b \|B'(t)\|_q \, dt, \\ \\ \left[\int_u^b \left(\int_u^t \|A(s)\|_p \, ds \right)^{\alpha} dt \right]^{1/\alpha} \left(\int_u^b \|B'(t)\|_q^{\beta} \, dt \right)^{1/\beta}, \\ \\ \int_u^b \left(\int_u^t \|A(s)\|_p \, ds \right) dt \sup_{t\in[u,b]} \|B'(t)\|_q, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^u \left(\int_t^u \|A(s)\|_p \, ds \right) \|B'(t)\|_q \, dt \\
&\leq \begin{cases} \sup_{t\in[a,u]} \left(\int_t^u \|A(s)\|_p \, ds \right) \int_a^u \|B'(t)\|_q \, dt \\ \\ \left[\int_a^u \left(\int_t^u \|A(s)\|_p \, ds \right)^{\alpha} dt \right]^{1/\alpha} \left(\int_a^u \|B'(t)\|_q^{\beta} \, dt \right)^{1/\beta} \\ \\ \int_a^u \left(\int_t^u \|A(s)\|_p \, ds \right) dt \sup_{t\in[a,u]} \|B'(t)\|_q, \end{cases} \\
&= \begin{cases} \left(\int_a^u \|A(s)\|_p \, ds \right) \int_a^u \|B'(t)\|_q \, dt, \\ \\ \left[\int_a^u \left(\int_t^u \|A(s)\|_p \, ds \right)^{\alpha} dt \right]^{1/\alpha} \left(\int_a^u \|B'(t)\|_q^{\beta} \, dt \right)^{1/\beta}, \\ \\ \int_a^u \left(\int_t^u \|A(s)\|_p \, ds \right) dt \sup_{t\in[a,u]} \|B'(t)\|_q. \end{cases}
\end{aligned}$$

By making use of (2.6) we deduce (2.2). □

Corollary 1. *With the assumptions of Theorem 3, we have*

$$\begin{aligned}(2.7)\quad &\left\| \left(\int_u^b A(s)\,ds \right) B(b) + \left(\int_a^u A(s)\,ds \right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_1 \\ &\le \int_u^b \|A(s)\|_p\,ds \int_u^b \|B'(t)\|_q\,dt + \int_a^u \|A(s)\|_p\,ds \int_a^u \|B'(t)\|_q\,dt \\ &\le \begin{cases} \max\left\{ \int_u^b \|A(s)\|_p\,ds, \int_a^u \|A(s)\|_p\,ds \right\} \int_a^b \|B'(t)\|_q\,dt \\ \\ \int_a^b \|A(s)\|_p\,ds \max\left\{ \int_u^b \|B'(t)\|_q\,dt, \int_a^u \|B'(t)\|_q\,dt \right\} \end{cases} \\ &\le \int_a^b \|A(s)\|_p\,ds \int_a^b \|B'(t)\|_q\,dt,\end{aligned}$$

for all $u \in [a, b]$.

The proof follows by the first branches in the bounds (2.2).

Remark 1. *If* $m \in (a, b)$ *is such that*

$$(2.8)\quad \int_a^u \|A(s)\|_p\,ds = \int_u^b \|A(s)\|_p\,ds = \frac{1}{2} \int_a^b \|A(s)\|_p\,ds,$$

then by (2.6) we get

$$\begin{aligned}(2.9)\quad &\left\| \left(\int_m^b A(s)\,ds \right) B(b) + \left(\int_a^m A(s)\,ds \right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_1 \\ &\le \frac{1}{2} \int_a^b \|A(s)\|_p\,ds \int_a^b \|B'(t)\|_q\,dt.\end{aligned}$$

Corollary 2. *With the assumptions of Theorem 3, we have*

$$\begin{aligned}(2.10)\quad &\left\| \left(\int_u^b A(s)\,ds \right) B(b) + \left(\int_a^u A(s)\,ds \right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_1 \\ &\le \left[\int_u^b (b-t)\,\|A(t)\|_p\,dt + \int_a^u (t-a)\,\|A(t)\|_p\,dt \right] \sup_{t\in[a,b]} \|B'(t)\|_q\end{aligned}$$

for all $u \in [a, b]$.

Proof. From the third branches in the bounds in (2.2) we have

$$\begin{aligned}(2.11)\quad &\left\| \left(\int_u^b A(s)\,ds \right) B(b) + \left(\int_a^u A(s)\,ds \right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_1 \\ &\le \int_u^b \left(\int_u^t \|A(s)\|_p\,ds \right) dt \sup_{t\in[u,b]} \|B'(t)\|_q \\ &+ \int_a^u \left(\int_t^u \|A(s)\|_p\,ds \right) dt \sup_{t\in[a,u]} \|B'(t)\|_q \\ &\le \sup_{t\in[a,b]} \|B'(t)\|_q \left[\int_u^b \left(\int_u^t \|A(s)\|_p\,ds \right) dt + \int_a^u \left(\int_t^u \|A(s)\|_p\,ds \right) dt \right].\end{aligned}$$

Using integration by parts, we have for $u \in [a,b]$ that

$$\begin{aligned}\int_u^b \left(\int_u^t \|A(s)\|_p \, ds\right) dt &= \left(\int_u^t \|A(s)\|_p \, ds\right) t\Big|_u^b - \int_u^b t \|A(t)\|_p \, dt \\ &= \left(\int_u^b \|A(s)\|_p \, ds\right) b - \int_u^b t \|A(t)\|_p \, dt \\ &= \int_u^b (b-t) \|A(t)\|_p \, dt\end{aligned}$$

and

$$\begin{aligned}\int_a^u \left(\int_t^u \|A(s)\|_p \, ds\right) dt &= \left(\int_t^u \|A(s)\|_p \, ds\right) t\Big|_a^u + \int_a^u t \|A(t)\|_p \, dt \\ &= -\left(\int_a^u \|A(s)\|_p \, ds\right) a + \int_a^u t \|A(t)\|_p \, dt \\ &= \int_a^u (t-a) \|A(t)\|_p \, dt,\end{aligned}$$

which, by (2.11) provides the desired result (2.10). □

Remark 2. *By making use of Hölder's integral inequality, we have for* α, $\beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ *that*

$$\begin{aligned}\int_u^b (b-t) \|A(t)\|_p \, dt &\le \begin{cases} \sup_{t\in[u,b]} (b-t) \int_u^b \|A(t)\|_p \, dt, \\ \left(\int_u^b (b-t)^\beta \, dt\right)^{1/\beta} \left(\int_u^b \|A(t)\|_p^\alpha \, dt\right)^{1/\alpha}, \\ \int_u^b (b-t) \, dt \sup_{t\in[u,b]} \|A(t)\|_p, \end{cases} \\ &= \begin{cases} (b-u) \int_u^b \|A(t)\|_p \, dt, \\ \frac{(b-u)^{1+1/\beta}}{(\beta+1)^{1/\beta}} \left(\int_u^b \|A(t)\|_p^\alpha \, dt\right)^{1/\alpha}, \\ \frac{1}{2}(b-u)^2 \sup_{t\in[u,b]} \|A(t)\|_p \end{cases}\end{aligned}$$

and

$$\begin{aligned}\int_a^u (t-a) \|A(t)\|_p \, dt &\le \begin{cases} \sup_{t\in[a,u]} (t-a) \int_a^u \|A(t)\|_p \, dt, \\ \left(\int_a^u (t-a)^\beta \, dt\right)^{1/\beta} \left(\int_a^u \|A(t)\|_p^\alpha \, dt\right)^{1/\alpha}, \\ \int_a^u (t-a) \, dt \sup_{t\in[a,u]} \|A(t)\|_p, \end{cases} \\ &= \begin{cases} (u-a) \int_a^u \|A(t)\|_p \, dt, \\ \frac{(u-a)^{1+1/\beta}}{(\beta+1)^{1/\beta}} \left(\int_a^u \|A(t)\|_p^\alpha \, dt\right)^{1/\alpha}, \\ \frac{1}{2}(u-a)^2 \sup_{t\in[a,u]} \|A(t)\|_p. \end{cases}\end{aligned}$$

By (2.10) we then get

$$\left\| \left(\int_u^b A(s)\,ds \right) B(b) + \left(\int_a^u A(s)\,ds \right) B(a) - \int_a^b A(t) B(t)\,dt \right\|_1 \tag{2.12}$$
$$\leq \sup_{t\in[a,b]} \|B'(t)\|_q$$
$$\times \begin{cases} \left[(b-u) \int_u^b \|A(t)\|_p\,dt + (u-a) \int_a^u \|A(t)\|_p\,dt \right], \\ \frac{1}{(\beta+1)^{1/\beta}} \left[(b-u)^{1+1/\beta} \left(\int_u^b \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha} \right. \\ \left. + (u-a)^{1+1/\beta} \left(\int_a^u \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha} \right] \\ \frac{1}{2} \left[(b-u)^2 \sup_{t\in[u,b]} \|A(t)\|_p + (u-a)^2 \sup_{t\in[a,u]} \|A(t)\|_p \right] \end{cases}$$

for all $u \in [a,b]$.

Observe that

$$(b-u) \int_u^b \|A(t)\|_p\,dt + (u-a) \int_a^u \|A(t)\|_p\,dt$$
$$\leq \max\{b-u, u-a\} \left[\int_u^b \|A(t)\|_p\,dt + \int_a^u \|A(t)\|_p\,dt \right]$$
$$= \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right] \int_a^b \|A(t)\|_p\,dt.$$

By using the elementary inequality for $a,\ b,\ c,\ d \geq 0$ and $\alpha,\ \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$,

$$(ab+cd) \leq (a^\alpha + c^\alpha)^{1/\alpha} \left(b^\beta + d^\beta \right)^{1/\beta}$$

we get

$$(b-u)^{1+1/\beta} \left(\int_u^b \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha} + (u-a)^{1+1/\beta} \left(\int_a^u \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha}$$
$$\leq \left[\left((b-u)^{1+1/\beta} \right)^\beta + \left((u-a)^{1+1/\beta} \right)^\beta \right]^{1/\beta}$$
$$\times \left[\left(\left(\int_u^b \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha} \right)^\alpha + \left(\left(\int_a^u \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha} \right)^\alpha \right]^{1/\alpha}$$
$$= \left[(b-u)^{\beta+1} + (u-a)^{\beta+1} \right]^{1/\beta} \left[\int_u^b \|A(t)\|_p^\alpha\,dt + \int_a^u \|A(t)\|_p^\alpha\,dt \right]^{1/\alpha}$$
$$= \left[(b-u)^{\beta+1} + (u-a)^{\beta+1} \right]^{1/\beta} \left(\int_a^b \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha}.$$

Also,

$$\begin{aligned}
&\frac{1}{2}\left[(b-u)^{2}\sup_{t\in[u,b]}\|A(t)\|_{p}+(u-a)^{2}\sup_{t\in[a,u]}\|A(t)\|_{p}\right]\\
&\leq\frac{1}{2}\left[(b-u)^{2}+(u-a)^{2}\right]\sup_{t\in[a,b]}\|A(t)\|_{p}\\
&=\left[\frac{1}{4}(b-a)+\left(u-\frac{a+b}{2}\right)^{2}\right]\sup_{t\in[a,b]}\|A(t)\|_{p}.
\end{aligned}$$

Then by (2.12) we get for $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *that*

$$\begin{aligned}
&\left\|\left(\int_{u}^{b}A(s)\,ds\right)B(b)+\left(\int_{a}^{u}A(s)\,ds\right)B(a)-\int_{a}^{b}A(t)B(t)\,dt\right\|_{1} \qquad (2.13)\\
&\leq\sup_{t\in[a,b]}\|B'(t)\|_{q}\\
&\times\begin{cases}
\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_{a}^{b}\|A(t)\|_{p}\,dt,\\
\frac{1}{(\beta+1)^{1/\beta}}\left[(b-u)^{\beta+1}+(u-a)^{\beta+1}\right]^{1/\beta}\left(\int_{a}^{b}\|A(t)\|_{p}^{\alpha}\,dt\right)^{1/\alpha},\\
\left[\frac{1}{4}(b-a)+\left(u-\frac{a+b}{2}\right)^{2}\right]\sup_{t\in[a,b]}\|A(t)\|_{p}
\end{cases}
\end{aligned}$$

for all $u\in[a,b]$.

We also have:

Corollary 3. *With the assumptions of Theorem 3, we have for all* $u\in[a,b]$,

$$\begin{aligned}
&\left\|\left(\int_{u}^{b}A(s)\,ds\right)B(b)+\left(\int_{a}^{u}A(s)\,ds\right)B(a)-\int_{a}^{b}A(t)B(t)\,dt\right\|_{1} \qquad (2.14)\\
&\leq\left[\left(\int_{u}^{b}\|A(s)\|_{p}\,ds\right)^{\alpha}(b-u)+\left(\int_{a}^{u}\|A(s)\|_{p}\,ds\right)^{\alpha}(u-a)\right]^{1/\alpha}\\
&\times\left(\int_{a}^{b}\|B'(t)\|_{q}^{\beta}\,dt\right)^{1/\beta}\\
&\leq(b-a)^{1/\alpha}\left[\left(\int_{u}^{b}\|A(s)\|_{p}\,ds\right)^{\alpha}+\left(\int_{a}^{u}\|A(s)\|_{p}\,ds\right)^{\alpha}\right]^{1/\alpha}\\
&\times\left(\int_{a}^{b}\|B'(t)\|_{q}^{\beta}\,dt\right)^{1/\beta}
\end{aligned}$$

for $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$

Proof. Observe that, by the elementary inequality for $a,\ b,\ c,\ d\geq0$ and $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$,

$$(ab+cd)\leq(a^{\alpha}+c^{\alpha})^{1/\alpha}\left(b^{\beta}+d^{\beta}\right)^{1/\beta},$$

we have

$$
\begin{aligned}
&\left[\int_u^b\left(\int_u^t\|A(s)\|_p\,ds\right)^\alpha dt\right]^{1/\alpha}\left(\int_u^b\|B'(t)\|_q^\beta\,dt\right)^{1/\beta}\\
&+\left[\int_a^u\left(\int_t^u\|A(s)\|_p\,ds\right)^\alpha dt\right]^{1/\alpha}\left(\int_a^u\|B'(t)\|_q^\beta\,dt\right)^{1/\beta}\\
&\le\left(\int_u^b\left(\int_u^t\|A(s)\|_p\,ds\right)^\alpha dt+\int_a^u\left(\int_t^u\|A(s)\|_p\,ds\right)^\alpha dt\right)^{1/\alpha}\\
&\times\left(\int_u^b\|B'(t)\|_q^\beta\,dt+\int_a^u\|B'(t)\|_q^\beta\,dt\right)^{1/\beta}\\
&=\left(\int_u^b\left(\int_u^t\|A(s)\|_p\,ds\right)^\alpha dt+\int_a^u\left(\int_t^u\|A(s)\|_p\,ds\right)^\alpha dt\right)^{1/\alpha}\\
&\times\left(\int_a^b\|B'(t)\|_q^\beta\,dt\right)^{1/\beta}
\end{aligned}
$$

$$
\begin{aligned}
&\le\left(\left(\int_u^b\|A(s)\|_p\,ds\right)^\alpha\int_u^b dt+\left(\int_a^u\|A(s)\|_p\,ds\right)^\alpha\int_a^u dt\right)^{1/\alpha}\\
&\times\left(\int_a^b\|B'(t)\|_q^\beta\,dt\right)^{1/\beta}\\
&=\left(\left(\int_u^b\|A(s)\|_p\,ds\right)^\alpha(b-u)+\left(\int_a^u\|A(s)\|_p\,ds\right)^\alpha(u-a)\right)^{1/\alpha}\\
&\times\left(\int_a^b\|B'(t)\|_q^\beta\,dt\right)^{1/\beta}\\
&\le(b-a)^{1/\alpha}\left(\left(\int_u^b\|A(s)\|_p\,ds\right)^\alpha+\left(\int_a^u\|A(s)\|_p\,ds\right)^\alpha\right)^{1/\alpha}\\
&\times\left(\int_a^b\|B'(t)\|_q^\beta\,dt\right)^{1/\beta},
\end{aligned}
$$

which proves (2.14). □

Remark 3. *If $m\in(a,b)$ is such that (2.8) is valid, then by (2.14) we get*

$$
\begin{aligned}
&\left\|\left(\int_m^b A(s)\,ds\right)B(b)+\left(\int_a^m A(s)\,ds\right)B(a)-\int_a^b A(t)B(t)\,dt\right\|_1\\
&\le\frac{1}{2}(b-a)^{1/\alpha}\int_a^b\|A(s)\|_p\,ds\left(\int_a^b\|B'(t)\|_q^\beta\,dt\right)^{1/\beta}.
\end{aligned}
\tag{2.15}
$$

Assume that $A,\, B : [a,b] \to \mathcal{B}$, are continuous and B is strongly differentiable on (a,b), then

$$\left\| \left(\int_{\frac{a+b}{2}}^{b} A(s)\,ds \right) B(b) + \left(\int_{a}^{\frac{a+b}{2}} A(s)\,ds \right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_1 \leq C_{p,q}(A,B), \tag{2.16}$$

where

$$C_{p,q}(A,B) := \int_{\frac{a+b}{2}}^{b} \left(\int_{\frac{a+b}{2}}^{t} \|A(s)\|_p\,ds \right) \|B'(t)\|_q\,dt + \int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} \|A(s)\|_p\,ds \right) \|B'(t)\|_q\,dt.$$

We also have the bounds

$$C_{p,q}(A,B) \leq \begin{cases} \int_{\frac{a+b}{2}}^{b} \|A(s)\|_p\,ds \int_{\frac{a+b}{2}}^{b} \|B'(t)\|_q\,dt, \\ \left[\int_u^b \left(\int_{\frac{a+b}{2}}^{t} \|A(s)\|_p\,ds \right)^{\alpha} dt \right]^{1/\alpha} \left(\int_{\frac{a+b}{2}}^{b} \|B'(t)\|_q^{\beta}\,dt \right)^{1/\beta}, \\ \int_{\frac{a+b}{2}}^{b} \left(\int_{\frac{a+b}{2}}^{t} \|A(s)\|_p\,ds \right) dt \sup_{t\in\left[\frac{a+b}{2},b\right]} \|B'(t)\|_q, \end{cases} + \begin{cases} \left(\int_a^{\frac{a+b}{2}} \|A(s)\|_p\,ds \right) \int_a^{\frac{a+b}{2}} \|B'(t)\|_q\,dt, \\ \left[\int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} \|A(s)\|_p\,ds \right)^{\alpha} dt \right]^{1/\alpha} \left(\int_a^{\frac{a+b}{2}} \|B'(t)\|_q^{\beta}\,dt \right)^{1/\beta}, \\ \int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} \|A(s)\|_p\,ds \right) dt \sup_{t\in\left[a,\frac{a+b}{2}\right]} \|B'(t)\|_q, \end{cases} \tag{2.17}$$

where $\alpha,\, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

From (2.7) we get

$$\begin{aligned} &\left\| \left(\int_{\frac{a+b}{2}}^{b} A(s)\,ds \right) B(b) + \left(\int_{a}^{\frac{a+b}{2}} A(s)\,ds \right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_1 \\ &\leq \int_{\frac{a+b}{2}}^{b} \|A(s)\|_p\,ds \int_{\frac{a+b}{2}}^{b} \|B'(t)\|_q\,dt + \int_a^{\frac{a+b}{2}} \|A(s)\|_p\,ds \int_a^{\frac{a+b}{2}} \|B'(t)\|_q\,dt \\ &\leq \begin{cases} \max\left\{ \int_{\frac{a+b}{2}}^{b} \|A(s)\|_p\,ds, \int_a^{\frac{a+b}{2}} \|A(s)\|_p\,ds \right\} \int_a^b \|B'(t)\|_q\,dt \\ \int_a^b \|A(s)\|_p\,ds \max\left\{ \int_{\frac{a+b}{2}}^{b} \|B'(t)\|_q\,dt, \int_a^{\frac{a+b}{2}} \|B'(t)\|_q\,dt \right\} \end{cases} \\ &\leq \int_a^b \|A(s)\|_p\,ds \int_a^b \|B'(t)\|_q\,dt, \end{aligned} \tag{2.18}$$

while from (2.13) we get

$$\left\| \left(\int_{\frac{a+b}{2}}^{b} A(s)\,ds \right) B(b) + \left(\int_{a}^{\frac{a+b}{2}} A(s)\,ds \right) B(a) - \int_{a}^{b} A(t) B(t)\,dt \right\|_1 \tag{2.19}$$

$$\leq \sup_{t\in[a,b]} \left\| B'(t) \right\|_q \times \begin{cases} \frac{1}{2}(b-a) \int_a^b \left\| A(t) \right\|_p dt, \\ \\ \frac{1}{2(\beta+1)^{1/\beta}} (b-a)^{1+1/\beta} \left(\int_a^b \left\| A(t) \right\|_p^{\alpha} dt \right)^{1/\alpha}, \\ \\ \frac{1}{4}(b-a) \sup_{t\in[a,b]} \left\| A(t) \right\|_p . \end{cases}$$

From (2.14) we also get

$$\left\| \left(\int_{\frac{a+b}{2}}^{b} A(s)\,ds \right) B(b) + \left(\int_{a}^{\frac{a+b}{2}} A(s)\,ds \right) B(a) - \int_{a}^{b} A(t) B(t)\,dt \right\|_1 \tag{2.20}$$

$$\leq \frac{(b-a)^{1/\alpha}}{2^{1/\alpha}} \left[\left(\int_{\frac{a+b}{2}}^{b} \left\| A(s) \right\|_p ds \right)^{\alpha} + \left(\int_{a}^{\frac{a+b}{2}} \left\| A(s) \right\|_p ds \right)^{\alpha} \right]^{1/\alpha} \times \left(\int_a^b \left\| B'(t) \right\|_q^{\beta} dt \right)^{1/\beta}.$$

If we consider the case when $A(t) = A$, $t \in [a,b]$, then by (2.1) we get for all $u \in [a,b]$,

$$\left\| A \left[(b-u) B(b) + (u-a) B(a) - \int_a^b B(t)\,dt \right] \right\|_1 \leq \left\| A \right\|_p C_q(B,u), \tag{2.21}$$

where

$$C_q(B,u) := \int_u^b (t-u) \left\| B'(t) \right\|_q dt + \int_a^u (u-t) \left\| B'(t) \right\|_q dt.$$

We also have the bounds

$$C_q(B,u) \leq \begin{cases} (b-u) \int_u^b \left\| B'(t) \right\|_q dt, \\ \\ \frac{(b-u)^{1+1/\alpha}}{(\alpha+1)^{1/\alpha}} \left(\int_u^b \left\| B'(t) \right\|_q^{\beta} dt \right)^{1/\beta}, \\ \\ \frac{1}{2}(b-u)^2 \sup_{t\in[u,b]} \left\| B'(t) \right\|_q , \end{cases} + \begin{cases} (u-a) \int_a^u \left\| B'(t) \right\|_q dt, \\ \\ \frac{(u-a)^{1+1/\alpha}}{(\alpha+1)^{1/\alpha}} \left(\int_a^u \left\| B'(t) \right\|_q^{\beta} dt \right)^{1/\beta}, \\ \\ \frac{1}{2}(u-a)^2 \sup_{t\in[a,u]} \left\| B'(t) \right\|_q , \end{cases} \tag{2.22}$$

for all $u \in [a,b]$, where $\alpha,\ \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

From (2.7) we get

$$\left\|(b-u)B(b)+(u-a)B(a)-\int_a^b B(t)\,dt\right\| \le (b-u)\int_u^b \|B'(t)\|\,dt+(u-a)\int_a^u \|B'(t)\|\,dt \le \left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_a^b \|B'(t)\|\,dt \tag{2.23}$$

for all $u\in[a,b]$.

From (2.14) we get

$$\left\|A\left[(b-u)B(b)+(u-a)B(a)-\int_a^b B(t)\,dt\right]\right\|_1 \le \left[(b-u)^{\alpha+1}+(u-a)^{\alpha+1}\right]^{1/\alpha}\|A\|_p\left(\int_a^b \|B'(t)\|_q^\beta\,dt\right)^{1/\beta} \tag{2.24}$$

for all $u\in[a,b]$.

3. p-Schatten Norm General Trapezoid Inequalities

We also have the p-Schatten norm inequalities:

Theorem 4. *Assume that A, $B:[a,b]\to\mathcal{B}_p(H)$ for $p>1$, are continuous and B is strongly differentiable on (a,b), then for all $u\in[a,b]$*

$$\left\|\left(\int_u^b A(s)\,ds\right)B(b)+\left(\int_a^u A(s)\,ds\right)B(a)-\int_a^b A(t)B(t)\,dt\right\|_p \le C_p(A,B,u), \tag{3.1}$$

where

$$C_p(A,B,u) := \int_u^b\left(\int_u^t \|A(s)\|_p\,ds\right)\|B'(t)\|_p\,dt+\int_a^u\left(\int_t^u \|A(s)\|_p\,ds\right)\|B'(t)\|_p\,dt.$$

We also have the bounds

$$C_p(A,B,u) \leq \begin{cases} \int_u^b \|A(s)\|_p ds \int_u^b \|B'(t)\|_p dt, \\ \left[\int_u^b \left(\int_u^t \|A(s)\|_p ds\right)^\alpha dt\right]^{1/\alpha} \left(\int_u^b \|B'(t)\|_p^\beta dt\right)^{1/\beta}, \\ \int_u^b \left(\int_u^t \|A(s)\|_p ds\right) dt \sup_{t\in[u,b]} \|B'(t)\|_p, \end{cases} + \begin{cases} \left(\int_a^u \|A(s)\|_p ds\right) \int_a^u \|B'(t)\|_p dt, \\ \left[\int_a^u \left(\int_t^u \|A(s)\|_p ds\right)^\alpha dt\right]^{1/\alpha} \left(\int_a^u \|B'(t)\|_p^\beta dt\right)^{1/\beta}, \\ \int_a^u \left(\int_t^u \|A(s)\|_p ds\right) dt \sup_{t\in[a,u]} \|B'(t)\|_p, \end{cases} \tag{3.2}$$

where α, $\beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Proof. Let $u \in [a,b]$. Taking the p-Schatten norm in (2.5), using the inequality (1.10) and the properties of the integral, we get

$$\begin{aligned} &\left\| \left(\int_u^b A(s)\,ds\right) B(b) + \left(\int_a^u A(s)\,ds\right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_p \\ &\leq \left\| \int_u^b \left(\int_u^t A(s)\,ds\right) B'(t)\,dt \right\|_p + \left\| \int_a^u \left(\int_t^u A(s)\,ds\right) B'(t)\,dt \right\|_p \\ &\leq \int_u^b \left\| \left(\int_u^t A(s)\,ds\right) B'(t) \right\|_p dt + \int_a^u \left\| \left(\int_t^u A(s)\,ds\right) B'(t) \right\|_p dt \\ &\leq \int_u^b \left\| \int_u^t A(s)\,ds \right\|_p \|B'(t)\|_p\,dt + \int_a^u \left\| \int_t^u A(s)\,ds \right\|_p \|B'(t)\|_p\,dt \\ &\leq \int_u^b \left(\int_u^t \|A(s)\|_p\,ds\right) \|B'(t)\|_p\,dt + \int_a^u \left(\int_t^u \left\|A(s)_p\right\| ds\right) \|B'(t)\|_p\,dt \\ &= C_p(A,B,u), \end{aligned}$$

which proves (3.2).

The rest follows as in the proof of Theorem 3 and we omit the details. □

We can also state the following particular inequalities of interest:

Corollary 4. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
(3.3)\qquad & \left\| \left(\int_u^b A(s)\,ds\right) B(b) + \left(\int_a^u A(s)\,ds\right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_p \\
&\le \int_u^b \|A(s)\|_p\,ds \int_u^b \|B'(t)\|_p\,dt + \int_a^u \|A(s)\|_p\,ds \int_a^u \|B'(t)\|_p\,dt \\
&\le \begin{cases} \max\left\{\int_u^b \|A(s)\|_p\,ds, \int_a^u \|A(s)\|_p\,ds\right\} \int_a^b \|B'(t)\|_p\,dt \\ \\ \int_a^b \|A(s)\|_p\,ds \max\left\{\int_u^b \|B'(t)\|_p\,dt, \int_a^u \|B'(t)\|_p\,dt\right\} \end{cases} \\
&\le \int_a^b \|A(s)\|_p\,ds \int_a^b \|B'(t)\|_p\,dt,
\end{aligned}$$

for all $u \in [a, b]$

and

Corollary 5. *With the assumptions of Theorem 4, we have for all* $u \in [a, b]$,

$$\begin{aligned}
(3.4)\qquad & \left\| \left(\int_u^b A(s)\,ds\right) B(b) + \left(\int_a^u A(s)\,ds\right) B(a) - \int_a^b A(t)\,B(t)\,dt \right\|_p \\
&\le \left[\left(\int_u^b \|A(s)\|_p\,ds\right)^\alpha (b-u) + \left(\int_a^u \|A(s)\|_p\,ds\right)^\alpha (u-a) \right]^{1/\alpha} \\
&\times \left(\int_a^b \|B'(t)\|_p^\beta\,dt\right)^{1/\beta} \\
&\le (b-a)^{1/\alpha} \left[\left(\int_u^b \|A(s)\|_p\,ds\right)^\alpha + \left(\int_a^u \|A(s)\|_p\,ds\right)^\alpha \right]^{1/\alpha} \\
&\times \left(\int_a^b \|B'(t)\|_{pq}^\beta\,dt\right)^{1/\beta}
\end{aligned}$$

for α, $\beta > 1$ *with* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

4. Some Examples

Consider the function $B(t) = \exp(tT)$, where $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. Then $B'(t) = T\exp(tT)$, for $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. By utilising (2.19) we get for p,

$q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned}
(4.1)\qquad & \left\| \left(\int_{\frac{a+b}{2}}^{b} A(s)\,ds \right) \exp(bT) + \left(\int_{a}^{\frac{a+b}{2}} A(s)\,ds \right) \exp(aT) \right. \\
& \left. - \int_{a}^{b} A(t)\exp(tT)\,dt_1 \right\|_1 \\
& \leq \sup_{t\in[a,b]} \left\| T\exp(tT) \right\|_q \times \begin{cases} \frac{1}{2}(b-a)\int_a^b \|A(t)\|_p\,dt, \\ \frac{(b-a)^{1+1/\beta}}{2(\beta+1)^{1/\beta}} \left(\int_a^b \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha}, \\ \frac{1}{4}(b-a)^2 \sup_{t\in[a,b]} \|A(t)\|_p \end{cases} \\
& \leq \|T\|_q \sup_{t\in[a,b]} \left\| \exp(tT) \right\|_q \times \begin{cases} \frac{1}{2}(b-a)\int_a^b \|A(t)\|_p\,dt, \\ \frac{(b-a)^{1+1/\beta}}{2(\beta+1)^{1/\beta}} \left(\int_a^b \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha}, \\ \frac{1}{4}(b-a)^2 \sup_{t\in[a,b]} \|A(t)\|_p, \end{cases}
\end{aligned}$$

provided that $A:[a,b]\to\mathcal{B}_p(H)$ and $T\in\mathcal{B}_q(H)$.

Since

$$\begin{aligned}
\|\exp(tT)\|_q &\leq \exp\left[|t|\,\|T\|_q\right] \leq \max\left\{ \exp\left[|a|\,\|T\|_q\right], \exp\left[|b|\,\|T\|_q\right] \right\} \\
&= \exp\left[\max\{|a|,|b|\}\,\|T\|_q \right],
\end{aligned}$$

then by (4.1) we derive

$$\begin{aligned}
(4.2)\qquad & \left\| \left(\int_{\frac{a+b}{2}}^{b} A(s)\,ds \right) \exp(bT) + \left(\int_{a}^{\frac{a+b}{2}} A(s)\,ds \right) \exp(aT) \right. \\
& \left. - \int_{a}^{b} A(t)\exp(tT)\,dt_1 \right\|_1 \\
& \leq \exp\left[\max\{|a|,|b|\}\,\|T\|_q \right] \times \begin{cases} \frac{1}{2}(b-a)\int_a^b \|A(t)\|_p\,dt, \\ \frac{(b-a)^{1+1/\beta}}{2(\beta+1)^{1/\beta}} \left(\int_a^b \|A(t)\|_p^\alpha\,dt \right)^{1/\alpha}, \\ \frac{1}{4}(b-a)^2 \sup_{t\in[a,b]} \|A(t)\|_p \end{cases}
\end{aligned}$$

provided that $A:[a,b]\to\mathcal{B}_p(H)$ and $T\in\mathcal{B}_q(H)$.

We use the follow inequality that follows by (2.19),

$$\left\| \left(\int_{\frac{1}{2}}^{1} A(s)\,ds \right) B(1) + \left(\int_{0}^{\frac{1}{2}} A(s)\,ds \right) B(0) - \int_{0}^{1} A(t)\,B(t)\,dt \right\|_{1} \tag{4.3}$$
$$\leq \sup_{t\in[0,1]} \left\| B'(t) \right\|_{q} \times \begin{cases} \frac{1}{2}\int_{0}^{1} \left\| A(t) \right\|_{p} dt, \\ \\ \frac{1}{2(\beta+1)^{1/\beta}} \left(\int_{0}^{1} \left\| A(t) \right\|_{p}^{\alpha} dt \right)^{1/\alpha}, \\ \\ \frac{1}{4}\sup_{t\in[0,1]} \left\| A(t) \right\|_{p}, \end{cases}$$

provided that $A:[0,1]\to\mathcal{B}_{p}(H)$, $B:[0,1]\to\mathcal{B}_{q}(H)$, $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, are continuous and B is strongly differentiable on $(0,1)$.

Further, let $C,\ D\in\mathcal{B}(H)$ such that $(1-t)\,C+tD$ is invertible for all $t\in[0,1]$. For this to happen, it is enough to assume that $C,\ D>0$ in the operator order of $\mathcal{B}(H)$. Consider the function $B(t):=((1-t)\,C+tD)^{-1}$, $t\in[0,1]$ and observe that

$$B'(t) = -\left((1-t)\,C+tD\right)^{-1}(D-C)\left((1-t)\,C+tD\right)^{-1},\ t\in[0,1].$$

If $D,\ C\in\mathcal{B}_{p}(H)$ then by utilising property (1.12) we get

$$\begin{aligned} \left\| B'(t) \right\|_{q} &\leq \left\| \left((1-t)\,C+tD\right)^{-1}(D-C)\left((1-t)\,C+tD\right)^{-1} \right\|_{q} \\ &\leq \left\| D-C \right\|_{q} \left\| \left((1-t)\,C+tD\right)^{-1} \right\|^{2} \end{aligned}$$

for $t\in(0,1)$.

By (4.3) we derive

$$\begin{aligned} &\left\| \left(\int_{\frac{1}{2}}^{1} A(s)\,ds \right) D^{-1} + \left(\int_{0}^{\frac{1}{2}} A(s)\,ds \right) C^{-1} \right. \\ &\left. - \int_{0}^{1} A(t)\left((1-t)\,C+tD\right)^{-1} dt \right\|_{1} \\ &\leq \left\| D-C \right\|_{q} \sup_{t\in[0,1]} \left\| \left((1-t)\,C+tD\right)^{-1} \right\|^{2} \\ &\times \begin{cases} \frac{1}{2}\int_{0}^{1} \left\| A(t) \right\|_{p} dt, \\ \\ \frac{1}{2(\beta+1)^{1/\beta}} \left(\int_{0}^{1} \left\| A(t) \right\|_{p}^{\alpha} dt \right)^{1/\alpha}, \\ \\ \frac{1}{4}\sup_{t\in[0,1]} \left\| A(t) \right\|_{p}, \end{cases} \end{aligned} \tag{4.4}$$

provided that $A:[0,1]\to\mathcal{B}_{p}(H)$ and $D,\ C\in\mathcal{B}_{p}(H)$.

Now, if $C,\ D\geq m>0$ with $D,\ C\in\mathcal{B}_{p}(H)$, then by (4.4) we derive

$$\begin{aligned} &\left\| \left(\int_{\frac{1}{2}}^{1} A(s)\,ds \right) D^{-1} + \left(\int_{0}^{\frac{1}{2}} A(s)\,ds \right) C^{-1} \right. \\ &\left. - \int_{0}^{1} A(t)\left((1-t)\,C+tD\right)^{-1} dt \right\|_{1} \end{aligned} \tag{4.5}$$

$$\leq \frac{1}{m^2} \left\| D - C \right\|_q \times \begin{cases} \frac{1}{2} \int_0^1 \left\| A(t) \right\|_p dt, \\ \\ \frac{1}{2(\beta+1)^{1/\beta}} \left(\int_0^1 \left\| A(t) \right\|_p^\alpha dt \right)^{1/\alpha}, \\ \\ \frac{1}{4} \sup_{t \in [0,1]} \left\| A(t) \right\|_p, \end{cases}$$

provided that $A : [0,1] \to \mathcal{B}_p(H)$ and $D, C \in \mathcal{B}_p(H)$.

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